2004 Mathematics

Advanced Higher

Finalised Marking Instructions
1. (a) \[ f(x) = \cos^2 x e^{\tan x} \]
\[ f'(x) = 2(-\sin x) \cos x e^{\tan x} + \cos^2 x \sec^2 x e^{\tan x} \]
1 for Product Rule
2 for accuracy
\[ f'\left(\frac{\pi}{4}\right) = \left(1 - \sin \frac{\pi}{2}\right) e^{\tan \pi/4} = 0. \]
1
(b) \[ g(x) = \frac{\tan^{-1}2x}{1 + 4x^2} \]
\[ g'(x) = \frac{\frac{2}{1+4x^2}(1 + 4x^2) - \tan^{-1}2x(8x)}{(1 + 4x^2)^2} \]
1 for Product Rule
2 for accuracy
\[ = \frac{2 - 8x \tan^{-1}2x}{(1 + 4x^2)^2} \]

2. \[ (a^2 - 3)^4 = (a^2)^4 + 4(a^2)^3(-3) + 6(a^2)^2(-3)^2 + 4(a^2)(-3)^3 + (-3)^4 \]
1 for binomial coefficients
\[ = a^8 - 12a^6 + 54a^4 - 108a^2 + 81 \]
1 for powers
1 for coefficients

3. \[ x = 5 \cos \theta \Rightarrow \frac{dx}{d\theta} = -5 \sin \theta \]
\[ y = 5 \sin \theta \Rightarrow \frac{dy}{d\theta} = 5 \cos \theta \]
1
\[ \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5 \cos \theta}{-5 \sin \theta} \]
1
When \[ \theta = \frac{\pi}{4}, \frac{dy}{dx} = -\frac{\frac{5}{\sqrt{2}}}{\frac{5}{\sqrt{2}}} = -1, \]
1
\[ x = \frac{5}{\sqrt{2}}, y = \frac{5}{\sqrt{2}} \]
1
so an equation of the tangent is
\[ y - \frac{5}{\sqrt{2}} = -\left(x - \frac{5}{\sqrt{2}}\right) \]
1
i.e. \[ x + y = 5\sqrt{2}. \]

page 2
4.

\[ z^2(z + 3) = (1 + 4i - 4)(1 + 2i + 3) \]
\[ = (-3 + 4i)(4 + 2i) \]
\[ = -20 + 10i \]

\[ z^3 + 3z^2 - 5z + 25 = z^2(z + 3) - 5z + 25 \]
\[ = -20 + 10i - 5 - 10i + 25 = 0 \]

Note: direct substitution of 1 + 2i into \( z^3 + 3z^2 - 5z + 25 \) was acceptable.

Another root is the conjugate of \( z \), i.e. \( 1 - 2i \).

The corresponding quadratic factor is \( (z - 1)^2 + 4 = z^2 - 2z + 5 \).

\[ z^3 + 3z^2 - 5z + 25 = (z^2 - 2z + 5)(z + 5) \]
\[ z = -5 \]

Note: any valid method was acceptable.

5.

\[ \frac{1}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2} \]
\[ = \frac{1}{5(x - 3)} - \frac{1}{5(x + 2)} \]

\[ \int_0^1 \frac{1}{x^2 - x - 6} \, dx = \frac{1}{5} \int_0^1 \frac{1}{|x - 3|} - \frac{1}{|x + 2|} \, dx \]

\[ = \frac{1}{5} \left[ \ln |x - 3| - \ln |x + 2| \right]_0^1 \]
\[ = \frac{1}{5} \left[ \ln 2 - \ln 3 \right] \]
\[ = \frac{1}{5} \ln \frac{2}{3} \approx -0.162 \]

6.

\[ M_1 = \begin{pmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ M_2M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \]

The transformation represented by \( M_2M_1 \) is reflection in \( y = -x \).
7. \[ f(x) = e^x \sin x \quad f(0) = 0 \]
\[ f'(x) = e^x \sin x + e^x \cos x \quad f'(0) = 1 \]
\[ f''(x) = e^x \sin x + e^x \cos x - e^x \sin x + e^x \cos x \quad f''(0) = 2 \]
\[ = 2e^x \cos x \]
\[ f'''(x) = 2e^x \cos x - 2e^x \sin x \]
\[ f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots \]
\[ e^x \sin x = x + x^2 + \frac{1}{3!}x^3 - \ldots \]

OR
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \]
\[ \sin x = x - \frac{x^3}{3!} + \ldots \]
\[ e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \right) \]
\[ = x - \frac{x^3}{6} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \ldots \]
\[ = x + x^2 + \frac{x^3}{3} - \ldots \]

8. \[ 231 = 13 \times 17 + 10 \quad 1 \text{ for method} \]
\[ 17 = 1 \times 10 + 7 \]
\[ 10 = 1 \times 7 + 3 \]
\[ 7 = 2 \times 3 + 1 \]
Thus the highest common factor is 1.
\[ 1 = 7 - 2 \times 3 \]
\[ = 7 - 2 \times (10 - 7) = 3 \times 7 - 2 \times 10 \quad 1 \text{ for method} \]
\[ = 3 \times (17 - 10) - 2 \times 10 = 3 \times 17 - 5 \times 10 \]
\[ = 3 \times 17 - 5 \times (231 - 13 \times 17) = 68 \times 17 - 5 \times 231. \quad 1 \]
So \(x = -5\) and \(y = 68\).

9. \[ x = (u - 1)^2 \Rightarrow dx = 2(u - 1)du \quad 1 \]
\[ \int \frac{1}{(1 + \sqrt{x})} \, dx = \int \frac{2(u - 1)}{u^3} \, du \quad 1 \]
\[ = 2 \int (u^2 - u^{-3}) \, du \quad 1 \]
\[ = 2 \left(-\frac{1}{u} + \frac{1}{2u^2}\right) + c \quad 1 \]
\[ = \left(\frac{1}{(1 + \sqrt{x})^2} - \frac{2}{(1 + \sqrt{x})}\right) + c \quad 1 \]
10. \( f(x) = x^4 \sin 2x \) so

\[
\begin{align*}
  f(-x) &= (-x)^4 \sin (-2x) \\
  &= -x^4 \sin 2x \\
  &= -f(x)
\end{align*}
\]

So \( f(x) = x^4 \sin 2x \) is an odd function.

Note: a sketch given with a comment and correct answer, give full marks. A sketch without a comment, gets a maximum of two marks.

11. \[ V = \int_a^b \pi y^2 \, dx \]

\[
= \pi \int_0^1 e^{-4x} \, dx \\
= \pi \left[ -\frac{e^{-4x}}{4} \right]_0^1 \\
= \pi \left[ -\frac{1}{4e^4} + \frac{1}{4} \right] \\
= \frac{\pi}{4} \left[ 1 - \frac{1}{e^4} \right] \approx 0.7706
\]

12. LHS = \( \frac{d}{dx} (xe^x) = xe^x + 1e^x = (x + 1)e^x \)

RHS = \( (x + 1)e^x \)

So true when \( n = 1 \).

