St Andrew's Academy
Mathematics Department

Higher Mathematics Notes

2018-19

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents acircle centre (-g, -f) and radius $\sqrt{g^2 + f^2 - c}$

The equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle centre (a, b) and radius $\boldsymbol{r}$

Scalar Product: $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$

or $\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Trigonometric Formulae:

$$
\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos A \mp \sin A \sin B \\
\sin 2A &= 2\sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A
\end{aligned}
$$

Table of Standard Derivatives:

$f(x)$	$f'(x)$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$

Table of Standard Integrals:

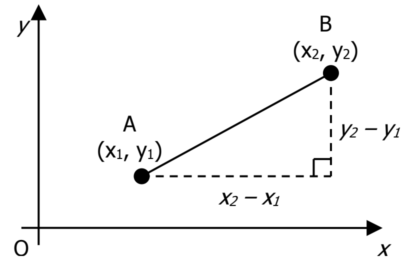
$f(x)$	$\int f(x)\,dx$
$\sin ax$	$-\frac{1}{a}\cos ax + C$
$\cos ax$	$\frac{1}{a}\sin ax + C$

The Straight Line

Revision from National 5

The graph of $y = mx + c$ is a **straight line**, where m is the gradient and c is the y-intercept.

Gradient is a measure of the steepness of a line. The gradient of the line joining points A (x_1, y_1) and B (x_2, y_2) is given by:



$$\mathbf{m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}}$$

Example 1: Find:

a) the gradient and y-intercept of the line $y = 2x + 5$

b) the equation of the line with gradient – 4 and y-intercept (0, -2)

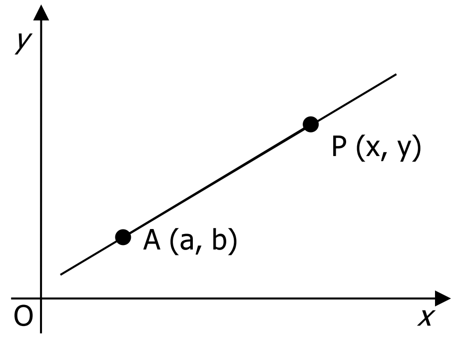
c) the gradient of the line joining P (-2, 4) and Q (3, -1)

d) the gradient of the line $3y + 4x - 11 = 0$

Equation of a Straight Line: *y* - b = m(*x* - a)

Points A (a, b) and P (x, y) both lie on a straight line.

The gradient of the line $m = \frac{y - b}{x - a}$. Rearranging this gives:



$$y - b = m(x - a)$$

NOTE: when you are asked to find the equation of a straight line, it is fine to leave it in this form (unless you are specifically asked to remove the brackets).

Example 2: Find the equations of the lines:

a) through (4, 5) with m = 2

b) joining (-1, -2) and (3, 10)

c) parallel to the line $x - 2y + 4 = 0$ and passing through the point (2, -3)

Equation of a Straight Line

$y = mx + c$ AND $Ax + By + C = 0$

Example 3: Find the equation of the line through (-5, -1) with m = $-\frac{2}{3}$, giving your answer in the form Ax + By + C = 0.

Example 4: Sketch the line 5x - 2y - 24 = 0 by finding the points where it crosses the x - and y - axes.

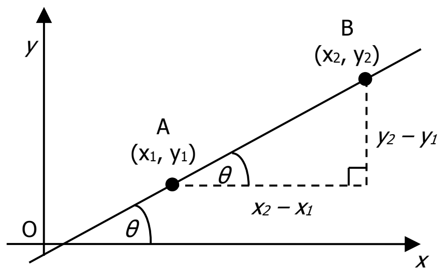
The Angle with the x-axis

The gradient of a line can also be described as the angle it makes with the ***positive*** direction of the x -axis.

As the y-difference is OPPOSITE the angle and the x-difference is ADJACENT to it, we get:

$$\mathbf{m_{AB} = \tan\theta}$$

(where θ is measured CLOCKWISE from the x-axis)



Example 5: Find the angle made with the positive direction of the x -axis and the lines:

a) $y = x - 1$

b) $y = 5 - \sqrt{3}x$

c) joining the points (3, -2) and (7, 4)

Gradients of straight lines can be summarised as follows:

a) lines sloping **up** from left to right have **positive** gradients and make **acute** angles with the positive direction of the x-axis
b) lines sloping **down** from left to right have **negative** gradients and make **obtuse** angles with the positive direction of the x-axis
c) lines with **equal** gradients are parallel
d) **horizontal** lines (parallel to the ***x* -axis**) have gradient **zero** and equation y = a
e) **vertical** lines (parallel to the ***y* -axis**) have gradient **undefined** and equation x = b

Collinearity

If three (or more) points lie on the same line, they are said to be **collinear.**

Example 6: Prove that the points D (-1, 5), E (0, 2) and F (4, -10) are collinear.

Perpendicular Lines

If two lines are perpendicular to each other (i.e. they meet at 90°), then: $m_1 m_2 = -1$

Example 7: Show whether these pairs of lines are perpendicular:

a) $x + y + 5 = 0$
$x - y - 7 = 0$

b) $2x - 3y = 5$
$3x = 2y + 9$

c) $y = 2x - 5$
$6y = 10 - 3x$

When asked to find the gradient of a line perpendicular to another, follow these steps:

1. **Find the gradient of the given line**
2. **Flip it upside down**
3. **Change the sign (e.g. negative to positive)**

Example 8: Find the gradients of the lines perpendicular to:

a) the line $y = 3x - 12$

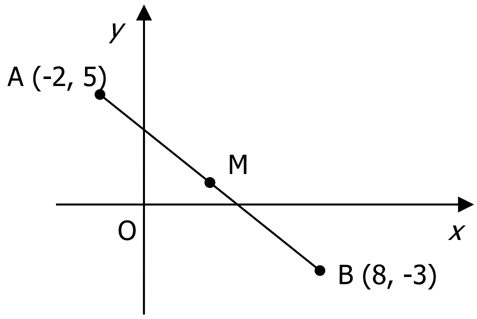
b) a line with gradient = -1.5

c) the line $2y + 5x = 0$

Example 9: Line L has equation $x + 4y + 2 = 0$. Find the equation of the line perpendicular to L which passes through the point (-2, 5).

Midpoints and Perpendicular Bisectors

The **midpoint** of a line lies exactly halfway along it. To find the coordinates of a midpoint, find halfway between the *x* – and *y* – coordinates of the points at each end of the line (see diagram).



The *x* – coordinate of M is halfway between -2 and 8, and its *y* – coordinate is halfway between 5 and -3.

In general, if M is the midpoint of A (x_1, y_1) and B (x_2, y_2):

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

The **perpendicular bisector** of a line passes through its midpoint at 90°.

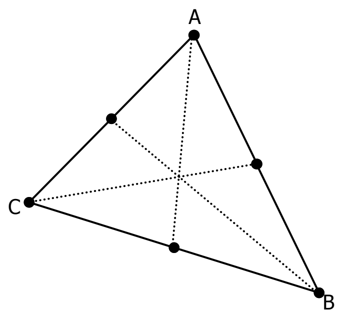
Example 10: Find the perpendicular bisector of the line joining F (-4, 2) and G (6, 8).

To find the equation of a perpendicular bisector:

- Find the gradient of the line joining the given points
- Find the perpendicular gradient (flip and make negative)
- Find the coordinates of the midpoint
- Substitute into $y - b = m(x - a)$

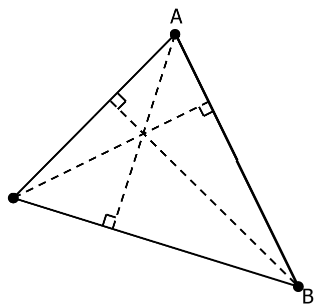
Lines Inside Triangles: Medians, Altitudes & Perpendicular Bisectors

In a triangle, a line joining a corner to the **midpoint** of the opposite side is called a **median.**



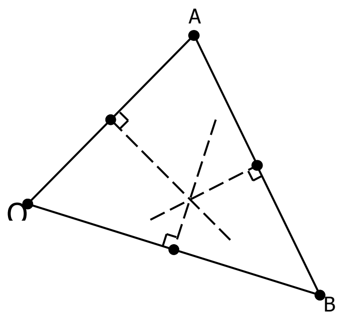
The medians are **concurrent** (i.e. meet at the same point) at the **centroid**, which divides each median in the ratio 2:1
The median always divides the area of a triangle in half.
A solid triangle of uniform density will balance on the centroid.

A line through a corner which is **perpendicular** to the opposite side is called an **altitude.**



The altitudes are concurrent at the **orthocentre.** The orthocentre isn't always located inside the triangle e.g. if the triangle is obtuse.

A line at 90° to the midpoint is called a **perpendicular bisector.**

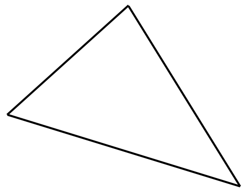


The perpendicular bisectors are concurrent at the **circumcentre.** The circumcentre is the centre of the circle touched by the vertices of the triangle.

For all triangles, the centroid, orthocentre and circumcentre are **collinear.**

Example 11: A triangle has vertices P (0, 2), Q (4, 4) and R (8, -6).

a) Find the equation of the median through P.



To find the equation of a median:

- Find the midpoint of the side opposite the given point
- Find the gradient of the line joining the given point and the midpoint
- Substitute into $y - b = m(x - a)$

b) Find the equation of the altitude through R.

To find the equation of an altitude:

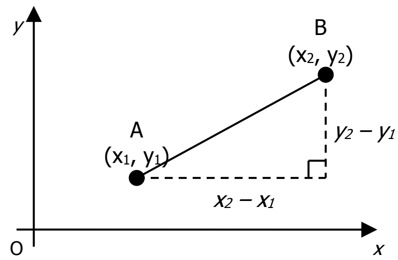
- Find the gradient of the side opposite the given point
- Find the perpendicular gradient (flip and make negative)
- Substitute into $y - b = m(x - a)$

Distance between Two Points

The distance between any two points A (x_1, y_1) and B (x_2, y_2) can be found easily by Pythagoras' Theorem.

If d is the distance between A and B, then:

$$\mathbf{d} = \sqrt{(\mathbf{x}_2 - \mathbf{x}_1)^2 + (\mathbf{y}_2 - \mathbf{y}_1)^2}$$



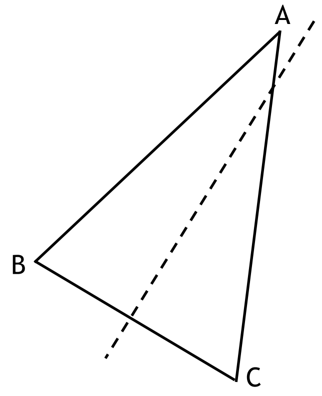
Example 12: Calculate the distance between:

a) A (-4, 4) and B (2, -4)

b) X (11, 2) and Y (-2, -5)

Example 13: A is the point (2, -1), B is (5, -2) and C is (7, 4). Show that BC = 2AB.

Past Paper Example 1: The vertices of triangle ABC are A(7, 9), B(-3, -1) and C(5, -5) as shown:
The broken line represents the perpendicular bisector of BC



a) Show that the equation of the perpendicular bisector of BC is $y = 2x - 5$

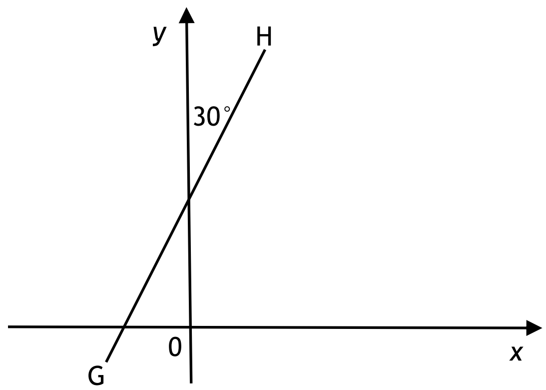
b) Find the equation of the median from C

c) Find the co-ordinates of the point of intersection of the perpendicular bisector of BC and the median from C.