Assume \( \frac{d^k}{dx^k} (xe^x) = (x + k)e^x \)

Consider \[
\frac{d^{k+1}}{dx^{k+1}} (xe^x) = \frac{d}{dx} \left( \frac{d^k}{dx^k} (xe^x) \right) \\
= \frac{d}{dx} ((x + k)e^x) \\
= e^x + (x + k)e^x \\
= (x + (k + 1))e^x
\]

So true for \( k \) means it is true for \( (k + 1) \), therefore it is true for all integers \( n \geq 1 \).
13. (a) 
\[ y = \frac{x - 3}{x + 2} = 1 - \frac{5}{x + 2} \]
Vertical asymptote is \( x = -2 \).
Horizontal asymptote is \( y = 1 \).

(b) 
\[ \frac{dy}{dx} = \frac{5}{(x + 2)^2} \]
\( \neq 0 \)

(c) 
\[ \frac{d^2y}{dx^2} = \frac{-10}{(x + 2)^3} \neq 0 \]
So there are no points of inflexion.

(d)

The asymptotes are \( x = 1 \) and \( y = -2 \).
The domain must exclude \( x = 1 \).

*Note: candidates are not required to obtain a formula for \( f^{-1} \).*

14. (a) 
\[ \overrightarrow{AB} = -i + 2j - 4k, \overrightarrow{AC} = 0i + j - 3k \]
\[ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & 2 & -4 \\ 0 & 1 & -3 \end{vmatrix} = -2i - 3j - k \]
\( \neq 0 \) for method
1 for accuracy

\(-2x - 3y - z = c (= -2 + 0 - 3 = -5)\)
i.e. an equation for \( \pi_1 \) is \( 2x + 3y + z = 5 \).

Let an angle be \( \theta \), then
\[ \cos \theta = \frac{(2i + 3j + k) \cdot (i + j - k)}{\sqrt{4 + 9 + 1} \sqrt{1 + 1 + 1}} \]
\[ = \frac{2 + 3 - 1}{\sqrt{14} \times 3} \]
\[ = \frac{4}{\sqrt{42}} \]
\[ \theta = 51.9^\circ \]

*Note: an acute angle is required.*
Let \( \frac{x - 11}{4} = \frac{y - 15}{5} = \frac{z - 12}{2} = t. \)

Then \( x = 4t + 11; y = 5t + 15; z = 2t + 12 \)

\[
(4t + 11) + (5t + 15) - (2t + 12) = 0
\]

\( 7t = -14 \Rightarrow t = -2 \)

\( x = 3; y = 5 \) and \( z = 8. \)

15.

(a) \[
x \frac{dy}{dx} - 3y = x^4
\]

\[
\frac{dy}{dx} - \frac{3}{x} y = x^3
\]

Integrating factor is \( e^{\int \frac{-3}{x} dx} = x^{-3}. \)

\[
\frac{1}{x^3} \frac{dy}{dx} - \frac{3}{x^4} y = 1
\]

\[
\frac{d}{dx} \left( \frac{1}{x^3} y \right) = 1
\]

\[
\frac{y}{x^3} = x + c
\]

\[
y = (x + c) x^3
\]

\( y = 2 \) when \( x = 1 \), so

\[
2 = 1 + c
\]

\[
c = 1
\]

\[
y = (x + 1) x^3
\]

(b) \[
y \frac{dy}{dx} - 3x = x^4
\]

\[
y \frac{dy}{dx} = x^4 + 3x
\]

\[
\int y \, dy = \int (x^4 + 3x) \, dx
\]

\[
\frac{y^2}{2} = \frac{x^5}{5} + \frac{3x^2}{2} + c'
\]

When \( x = 1, y = 2 \) so \( c' = 2 - \frac{1}{5} - \frac{3}{2} = \frac{3}{10} \) and so

\[
y = \sqrt{2 \left( \frac{x^5}{5} + \frac{3x^2}{2} + \frac{3}{10} \right)}.
\]
16. (a) The series is arithmetic with \( a = 8, d = 3 \) and \( n = 17 \).

\[
S = \frac{n}{2} \{2a + (n - 1)d\} = \frac{17}{2} \{16 + 16 \times 3\} = 17 \times 32 = 544
\]

(b) \( a = 2, S_3 = a + ar + ar^2 = 266 \). Since \( a = 2 \)

\[
r^2 + r + 1 = 133
\]

\[
r^2 + r - 132 = 0
\]

\[
(r - 11)(r + 12) = 0
\]

\( r = 11 \) (since terms are positive).

Note: other valid equations could be used.

(c) 

\[
2(2a + 3 \times 2) = a \left(1 + 2 + 2^2 + 2^3\right)
\]

\[
4a + 12 = 15a
\]

\[
11a = 12
\]

\[
a = \frac{12}{11}
\]

The sum \( S_B = \frac{12}{11} (2^n - 1) \) and \( S_A = \frac{24}{11} + 2(n - 1) = n \left(\frac{4}{11} + n\right) \). 

<table>
<thead>
<tr>
<th>( n )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>( S_B )</td>
<td>( \frac{180}{11} )</td>
<td>( \frac{222}{11} )</td>
<td>( \frac{266}{11} )</td>
<td>( \frac{3524}{11} )</td>
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<tr>
<td>( S_A )</td>
<td>( \frac{180}{11} )</td>
<td>( \frac{280}{11} )</td>
<td>( \frac{402}{11} )</td>
<td>( \frac{446}{11} )</td>
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The smallest \( n \) is 7.
2004 Applied Mathematics

Advanced Higher – Section A

Finalised Marking Instructions
Advanced Higher Applied Mathematics 2004
Solutions for Section A (Statistics 1 and 2)

A1. (a) Stratified
and Quota [or Quota (convenience)]
(b) Approach (a) should be best
since (b) is not random (other forms e.g. Glasgow not typical, biased)

A2. (a) $F \sim Bin(192, 0.002)$. 
1 for distribution
1 for parameters
(b) $P(F \geq 3) = 1 - P(F \leq 2)$
$= 1 - (0.6809 + 0.2620 + 0.0501)$
$= 0.0070$
Notes: applying a Poisson distribution loses (at least) one mark; a Normal
distribution loses two marks.
(c) Approximate using the $Poi(0.384)$
1 for distribution
1 for parameters

A3. Assume that yields are normally distributed .
Random or independent will not do.}
$\bar{x} = 404.2; s = 10.03$
$t = 2.776$
A 95% confidence interval for the mean yield, $\mu$, is given by:-
$\bar{x} \pm t \frac{s}{\sqrt{n}}$
$404.2 \pm 2.776 \frac{10.03}{\sqrt{5}}$
$404.2 \pm 12.45$
or (391.75, 416.65).
The fact that the confidence interval does not include 382
provides evidence, at the 5% level of significance, of a
change in the mean yield. (Stating it is changed loses one mark.)
Note: the third and fourth marks are lost if a z interval is used.