Past Paper Example 2:

The line GH makes an angle of 30° with the y-axis as shown in the diagram opposite.

What is the gradient of GH?



Sets and Functions

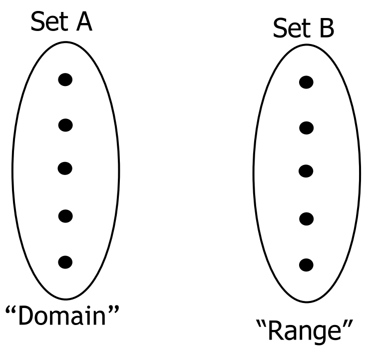
A **set** is a group of numbers which share common properties. Some common sets are:

Natural Numbers	N = {1, 2, 3, 4, 5,……}
Whole Numbers	W = {0, 1, 2, 3, 4, 5,…..}
Integers	Z = {…..,-3, -2, -1, 0, 1, 2, 3,…..}
Rational Numbers	Q = all integers **and** fractions of them (e.g. ¾, -⅝, etc)
Real Numbers	R = all rational **and** irrational numbers (e.g. $\sqrt{2}, \pi$, etc.)

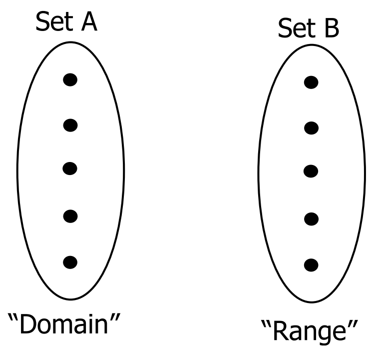
Sets are written inside curly brackets. The set with no members "{ }" is called the **empty set.**

$\in$ means "is a member of", e.g. $5 \in \{3, 4, 5, 6, 7\}$ $\notin$ means "is not a member of", e.g. $5 \notin \{6, 7, 8\}$

A **function** is a rule which links an element in Set A to **one and only one** element in Set B.



This shows a function



This does **not** show a function

The set that the function works on is called the **domain**; the values produced are called the **range.** For graphs of functions, we can think of the **domain** as the ***x* - values**, and the **range** as the ***y* - values**.

This means that any operation which produces **more than one answer** is **not** considered a function. For example, since √4 = 2 **and** -2, "f(x) = √x" is **not** considered a function.

Example 1: Each function below is defined on the set of real numbers. State the **range** of each.

a) $f(x) = \sin x°$ b) $g(x) = x^2$ c) $h(x) = 1 - x^2$

When choosing the domain, two cases MUST be avoided:	**a) Denominators can't be zero** **b) Can't find the square root of a negative value**

e.g. For $f(x) = \dfrac{1}{x+5}$, $x \neq -5$, i.e. $\{x \in R : x \neq -5\}$ e.g. For $g(x) = \sqrt{x-3}$, $x \geq 3$, i.e. $\{x \in R : x \geq 3\}$

Example 2: For each function, state a suitable domain.

a) $g(x) = \sqrt{3x-2}$ b) $p(\theta) = \dfrac{2}{5-\theta}$ c) $f(y) = \dfrac{y^2}{\sqrt{y-1}}$

Composite Functions

In the linear function $y = 3x - 5$, we get y by doing **two** acts: (i) multiply x by 3; (ii) then subtract 5. This is called a **composite function**, where we "do" a function to the range of another function.

e.g. If $h(x)$ is the composite function obtained by performing $f(x)$ on $g(x)$, then we say

$h(x) = f(g(x))$ ("f of g of x")

Example 3: $f(x) = 5x + 1$ and $g(x) = 3x^2 + 2x$.

a) Find $f(g(-1))$

b) Find $f(g(x))$

c) Find $f(f(x))$

d) Find $g(f(x))$

NOTE: Usually, $f(g(x))$ and $g(f(x))$ are NOT the same!

Example 4: $f(x) = 2x + 1$, $g(x) = x^2 + 6$

a) Find formulae for:

(i) $f(g(x))$

(ii) $g(f(x))$

b) Solve the equation $f(g(x)) = g(f(x))$

Example 5: $f(x) = \frac{3}{x+1}$, $x \neq -1$. Find an expression for $f(f(x))$, as a fraction in its simplest form.

Past Paper Example: Functions *f* and *g* are defined on a set of real numbers by

$$f(x) = x^2 + 3 \qquad\qquad g(x) = x + 4$$

a) Find expressions for:

(i) $f(g(x))$

(ii) $g(f(x))$

b) Show that $f(g(x)) + g(f(x)) = 0$ has no real roots

Recurrence Relations

A **recurrence relation** is a rule which produces a sequence of numbers where each term is obtained from the previous one. Recurrence relations can be used to solve problems involving systems which grow or shrink by the same amount at regular intervals (e.g. the amount of money in a savings account which grows by 3.5% p/a, the volume of water left in a pool if 10% evaporates each day, etc).

Recurrence relations are generally written in one of two forms:

$$\mathbf{U_{n+1} = aU_n + b}$$

OR

$$\mathbf{U_n = aU_{n-1} + b}$$

In both cases, a term is found by multiplying the previous term by a constant *a*, then adding (or subtracting) another constant *b*.

$\mathbf{U_n}$ means the **nth term** in the sequence (i.e. U_7 would be the 7th term, etc). $\mathbf{U_0}$ ("U zero") is the starting point of the sequence, e.g. the amount of money put into an account before interest is added.

Example 1: A sequence is defined by the recurrence relation $U_{n+1} = 3U_n + 2$, $U_0 = 4$.

Find the value of U_4.

Example 2: A sequence is defined by the recurrence relation $U_n = 4U_{n-1} - 3$, where $U_0 = a$.

Find an expression for U_2 in terms of a.