A4. TNE = 3% of 500 = 15
With maximum allowable standard deviation
$P(weight < 485) = 0.025$
$\Rightarrow 485 - 505\sigma = -1.96$
$\Rightarrow \sigma = \frac{20}{1.96} = 10.2$

There will be a small probability of obtaining a content
weight less than 470g with the normal model.
A5. Assume that the distributions of times Before and After have the same shape. 

Notes: a Normal distribution with the same shape is a valid comment. 
Independent, random, Normal (without shape) are not valid.

Null hypothesis $H_0$: Median After = Median Before
Alternative hypothesis $H_1$: Median After < Median Before

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<th>Time</th>
<th>19</th>
<th>29</th>
<th>31</th>
<th>35</th>
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<th>42</th>
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<th>45</th>
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<th>59</th>
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<td>3</td>
<td>4</td>
<td>5</td>
<td>6.5</td>
<td>6.5</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
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</table>

Rank sum for After times = 37.5

$W - \frac{1}{2}n (n + 1) = 37.5 - 28 = 9.5$

$P(W - \frac{1}{2}n (n + 1) < 10) = 0.036$

Since this value is less than 0.05 the null hypothesis would be rejected in favour of the alternative, indicating evidence of improved performance.

Notes:
As the computed value, 9.5, is not in the tables, a range of values for the probability was acceptable.
A Normal approximation was accepted.

A6. | Cream | A | B | C |
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<tr>
<td>Obs. No. of purchasers</td>
<td>66</td>
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<td>75</td>
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<tr>
<td>Exp. No. of purchasers</td>
<td>80</td>
<td>80</td>
<td>80</td>
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</table>

$X^2 = \sum \frac{(O - E)^2}{E}$

$= \frac{(66 - 80)^2}{80} + \frac{(99 - 80)^2}{80} + \frac{(75 - 80)^2}{80}$

$= 2.45 + 4.5125 + 0.3125 = 7.275$

with 2 d.f.

The critical value of chi-squared at the 5% level is 5.991 so the null hypothesis would be rejected.

i.e. there is evidence of a preference.

The fact that the p-value is less than 0.05 confirms rejection of the null hypothesis at the 5% level of significance.

Note: using a two-tail test loses a mark.
A7. (a) The fitted value is 13.791 with residual 10.209. 

(b) The wedge-shaped plot casts doubt on the assumption of constant variance of $Y_i$ (i.e. variance not constant).

(c) Satisfactory now since variance seems to be more constant. 

Note: A phrase such as 'more randomly scattered' is acceptable.

(d) The residuals are normally distributed.

A8. (a) 

<table>
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<td>1</td>
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<td>1</td>
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</table>

Assume that differences are independent.

$H_0$: Median (Post − Pre) = 0 [or $\eta_d = 0$]

$H_1$: Median (Post − Pre) > 0 [or $\eta_d > 0$]

Under $H_0$ the differences Bin (11,0.5) with $b = 2$.

$P(B \leq 2) = (C_{11}^{11} + C_{11}^{11} + C_{11}^{11})0.5^{11}$

$= (1 + 11 + 55)0.5^{11} = 0.0327$.

Since 0.0327 < 0.05 the null hypothesis is rejected and there is evidence that the median PCS-12 score has gone up.

Note: applying a two-tailed test loses a mark.

(b) $H_0 : \mu_{Post} = 50$

$H_1 : \mu_{Post} \neq 50$

$\bar{x} = 48.42$

$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{48.42 - 50}{10 / \sqrt{12}} = -0.55.$

The critical region is $|z| > 1.96$ at the 5% level of significance.

Since −0.55 is not in the critical region, the null hypothesis is accepted indicating that the Post-operation scores are consistent with a population mean of 50.

Note: a correct use of probability comparisons gets full marks.
A9. (a) \( P(\text{Alaskan fish classified as Canadian}) \)
\[ = P(X > 120 \mid X \sim N(100, 20^2)) \]
\[ = P\left( Z > \frac{120 - 100}{20} \right) \]
\[ = P(Z > 1) \]
\[ = 0.1587 \]

(b) The probability is the same as in (a) because of symmetry.

(c) \( P(\text{Canadian origin} \mid \text{Alaskan predicted}) \)
\[ = \frac{P(\text{Alaskan predicted and Canadian origin})}{P(\text{Alaskan predicted})} \]
\[ = \frac{P(\text{Alas pred and Alaskan}) + P(\text{Ala pred but Canadian})}{0.4 \times 0.1587} \]
\[ = \frac{0.6 \times 0.8413 + 0.4 \times 0.1587}{0.06348} \]
\[ = 0.50478 + 0.06348 = 0.5682 \]

Note: Alternative methods acceptable e.g. Venn or Tree Diagrams

A10. The number, \( X \), of inaccurate invoices in samples of \( n \) will have the Bin \((n, p)\) distribution so
\[ V(X) = npq \]
\[ = np(1 - p) \]
\[ \Rightarrow V(\text{Proportion}) = V\left( \frac{1}{n} X \right) = \frac{1}{n^2} V(X) \]
\[ = \frac{p(1 - p)}{n} \]
\[ \Rightarrow \text{Standard deviation of Proportion} = \sqrt{\frac{p(1 - p)}{n}}. \]

(a) UCL = \( p + 3\sqrt{\frac{p(1 - p)}{n}} \)
\[ = 0.12 + 3\sqrt{\frac{0.12 \times 0.88}{150}} \]
\[ = 0.12 + 0.08 = 0.20. \]

LCL = \( 0.12 - 0.08 = 0.04 \)

(b) The fact that the point for Week 30 falls below the lower chart limit provides evidence of a drop in the proportion of inaccurate invoices. or: 8 consecutive points fell below the centre line.

(c) A new chart should be constructed (or set new limits) using an estimate of \( p \) for calculation of limits which is based on data collected since the process change.
2004 Applied Mathematics

Advanced Higher – Section B

Finalised Marking Instructions
B1. \( f(x) = \ln(2 - x) \) \( f'(x) = \frac{-1}{2 - x} \) \( f''(x) = \frac{-1}{(2 - x)^2} \) \( f'''(x) = \frac{-2}{(2 - x)^3} \)

Taylor polynomial is

\[ p(1 + h) = \ln 1 - h - \frac{h^2}{2} - \frac{h^3}{6} \]

For \( \ln 1 \cdot 1 \), \( p(0 \cdot 9) = 0 \cdot 1 - 0 \cdot 005 + 0 \cdot 00033 = 0 \cdot 0953 \).

Hence expect \( f(x) \) to be more sensitive in \( I_2 \) since coefficient of \( h \) is much larger.