Finding a Formula

Recurrence relations can be used to describe situations seen in real life where a quantity changes by the same percentage at regular intervals. The first thing to do in most cases is find a formula to describe the situation.

Example: Jennifer puts £5000 into a high-interest savings account which pays 7.5% p/a. Find a recurrence relation for the amount of money in the savings account.

Solution: Starting amount = £5000
After 1 year: amount = starting amount + 7.5% (i.e. 107.5% of starting amount)
= 1.075 x starting amount

Recurrence relation is: $\mathbf{U_{n+1} = 1.075U_n}$ $\mathbf{(U_0 = 5000)}$

Example 3: Find a recurrence relation to describe:

a) The amount left to pay on a loan of £10000, with interest charged at 1.5% per month and fixed monthly payments of £250.

b) The amount of water in a swimming pool of volume 750,000 litres if 0.05% per day is lost to evaporation, but 350 litres extra is added daily.

Example 4: Bill puts lottery winnings of £120000 in a bank account which pays 5% interest p/a. After a year, he decides to spend £20000 per year from the money in the account.

a) Find a recurrence relation to describe the amount of money left each year.

b) How much money will there be in the account after five years?

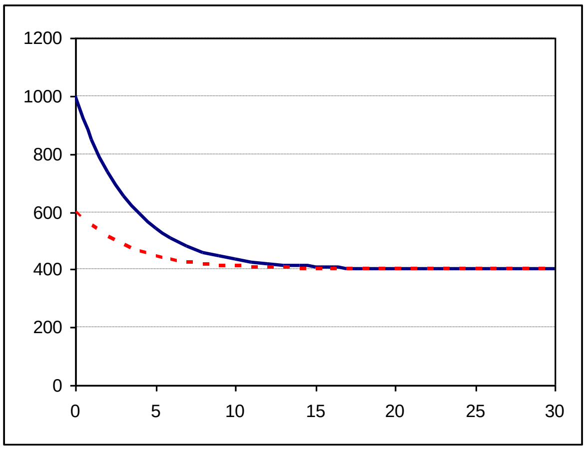
c) After how many years will Bill's money run out?

Limits of Recurrence Relations

Some recurrence relations produces sequences which tend towards a **limit**, i.e. as the number of terms increases, the sequence gets closer and closer to a certain value without actually meeting it.

The graph opposite shows two sequences generated by the recurrence relation $U_{n+1} = 0.75U_n + 100$, one where $U_0 = 1200$, the other where $U_0 = 600$.

In both cases, irrespective of the value of U_0, as U_n approaches infinity, the sequences tend to a limit of 400. How can we find the value of these limits without plotting a graph?



For $U_{n+1} = aU_n + b$, a limit exists when $-1 < a < 1$

If a limit exists, its value is independent of the value of U_0

Example 5: For the recurrence relation $U_{n+1} = 0.6U_n - 20$,

a) State whether a limit exists, and if so
b) Find the limit.

Example 6: A man plants some trees as a boundary between his house and the house next door. Each year, the trees are expected to grow by 0.5m. To counter this, he decides to trim them by 20% per year.

a) To what height will the trees eventually grow?

b) His neighbour is unhappy that the trees are too tall, and insists they grow no taller than 2m high. What is the **minimum** percentage they must be trimmed each year to meet this condition?

Solving Recurrence Relations to Find a and b

If we have three consecutive terms in a sequence, we can find the values of a and b in the recurrence relation which generated the sequence using simultaneous equations.

Example 7: A sequence is generated by a recurrence relation of the form $U_{n+1} = aU_n + b$. In this sequence, $U_1 = 28$, $U_2 = 32$ and $U_3 = 38$. Find the values of a and b.

Past Paper Example: Marine biologists calculate that when the concentration of a particular chemical in a loch reaches 5 milligrams per litre (mg/L) the level of pollution endangers the lives of the fish.

A factory wishes to release waste containing this chemical into the loch, and supplies the Scottish Environmental Protection Agency with the following information:

1. The loch contains none of the chemical at present.
2. The company will discharge waste once per week which will result in an increase in concentration of 2.5 mg/L of the chemical in the loch.
3. The natural tidal action in the loch will remove 40% of the loch every week.

a) After how many weeks at this level of discharge will the lives of the fish become endangered?

b) The company offers to install a cleaning process which would result in an increase in concentration of only 1.75 mg/L of the chemical in the loch, and claim this will not endanger the lives of the fish in the long term.

Should permission be given to allow the company to discharge waste into the loch using this revised process? Justify your answer.

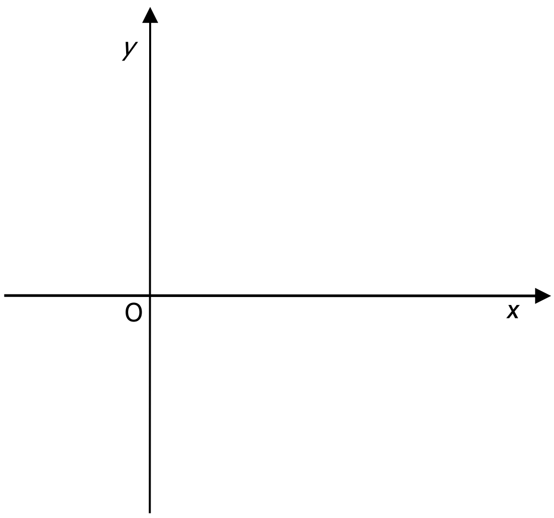
Graphs of Functions

Sketching a Quadratic Graph (Revision)

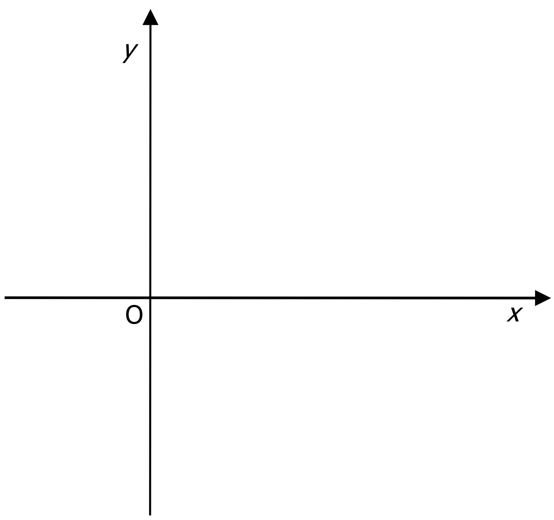
To sketch a quadratic graph:

- **Find the roots (set $y = 0$)**
- **Find the y – intercepts (set $x = 0$)**
- **Find the turning point (x value is halfway between roots; sub. into formula to find y)**

Example 1: Sketch and annotate the graph of $y = x^2 - 2x - 15$



Example 2: Sketch and annotate the graph of $y = x^2 - 4x + 4$



Note: when quickly sketching a quadratic graph, the roots and shape (“happy” or “sad” face) are enough.