B2. \( L(2.5) \)

\[ = \frac{(2.5 - 1.5)(2.5 - 3.0)(2.5 - 4.5)}{(0.5 - 1.5)(0.5 - 3.0)(0.5 - 4.5)} \times 1.737 + \frac{(2.5 - 0.5)(2.5 - 3.0)(2.5 - 4.5)}{(1.5 - 0.5)(1.5 - 3.0)(1.5 - 4.5)} \times 2.412 \]

\[ + \frac{(2.5 - 0.5)(2.5 - 1.5)(2.5 - 4.5)}{(3.0 - 0.5)(3.0 - 1.5)(3.0 - 4.5)} \times 3.284 + \frac{(2.5 - 0.5)(2.5 - 1.5)(2.5 - 3.0)}{(4.5 - 0.5)(4.5 - 1.5)(4.5 - 3.0)} \times 2.797 \]

\[ = -\frac{1 \times 1.737}{10} + \frac{2 \times 2.412}{4.5} + \frac{4 \times 3.284}{2.5 \times 2.25} - \frac{1 \times 2.797}{18} \]

\[ = -0.1737 + 0.0720 + 0.3353 - 0.1554 = 0.078 \]

B3. \( \Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0 \)

Maximum rounding error \( = \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon \).

\( \Delta^2 f_0 = 2.618 - 2 \times 2.369 + 2.124 = 0.004 \)

and \( 4\varepsilon = 4 \times 0.0005 = 0.002 \).

\( \Delta^2 f_0 \) appears to be significantly different from 0.
B4. (a) Difference table is:

<table>
<thead>
<tr>
<th>i</th>
<th>x</th>
<th>f(x)</th>
<th>diff1</th>
<th>diff2</th>
<th>diff3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.023</td>
<td>352</td>
<td>−95</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>1.375</td>
<td>257</td>
<td>−92</td>
<td>−4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.632</td>
<td>165</td>
<td>−96</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>1.797</td>
<td>69</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1.866</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) \( p = 0.3 \)

\[
f(0.65) = 1.375 + 0.3(0.257) + \frac{(0.3)(-0.7)}{2} (-0.092)
\]

\[
= 1.375 + 0.077 + 0.010 = 1.462
\]

(or, with \( p = 1.3, 1.023 + 0.458 - 0.019 \).

B5. \( f(x) = (((x - 1.1)x + 1.7)x) - 3.2 \) and \( f(1.3) = 0.1124 \).

Since \( f \) is positive and increasing at \( x = 1.3 \), root appears to occur for \( x < 1.3 \).

\[
f(x)_{\text{min}} = (((x - 1.15)x + 1.65)x) - 3.25
\]

\[
f(1.3)_{\text{min}} = -0.132 \text{ (opposite sign)}, \text{ so root may occur for } x > 1.3.
\]

B6. In diagonally dominant form,

\[
\begin{align*}
4x_1 & - 0.3x_2 + 0.5x_3 = 6.1 \\
0.5x_1 & - 7x_2 + 0.7x_3 = 3.7 \\
0.3x_1 & + 2x_3 = 8.6.
\end{align*}
\]

The diagonal coefficients of \( x \) are large relative to the others, so system is likely to be stable. (Or, this implies equations are highly linearly independent, or, determinant of system is large.)

Rewritten equations are:

\[
\begin{align*}
x_1 &= (6.1 + 0.3x_2 - 0.5x_3)/4 \\
x_2 &= (-3.7 + 0.5x_2 + 0.7x_3)/7 \\
x_3 &= (8.6 - 0.3x_1)/2
\end{align*}
\]

Gauss Seidel table is:

\[
\begin{array}{ccc}
x_1 & x_2 & x_3 \\
0 & 0 & 0 \\
1.525 & -0.420 & 4.051 \\
0.985 & -0.051 & 4.152 \\
1.002 & -0.042 & 4.150 \\
1.003 & -0.042 & \\
\end{array}
\]

Hence (2 decimal places) \( x_1 = 1.00; x_2 = -0.04; x_3 = 4.15 \).
B7. Tableau is:
\[
\begin{pmatrix}
2.6 & 0 & 1.622 & 0.742 & 0.479 & 0 \\
0 & 6.469 & 1.923 & -0.538 & 1 & 0 \\
0 & 0 & 3.604 & -0.415 & 0.128 & 1
\end{pmatrix}
\]
\[\sim \begin{pmatrix}
2.6 & 0 & 0 & 0.929 & 0.421 & -0.450 \\
0 & 6.469 & 0 & -0.317 & 0.932 & -0.534 \\
0 & 0 & 3.604 & -0.415 & 0.128 & 1
\end{pmatrix} \quad (R_1 - 1.622R_3 / 3.604)
\]
\[\sim \begin{pmatrix}
1 & 0 & 0 & 0.357 & 0.162 & -0.173 \\
0 & 1 & 0 & -0.049 & 0.144 & -0.083 \\
0 & 0 & 1 & -0.115 & 0.036 & 0.277
\end{pmatrix} \quad \text{(dividing by diagonal elements)}
\]
Hence \(A^{-1} = \begin{pmatrix}
0.36 & 0.16 & -0.17 \\
-0.05 & 0.14 & -0.08 \\
-0.12 & 0.04 & 0.28
\end{pmatrix}\).}

B8. (a) \[
x & y & f(x, y) & hf(x, y) \\
1 & 1 & 0.414 & 0.041 \\
1.1 & 1.041 & 0.514 & 0.051 \\
1.2 & 1.092 & 0.620 & 0.062 \\
1.3 & 1.154 &
\]
Global truncation error is first order.

(b) Predictor-corrector calculation (with one corrector application) is:
\[
x & y & y' = \sqrt{x^2 + 2y - 1} - 1 & y_P & y_P' & \frac{1}{2}h(y' + y_P') \\
1 & 1 & 0.4142 & 1.0414 & 0.5142 & 0.0464 \\
1.1 & 1.0464 &
\]
The difference in the (rounded) second decimal place between the values of \(x(1.1)\) in the two calculations suggests that the second decimal place cannot be relied upon in the first calculation.

\text{accuracy 1}
B9. Trapezium rule calculation is:

<table>
<thead>
<tr>
<th></th>
<th>$f(x)$</th>
<th>$m$</th>
<th>$mf_1(x)$</th>
<th>$mf_2(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2690</td>
<td>1</td>
<td>1.2690</td>
<td>1.2690</td>
</tr>
<tr>
<td>1.25</td>
<td>1.1803</td>
<td>2</td>
<td>2.3606</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0.9867</td>
<td>2</td>
<td>1.9734</td>
<td>1.9734</td>
</tr>
<tr>
<td>1.75</td>
<td>0.6839</td>
<td>2</td>
<td>1.3678</td>
<td>1.3678</td>
</tr>
<tr>
<td>2</td>
<td>0.2749</td>
<td>1</td>
<td>0.2749</td>
<td>0.2749</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\frac{3.5173}{7.2457}
\end{align*}
\]

Hence $I_1 = 3.5173 \times 0.5/2 = 0.8793$ and $I_2 = 7.2457 \times 0.25/2 = 0.9057$.

Difference table is:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>−887</td>
<td>−1049</td>
<td></td>
</tr>
<tr>
<td>−1936</td>
<td>−1092</td>
<td></td>
</tr>
<tr>
<td>−3028</td>
<td>−1062</td>
<td></td>
</tr>
<tr>
<td>−4090</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$|\text{max truncation error}| = 1 \times 0.1092 / 12 \approx 0.009$

Hence $I_2 = 0.91$ or 0.9.

Expect to reduce error by factor 4.

With $n$ strips and step size $2h$, Taylor series for expansion of an integral $I$ approximated by the trapezium rule is:

\[
I = I_n + C(2h)^2 + D(2h)^4 + \ldots = I_n + 4Ch^2 + 16Dh^4 + \ldots \quad (a)
\]

With $2n$ strips and step size $h$, we have:

\[
I = I_{2n} + Ch^2 + Dh^4 + \ldots \quad (b)
\]

4(b) − (a) gives

\[
3I = 4I_{2n} - I_n - 12Dh^4 + \ldots
\]

i.e.

\[
I = (4I_{2n} - I_n)/3 = I_{2n} + (I_{2n} - I_n)/3
\]

$IR = (4 \times 0.9057 - 0.8793)/3 = 0.914$
**B10.** Gradient of \( y = f(x) \) at \( x_0 \) is \( f'(x_0) = \frac{f(x_0)}{x_0 - x_1} \).

Hence \( x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)} \), i.e. \( x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \).

Likewise \( x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \) and in general \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \).

\[
f(x) = e^{-x} + x^4 - 2x^3 - 5x^2 - 1 \quad \text{and} \quad f'(x) = -e^{-x} + 4x^3 - 6x^2 - 10x; \quad x_0 = 3.5
\]

Root is 3.47 (2 decimal places).

In a situation such as diagrammed, the Newton-Raphson method depends for convergence on the point of intersection of tangent with \( x \)-axis being closer to the root than the initial point. In the interval \([-0.3, 0]\) there must be a TV of \( f(x) \) so that \( f'(x) = 0 \) and the point of intersection may be far from initial point; so iteration may lead to a different root.

For bisection, \( f(-1.1) = 0.080; \quad f(-1) = -0.281 \)

\[
f(-1.05) = -0.124 \\
f(-1.075) = -0.208;
\]

\[
f(-1.0875) = 0.024
\]

Hence root lies in \([-1.0875, -1.075]\).
C1. \[
\mathbf{r}(t) = (2t^2 - t)\mathbf{i} - (3t + 1)\mathbf{j}
\]
\[\Rightarrow \mathbf{v}(t) = (4t - 1)\mathbf{i} - 3\mathbf{j}\]
\[\Rightarrow |\mathbf{v}(t)| = \sqrt{(4t - 1)^2 + 9}\]
When the speed is 5,
\[(4t - 1)^2 + 9 = 25\]
\[(4t - 1)^2 = 16\]
\[4t - 1 = \pm 4\]
\[t = \frac{5}{4} \text{ seconds (as } t > 0)\].

C2. (a) \[
\mathbf{v}_F = 25\sqrt{2} (\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j})
\]
\[= 25 (\mathbf{i} + \mathbf{j})\]
\[
\mathbf{r}_F = 25t (\mathbf{i} + \mathbf{j}) \quad \text{as } \mathbf{r}_F(0) = \mathbf{0}
\]
\[\mathbf{v}_L = 20\mathbf{j}\]
\[\mathbf{r}_L = 20t\mathbf{j} + \mathbf{c}\]
But \[\mathbf{r}_L(0) = 10\mathbf{i}\] so \[\mathbf{r}_L = 10\mathbf{i} + 20t\mathbf{j}\]
The position of the ferry relative to the freighter is
\[
\mathbf{r}_F - \mathbf{r}_L = (25t - 10)\mathbf{i} + 5t\mathbf{j}
\]
(b) When \[t = 1\]
\[|\mathbf{r}_F - \mathbf{r}_L| = \sqrt{15^2 + 5^2}
\]
\[= \sqrt{250} = 5\sqrt{10} \text{ km}\]

C3. (a) Using \[T = \frac{2\pi}{\omega} \Rightarrow 8\pi = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{1}{4}\].
Maximum acceleration \[\omega^2 a\]
\[\frac{1}{4} = \frac{1}{16} a \Rightarrow a = 4\]
(b) Maximum speed \[\omega a = \frac{1}{4} \times 4 = 1\].
Using
\[v^2 = \omega^2 (a^2 - x^2)\]
\[
\left(\frac{1}{2}\right)^2 = \frac{1}{16} (16 - x^2)
\]
\[4 = 16 - x^2\]
\[x^2 = 12\]
\[x = \pm 2\sqrt{3} \text{ m}\]
C4.

Resolving perp. to plane: $R = mg \cos \theta$

Parallel to the plane (by Newton II)

$$ma = -\mu R - mg \sin \theta$$

$$= -\mu mg \cos \theta - mg \sin \theta$$

$$a = -g (\mu \cos \theta + \sin \theta)$$

$$= \frac{- (5 + 12\mu)}{13} \, g$$

Using $v^2 = u^2 + 2as$

$$0 = gL - \frac{2 (5 + 12\mu)gL}{13}$$

$$gL = \frac{2 (5 + 12\mu)gL}{13}$$

$$10 + 24\mu = 13 \Rightarrow \mu = \frac{1}{8}$$

C5.

Combined mass = $M + 0.01M = 1.01M$.

By Newton II

$$1.01Ma = (P + 0.05P) - 1.01Mg$$

$$1.01Ma = 1.05Mg - 1.01Mg$$

$$1.01a = 0.04g$$

$$a = \frac{4}{101} g (\approx 0.3) \, m \, s^{-2}$$
C6.  
(i) By conservation of energy, the speed of block A \( (v_A) \) immediately before the collision is given by

\[ v_A = \sqrt{2gh}. \]

By conservation of momentum, the speed of the composite block \( (v_C) \) after the collision is given by

\[ 2mv_C = mv_A \]

\[ v_C = \frac{1}{2}\sqrt{2gh} \]

(ii) By the work/energy principle

Work done against friction = Loss of KE + Change in PE

\[ F \times h = \frac{1}{2}(2m) \times \frac{1}{2}gh + 2mg \times \frac{1}{2}h \]

\[ F = \frac{mg}{2} + mg \]

\[ F = \frac{3}{2}W \text{ since } W = mg. \]

C7.  
(a) The equations of motion give

\[ \ddot{y} = -g \]

\[ v(0) = V \cos \alpha \hat{i} + V \sin \alpha \hat{j} \]

\[ \dot{y} = -gt + V \sin \alpha \]

\[ y = V \sin \alpha t - \frac{1}{2}gt^2 \]

Maximum height when \( \dot{y} = 0 \) \( \Rightarrow \) \( t = \frac{V}{g} \sin \alpha \), and so

\[ H = V \sin \alpha \times \frac{V}{g} \sin \alpha - \frac{1}{2}g \times \frac{V^2}{g^2} \sin^2 \alpha \]

\[ = \frac{V^2}{2g} \sin^2 \alpha \]