Graphs of Functions

Sketching Graphs (Revision)

In the exam, diagrams are provided whenever the question involves a graph. However, this is not the case when working from the textbook: it is therefore important that we are able to sketch basic graphs where necessary, as often the question becomes simpler when you can see it.

Example 3: in the spaces provided, make a **basic** sketch of the graph(s) of the function(s) stated.

a) $y = 2x + 1$

b) $3x + 4y - 12 = 0$

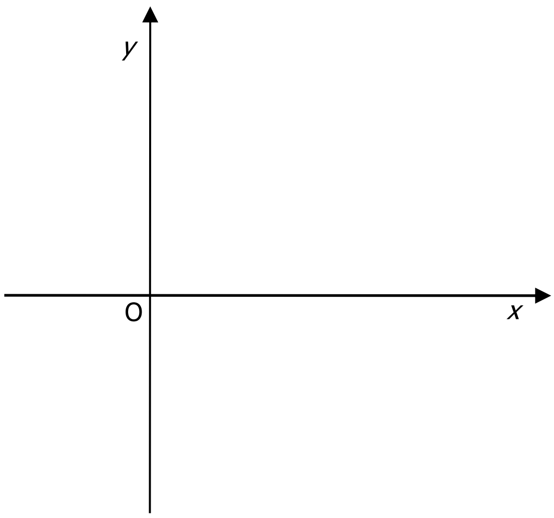
c) $y = -1$ **and** $x = 5$

d) $y = x^2$ **and** $y = 4$

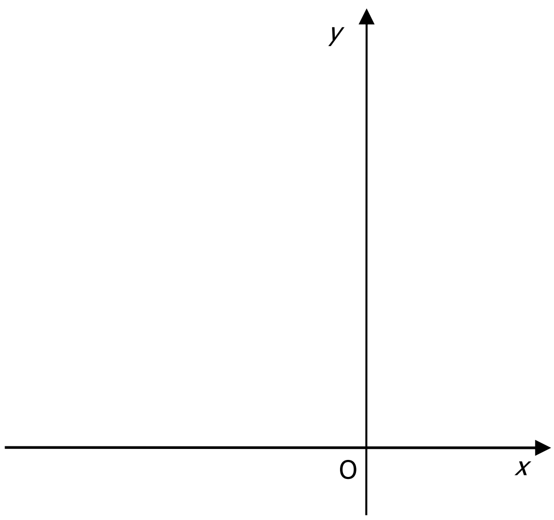
e) $y = x^2 - 4$

f) $y = (x - 2)^2$ **and** $y = 2x - x^2$

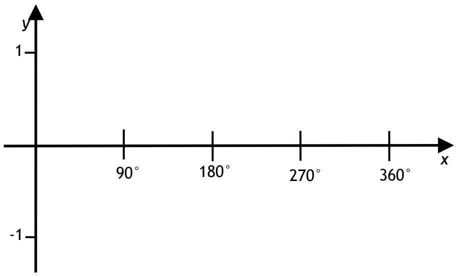
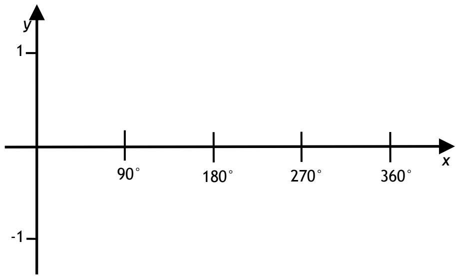
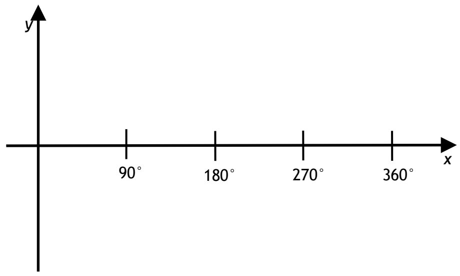
Example 4: Sketch and annotate the graph of
$y = x^2 - 2x - 8$



Example 5: Sketch and annotate the graph of
$y = (x + 3)^2 + 1$



Example 6: Sketch the graphs of y = sin$x$ °, y = cos$x$ ° and y = tan$x$ ° below.

		
***y* = sin*x* °**	***y* = cos*x* °**	***y* = tan*x* °**

For trig graphs, how soon the graph repeats itself horizontally is known as the **period**, and half of the vertical height is known as the **amplitude.**

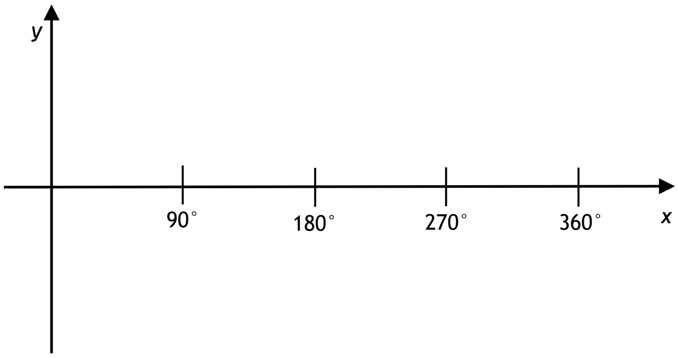
Function	Period	Amplitude
y = sin$x$ °		
y = cos$x$ °		
y = tan$x$ °		