(b) (i)

\[ h = \frac{V^2}{2g} \sin^2 2\alpha \]

\[ = \frac{V^2}{2g} 4 \sin^2 \alpha \cos^2 \alpha \]

\[ = \frac{2V^2}{g} \sin^2 \alpha (1 - \sin^2 \alpha) \]

\[ = 4H \left(1 - \frac{2gH}{V^2}\right) \text{ since } \sin^2 \alpha = \frac{2gH}{V^2} \]

(ii) Since \( h = 3H \)

\[ 3H = 4H \left(1 - \sin^2 \alpha\right) \]

\[ \frac{3}{4} = 1 - \sin^2 \alpha \]

\[ \sin^2 \alpha = \frac{1}{4} \]

\[ \sin \alpha = \pm \frac{1}{2} \]

\[ \Rightarrow \alpha = \frac{\pi}{6} \text{ and so } 2\alpha = \frac{\pi}{3} \]
C8. (a) Radius of horizontal circle \( r = L \sin 60^\circ = \frac{\sqrt{3}}{2} L \).

\[
AB = \frac{r}{\sin 30^\circ} = 2 \times \frac{\sqrt{3}}{2} L = \sqrt{3} L
\]

Extension of \( AB, x = (\sqrt{3} - 1) L \)

Tension in \( AB, T_1 = \frac{\lambda x}{L} \)

\[
= 2 (\sqrt{3} - 1) mg.
\]

(b) Resolving vertically (where \( T_2 \) is the tension in \( BC \))

\[
T_1 \cos 30^\circ = mg + T_2 \cos 60^\circ
\]

\[
\frac{\sqrt{3}}{2} \times 2 (\sqrt{3} - 1) mg = mg + \frac{1}{2} T_2
\]

\[
T_2 = (6 - 2\sqrt{3} - 2) mg
\]

\[
= 2 (2 - \sqrt{3}) mg
\]

(c) Resolving horizontally (using \( L = 1 \))

\[
T_1 \sin 30^\circ + T_2 \sin 60^\circ = m \left( \frac{\sqrt{3}}{2} \right) \omega^2
\]

\[
\frac{1}{2} \times 2 (\sqrt{3} - 1) mg + \frac{\sqrt{3}}{2} \times 2 (2 - \sqrt{3}) mg = m \left( \frac{\sqrt{3}}{2} \right) \omega^2
\]

\[
(2\sqrt{3} - 2 + 4\sqrt{3} - 6) g = \sqrt{3} \omega^2
\]

\[
(6\sqrt{3} - 8) g = \sqrt{3} \omega^2
\]

\[
\omega^2 = \frac{2 (3\sqrt{3} - 4) g}{\sqrt{3}}
\]

\[
\omega = \frac{\sqrt{2 (3\sqrt{3} - 4) g}}{\sqrt{3}}
\]
C9.

(i) \[ \frac{dv}{dt} = -mkv^3 \]
\[ v \frac{dv}{dx} = -kv^3 \]
\[ \frac{dv}{dx} = -kv^2 \]

Separating the variables and integrating gives
\[ \int v^{-2} dv = \int -k \, dx \]
\[ \Rightarrow -v^{-1} = -kx + c \]

At \( x = 0, v = U \)
\[ -U^{-1} = c \]

so
\[ v^{-1} = kx + U^{-1} \]
\[ v = \frac{U}{1 + kUx} \].

(ii) Now \( v = \frac{dx}{dt} \), so
\[ \frac{dx}{dt} = \frac{U}{1 + kUx} \]
\[ \int (1 + kUx) \, dx = \int U \, dt \]
\[ x + \frac{1}{2}kUx^2 = Ut + c_1 \]

Since \( x = 0 \) when \( t = 0 \), then \( c_1 = 0 \)
\[ kUx^2 + 2x = 2Ut \]

(iii)
\[ V = \frac{1}{2}U \Rightarrow \frac{1}{2}U (1 + kUx) = U \]
\[ \Rightarrow 1 + kUx = 2 \Rightarrow x = \frac{1}{kU} \]

The time taken
\[ 2Ut = kU \left( \frac{1}{k^2U^2} + \frac{2}{kU} \right) = \frac{3}{kU} \]
\[ \Rightarrow t = \frac{3}{2kU^2} \]
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Advanced Higher – Section D

Finalised Marking Instructions
D1. \[(4x - 5y)^4 = (4x)^4 - 4 \times (4x)^3 (5y) + 6 \times (4x)^2 (5y)^2 - 4 \times (4x)(5y)^3 + (5y)^4\]  
\[= 256x^4 - 1280x^3 y + 2400x^2 y^2 - 2000xy^3 + 625y^4.\]
When \(y = \frac{1}{x}\), the term independent of \(x\) is 2400.

D2. 
\[y = x^2 \ln x\]
\[\frac{dy}{dx} = 2x \ln x + x \cdot \frac{1}{x} = 2x \ln x + x\]
\[\frac{d^2y}{dx^2} = 2 \ln x + 2x \cdot \frac{1}{x} + 1 = 2 \ln x + 3\]
\[x \frac{d^2y}{dx^2} - \frac{dy}{dx} = 2x \ln x + 3x - 2x \ln x - x = 2x\]
Thus \(k = 2\).

D3. (a)

\[
\begin{array}{ccc|c}
1 & 1 & -2 & -6 \\
3 & -1 & 1 & 7 \\
2 & 1 & -\lambda & -2 \\
\hline
1 & 1 & -2 & -6 \\
0 & -4 & 7 & 25 \Rightarrow R_2 - 3R_1 \\
0 & -1 & 4 - \lambda & 10 \Rightarrow R_3 - 2R_1 \\
\hline
1 & 1 & -2 & -6 \\
0 & -4 & 7 & 25 \\
0 & 0 & 9 - 4\lambda & 15 \Rightarrow 4R_3 - R_2 \\
\end{array}
\]

There is no solution when \(\lambda = \frac{9}{4}\).

(b) When \(\lambda = 1\),

\[5c = 15 \Rightarrow c = 3\]
\[-4b + 21 = 25 \Rightarrow b = -1\]
\[a - 1 - 6 = -6 \Rightarrow a = 1\]
i.e. \(a = 1, b = -1, c = 3\)
D4. \( x + 1 = u \Rightarrow dx = du \)
\( x = u - 1 \Rightarrow x^2 + 2 = u^2 - 2u + 3. \)
\[
\int \frac{x^2 + 2}{(x + 1)^2} \, dx = \int \frac{u^2 - 2u + 3}{u^2} \, du
\]
\[
= \int \left(1 - \frac{2}{u} + 3u^{-2}\right) \, du
\]
\[
= u - 2 \ln|u| - 3u^{-1} + c
\]
\[
= x - 2 \ln|x + 1| - \frac{3}{x + 1} + c
\]

D5. (a) \[
\frac{(x - 1)(x - 4)}{x^2 + 4} = A + \frac{Bx + C}{x^2 + 4}
\]
\[
x^2 - 5x + 4 = Ax^2 + 4A + Bx + C
\]
\[
A = 1, \quad B = -5, \quad C = 0
\]
i.e. \[
f(x) = 1 - \frac{5x}{x^2 + 4}
\]

(b) As \( x \to \pm \infty, y \to 1. \)
[No vertical asymptotes since \( x^2 + 4 \neq 0. \)]

(c) \[
f(x) = 1 - \frac{5x}{x^2 + 4}
\]
\[
f'(x) = -\frac{5(x^2 + 4) - 10x^2}{(x^2 + 4)^2} = 0 \text{ at S.V.}
\]
\[
\Rightarrow 20 - 5x^2 = 0 \Rightarrow x = \pm 2
\]
\[
\Rightarrow (2, -\frac{1}{2}) \text{ and } (-2, 2\frac{1}{2})
\]

(d) \( y = 0 \Rightarrow x = 1 \) or \( x = 4. \)
Area = \(-\int_1^4 \left(1 - \frac{5x}{x^2 + 4}\right) \, dx\)
\[
= -\left[x - \frac{5}{2} \ln(x^2 + 4)\right]_1^4
\]
\[
= -\left[4 - \frac{5}{2} \ln 20\right] + \left[1 - \frac{5}{2} \ln 5\right]
\]
\[
= \frac{5}{2} \ln 4 - 3 = 5 \ln 2 - 3 \text{ (acceptable but not required)}
\]
\[
\approx 0.47 \text{ (acceptable but not required)}
\]
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Advanced Higher – Section E

Finalised Marking Instructions
E1. (a) Stratified and Quota [or Quota (convenience)]
(b) Approach (a) should be best since (b) is not random (other forms e.g. Glasgow not typical, biased)

E2. (a) $F \sim \text{Bin}(192, 0.002)$. \hfill 1 for distribution
\hfill 1 for parameters
(b) $P(F \geq 3) = 1 - P(F \leq 2)$
\hfill 1
$= 1 - (0.6809 + 0.2620 + 0.0501)$
\hfill 1
$= 0.0070$
\hfill 1

Notes: applying a Poisson distribution loses (at least) one mark; a Normal distribution loses two marks.

(c) Approximate using the $\text{Poi}(0.384)$ \hfill 1 for distribution
\hfill 1 for parameters

E3. Assume that yields are normally distributed. \hfill 1
Assume that the standard deviation is unchanged. \hfill 1
$\bar{x} = 404.2.$
A 95% confidence interval for the mean yield, $\mu$, is given by:

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$$
\hfill 1

$404.2 \pm 1.96 \frac{10}{\sqrt{5}}$
$404.2 \pm 8.8$
or $(395.4, 413.0)$.
The fact that the confidence interval does not include 382 provides evidence, at the 5% level of significance, of a change in the mean yield. \hfill 1

E4. TNE = 3% of 500 = 15 \hfill 1
With maximum allowable standard deviation
$P(\text{weight} < 485) = 0.025$ \hfill 1
$\Rightarrow \frac{485 - 505}{\sigma} = -1.96$ \hfill 1,1
$\Rightarrow \sigma = \frac{20}{1.96} = 10.2$ \hfill 1

There will be a small probability of obtaining a content weight less than 470g with the normal model. \hfill 1
E5.  (a) \[ P(\text{Alaskan fish classified as Canadian}) \]
\[= P(X > 120 \mid X \sim N(100, 20^2)) \]
\[= P\left(Z > \frac{120 - 100}{20}\right) \]
\[= P(Z > 1) \]
\[= 0.1587 \]

(b) The probability is the same as in (a) because of symmetry.

(c) \[ P(\text{Canadian origin} \mid \text{Alaskan predicted}) \]
\[= \frac{P(\text{Alaskan predicted and Canadian origin})}{P(\text{Alaskan predicted})} \]
\[= \frac{P(\text{Alaskan predicted and Canadian})}{P(\text{Alaskan predicted})} \]
\[= \frac{P(\text{Alaska pred and Alaskan}) + P(\text{Alaska pred but Canadian})}{0.4 \times 0.1587} \]
\[= \frac{0.4 \times 0.1587}{0.6 \times 0.8413 + 0.4 \times 0.1587} \]
\[= \frac{0.06348}{0.50478 + 0.06348} \]
\[= 0.112. \]

Note: Alternative methods acceptable e.g. Venn or Tree Diagrams
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Advanced Higher – Section F

Finalised Marking Instructions
Advanced Higher Applied Mathematics 2004
Solutions for Section F (Numerical Analysis 1)

F1. \[ f(x) = \ln(2 - x) \quad f'(x) = \frac{-1}{(2 - x)} \quad f''(x) = \frac{-1}{(2 - x)^2} \quad f'''(x) = \frac{-2}{(2 - x)^3} \]

Taylor polynomial is
\[ p(1 + h) = \ln 1 + h - \frac{h^2}{2} - \frac{2h^3}{6} \]

For \( h = 1 \cdot 1 \), \( p(0.9) = 0.1 - 0.005 + 0.00033 = 0.0953 \).

Hence expect \( f(x) \) to be more sensitive in \( l_2 \) since coefficient of \( h \) is much larger.

F2. \( L(2.5) \)
\[ \begin{align*}
&= \frac{(2.5 - 1.5)(2.5 - 3.0)(2.5 - 4.5)}{(0.5 - 1.5)(0.5 - 3.0)(0.5 - 4.5)} \cdot 1.737 + \frac{(2.5 - 0.5)(2.5 - 3.0)(2.5 - 4.5)}{(1.5 - 0.5)(1.5 - 3.0)(1.5 - 4.5)} \cdot 2.412 \\
&+ \frac{(2.5 - 0.5)(2.5 - 1.5)(2.5 - 4.5)}{(3.0 - 0.5)(3.0 - 1.5)(3.0 - 4.5)} \cdot 3.284 + \frac{(2.5 - 0.5)(2.5 - 1.5)(2.5 - 3.0)}{(4.5 - 0.5)(4.5 - 1.5)(4.5 - 3.0)} \cdot 2.797 \\
&= -\frac{1 \times 1.737}{10} + \frac{2 \times 2.412}{4.5} + \frac{4 \times 3.284}{2.5 \times 2.25} - \frac{1 \times 2.797}{18} \\
&= -0.1737 + 1.0720 + 2.3353 - 0.1554 = 3.078
\]

F3. \[ \Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0 \]

Maximum rounding error = \( \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon \).

\[ \Delta^2 f_0 = 2.618 - 2 \times 2.369 + 2.124 = 0.004 \]

and \( 4\varepsilon = 4 \times 0.0005 = 0.002 \).