For the graphs of:

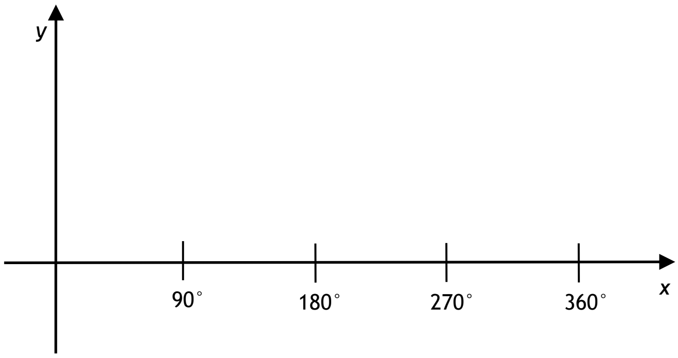
$y = a$ sinbx ° + c and $y = a$ cosbx ° + c :	***a* = amplitude** ***b* = waves in 360°** ***c* = vertical shift**	$y = a$ tanbx ° + c:	***b* = "waves" in 180°** ***c* = vertical shift**

Example 7: Sketch the graphs of:

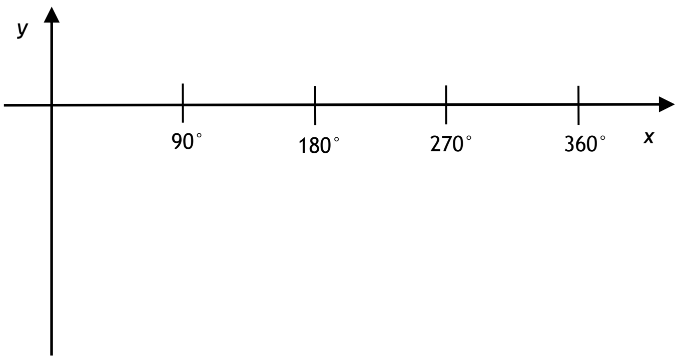
a) $y = \sin 2x°$



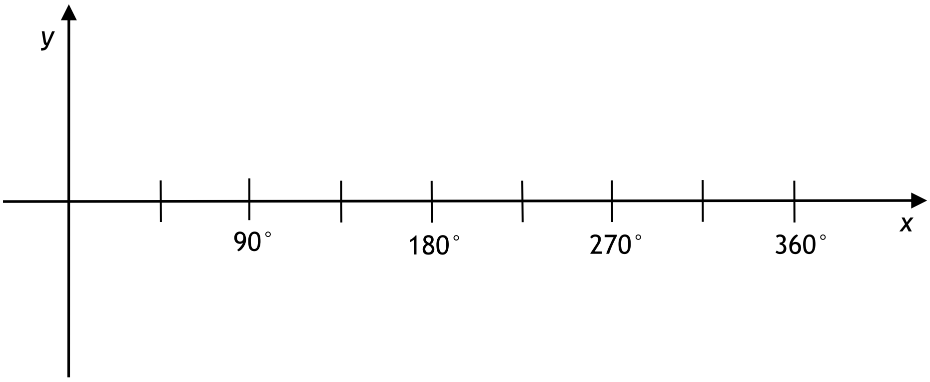
b) y = 5cos2x ° + 3



c) y = -3sin3x ° – 2



A compound angle is one containing two parts, e.g. $(x - 60)^\circ$. The graphs of compound angles can be thought of as the trig version of $y = f\,(x - a)$, i.e. shifted left or right by a units.



Example 8: On the axes opposite, sketch:

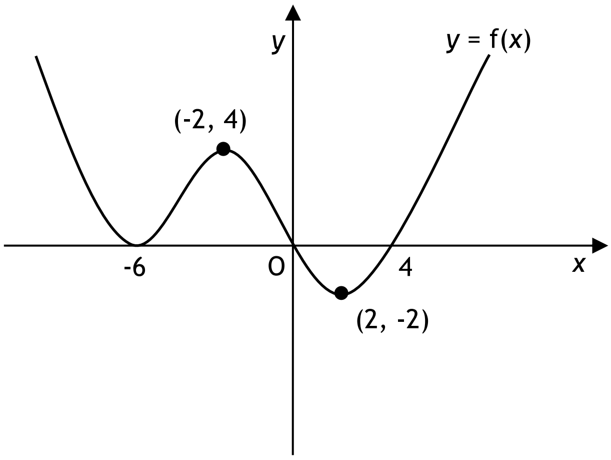
a) $y = \sin x\ ^\circ$

b) $y = \sin\,(x - 45)^\circ$

Transformation of Graphs

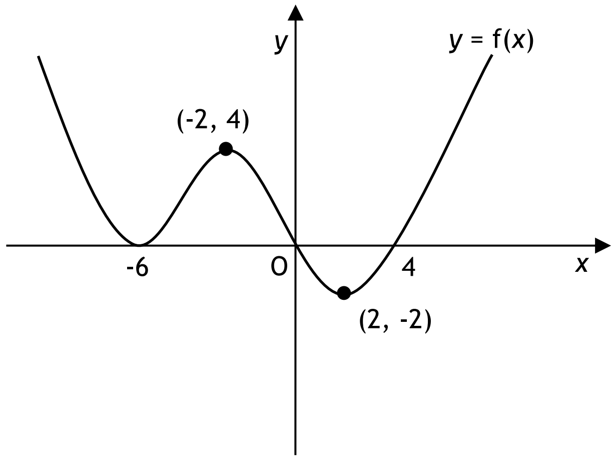
We have seen how the graph of y = sin(x) is different to that of y = sin(2x), and how $y = x^2$ differs from $y = (x - 1)^2$. The six operations below are used to transform the graph of a function:

y = f(*x*) + a



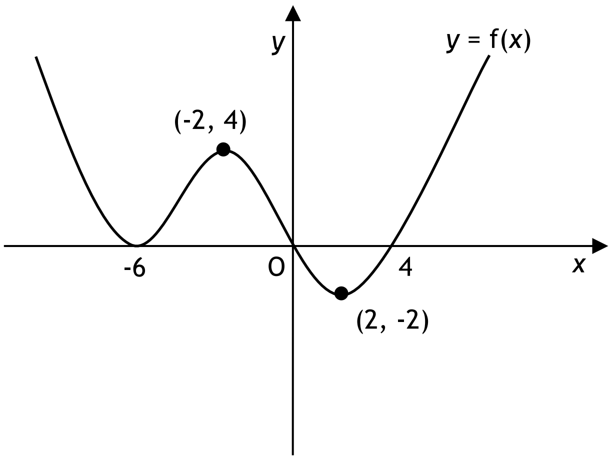
y = f(*x*) + a is obtained by **sliding** y = f(*x*):
Vertically upwards if a > 0
Vertically downwards if a < 0

y = f(*x* + a)



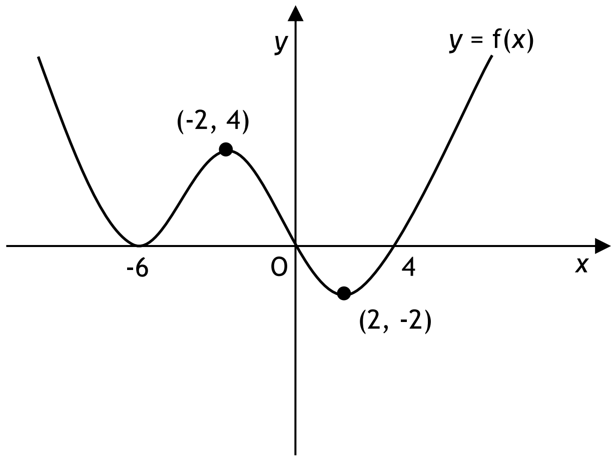
y = f(*x* + a) is obtained by **sliding** y = f(*x*):
Horizontally left if a > 0
Horizontally right if a < 0

y = k f(*x*)



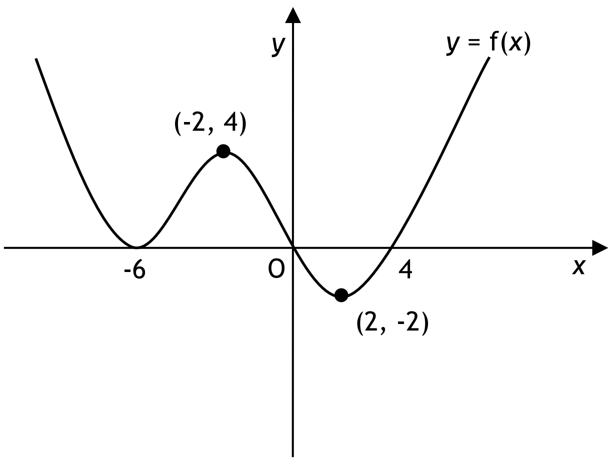
y = k f(*x*) is obtained by **vertically:**
stretching *y* = f(*x*) if k > 1
compressing *y* = f(*x*) if 0 < k < 1

y = f(k*x*)



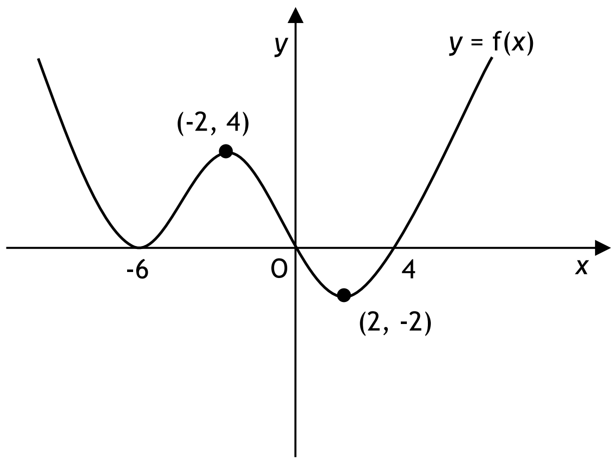
y = f(k*x*) is obtained by **horizontally:**
compressing *y* = f(*x*) if k > 1
stretching *y* = f(*x*) if 0 < k < 1

y = - f(*x*)



y = **-** f(*x*) is obtained by **reflecting** y = f(*x*):
in the *x* - axis

y = f(-*x*)



y = f(-*x*) is obtained by **reflecting** y = f(*x*):
in the *y* - axis

Multiple Transformations

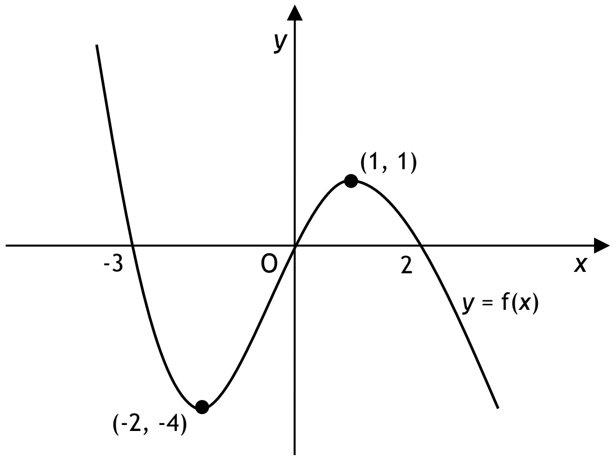
Often, are asked to perform more than one transformation on a graph.
Where appropriate, always leave sliding vertically to last.