\( \Delta^2 f_0 \) appears to be significantly different from 0.
(a) Difference table is:

\[
\begin{array}{cccccc}
\text{\(i\)} & \text{\(x\)} & \text{\(f(x)\)} & \text{\(\text{diff1}\)} & \text{\(\text{diff2}\)} & \text{\(\text{diff3}\)} \\
0 & 0 & 1.023 & 352 & -95 & 3 \\
1 & 0.5 & 1.375 & 257 & -92 & -4 \\
2 & 1 & 1.632 & 165 & -96 & \\
3 & 1.5 & 1.797 & 69 & & \\
4 & 2 & 1.866 & & & \\
\end{array}
\]

(b) \( p = 0.3 \)

\[
f(0.65) = 1.375 + 0.3(0.257) + \frac{(0.3)(-0.7)}{2}(-0.092) = 1.375 + 0.077 + 0.010 = 1.462
\]

(or, with \( p = 1.3, 1.023 + 0.458 - 0.019 \).)

F5. Trapezium rule calculation is:

\[
\begin{array}{cccccc}
\text{\(x\)} & \text{\(f(x)\)} & \text{\(m\)} & \text{\(mf_1(x)\)} & \text{\(mf_2(x)\)} & \\
1 & 1.2690 & 1 & 1.2690 & 1.2690 & \\
1.25 & 1.1803 & 2 & 2.3606 & & \\
1.5 & 0.9867 & 2 & 1.9734 & 1.9734 & \\
1.75 & 0.6839 & 2 & 1.3678 & & \\
2 & 0.2749 & 1 & 0.2749 & 0.2749 & \\
\hline
\end{array}
\]

Hence \( I_1 = 3.5173 \times 0.5/2 = 0.8793 \) and \( I_2 = 7.2457 \times 0.25/2 = 0.9057 \).

Difference table is:

\[
\begin{array}{cccc}
\text{\(|\text{max truncation error}|\)} & -887 & -1049 & \\
& -1936 & -1092 & \\
& -3028 & -1062 & \\
& -4090 & \\
\end{array}
\]

\( | \text{max truncation error} | = 1 \times 0.1092 / 12 = 0.009 \)

Hence \( I_2 = 0.91 \) or 0.9.

Expect to reduce error by factor 4.

With \( n \) strips and step size \( 2h \), Taylor series for expansion of an integral \( I \) approximated by the trapezium rule is:

\[
I = I_n + C(2h)^2 + D(2h)^4 + \ldots = I_n + 4Ch^2 + 16Dh^4 + \ldots \\
\]

(a)

With 2\( n \) strips and step size \( h \), we have:

\[
I = I_{2n} + Ch^2 + Dh^4 + \ldots \\
\]

(b)

4(b) – (a) gives \( 3I = 4I_{2n} - I_n - 12Dh^4 + \ldots \)

i.e. \( I = (4I_{2n} - I_n)/3 = I_{2n} + (I_{2n} - I_n)/3 \)

\( I_R = (4 \times 0.9057 - 0.8793)/3 = 0.914 \)

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Advanced Higher – Section G

Finalised Marking Instructions
G1. \[ \mathbf{r}(t) = (2t^2 - t)i - (3t + 1)j \]
\[ \Rightarrow \mathbf{v}(t) = (4t - 1)i - 3j \]
\[ \Rightarrow |\mathbf{v}(t)| = \sqrt{(4t - 1)^2 + 9} \]

When the speed is 5,
\[ (4t - 1)^2 + 9 = 25 \]
\[ (4t - 1)^2 = 16 \]
\[ 4t - 1 = \pm 4 \]
\[ t = \frac{5}{4} \text{ seconds (as } t > 0). \]

G2. (a)
\[ \mathbf{v}_F = 25\sqrt{2}(\cos 45^\circ i + \sin 45^\circ j) \]
\[ = 25(i + j) \]
\[ \mathbf{r}_F = 25t(i + j) \quad \text{as } \mathbf{r}_F(0) = 0 \]
\[ \mathbf{v}_L = 20j \]
\[ \mathbf{r}_L = 20tj + \mathbf{c} \]

But \( \mathbf{r}_L(0) = 10i \) so \( \mathbf{r}_L = 10i + 20tj \)

The position of the ferry relative to the freighter is
\[ \mathbf{r}_F - \mathbf{r}_L = (25t - 10)i + 5tj \]

(b) When \( t = 1 \)
\[ |\mathbf{r}_F - \mathbf{r}_L| = \sqrt{15^2 + 5^2} \]
\[ = \sqrt{250} = 5\sqrt{10} \text{ km} \]

G3.

Combined mass = \( M + 0.01M = 1.01M \).

By Newton II
\[ 1.01Ma = (P + 0.05P) - 1.01Mg \]
\[ 1.01Ma = 1.05Mg - 1.01Mg \]
\[ 1.01a = 0.04g \]
\[ a = \frac{4}{101}g \approx 0.3 \text{ m s}^{-2} \]
Resolving perp. to plane: \( R = mg \cos \theta \)
Parallel to the plane (by Newton II)

\[
ma = -\mu R - mg \sin \theta = -\mu mg \cos \theta - mg \sin \theta
\]

\[
a = -g(\mu \cos \theta + \sin \theta) = \frac{-(5 + 12\mu)g}{13}
\]

Using \( v^2 = u^2 + 2as \)

\[
0 = gL - \frac{2(5 + 12\mu)gL}{13}
\]

\[
gL = \frac{2(5 + 12\mu)gL}{13}
\]

\[
10 + 24\mu = 13 \Rightarrow \mu = \frac{1}{8}
\]
G5.

(a) The equations of motion give

\[ \ddot{y} = -g \quad \mathbf{v}(0) = V \cos \alpha \mathbf{i} + V \sin \alpha \mathbf{j} \]

\[ \dot{y} = -gt + V \sin \alpha \]

\[ y = V \sin \alpha t - \frac{1}{2}gt^2 \]

Maximum height when \( \dot{y} = 0 \) \( \Rightarrow t = \frac{V}{g} \sin \alpha \), and so

\[ H = V \sin \alpha \times \frac{V}{g} \sin \alpha - \frac{1}{2} \frac{V}{g} \frac{V^2}{g^2} \sin^2 \alpha \]

\[ = \frac{V^2}{2g} \sin^2 \alpha \]

(b) (i)

\[ h = \frac{V^2}{2g} \sin^2 2\alpha \]

\[ = \frac{V^2}{2g} \cdot 4 \sin^2 \alpha \cos^2 \alpha \]

\[ = \frac{2V^2}{g} \sin^2 \alpha (1 - \sin^2 \alpha) \]

\[ = 4H \left(1 - \frac{2gH}{V^2}\right) \quad \text{since} \quad \sin^2 \alpha = \frac{2gH}{V^2} \]

(ii) Since \( h = 3H \)

\[ 3H = 4H \left(1 - \sin^2 \alpha\right) \]

\[ \frac{3}{4} = 1 - \sin^2 \alpha \]

\[ \sin^2 \alpha = \frac{1}{4} \]

\[ \sin \alpha = \pm \frac{1}{2} \]

\[ \Rightarrow \alpha = \frac{\pi}{6} \quad \text{and so} \quad 2\alpha = \frac{\pi}{3} \]