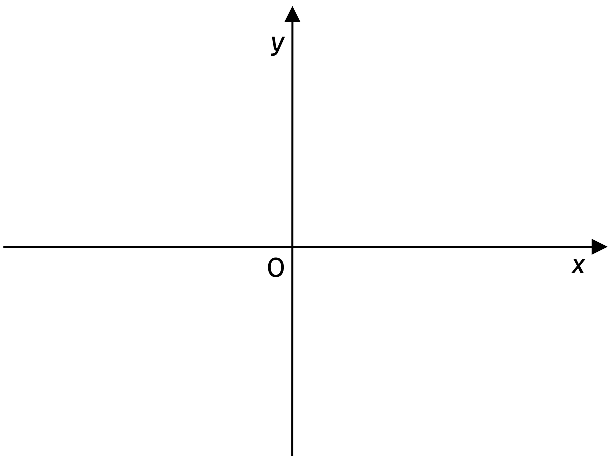
Example 9: Part of the graph of $y = f(x)$ is shown.

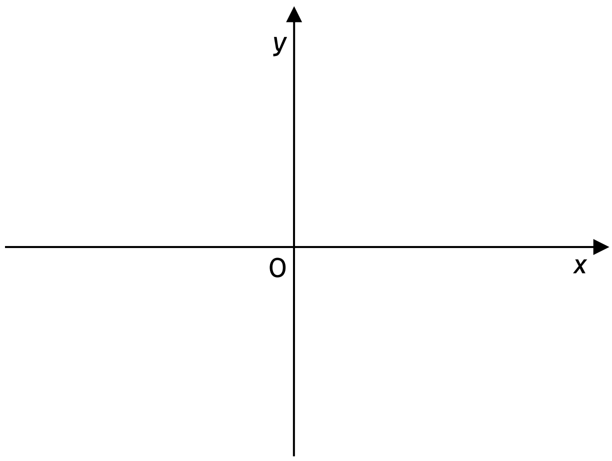
On separate diagrams, sketch:

a) $y = f(-x) + 2$

b) $y = -\frac{1}{2} f(x + 1)$





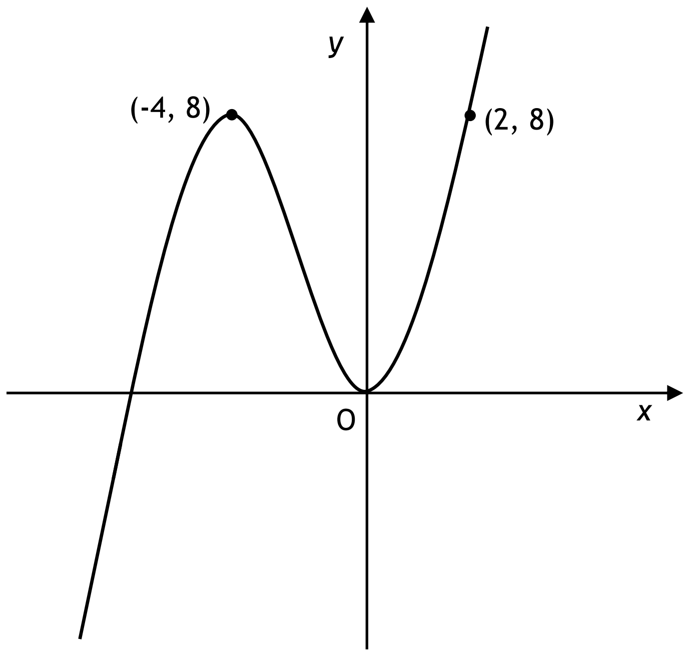


Past Paper Example: The diagram shows a sketch of the function $y = f(x)$.

To the diagram, add the graphs of:

a) $y = f(2x)$

b) $y = 1 - f(2x)$.



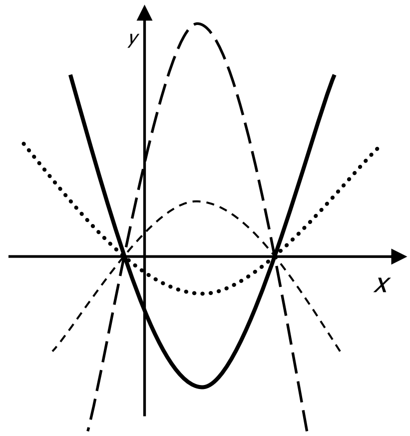
Quadratic Functions

Finding the Equation of a Quadratic Function From Its Graph: $y = k(x - a)(x - b)$

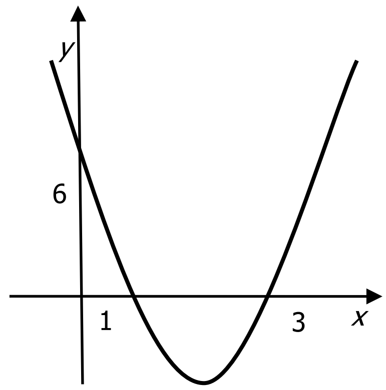
If the graph of a quadratic function has roots at $x = -1$ and $x = 5$, a reasonable guess at its equation would be $y = x^2 - 4x - 5$, i.e. from $y = (x + 1)(x - 5)$.

However, as the diagram shows, there are many parabolas which pass through these points, all of which belong to the **family** of functions $\boldsymbol{y = k\,(x + 1)\,(x - 5)}$.

To find the equation of the original function, we need the roots and one other point on the curve (to allow us to determine the value of k).



Example 1: State the equation of the graph below in the form $y = ax^2 + bx + c$.



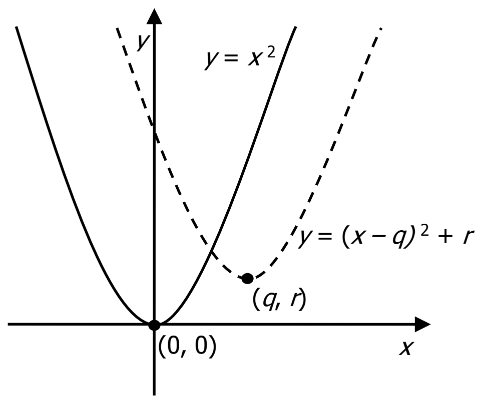
Completing the Square (Revision)

The diagram shows the graphs of two quadratic functions.

If the graph of $y = x^2$ is shifted q units to the right, followed by r units up, then the graph of $y = (x - q)^2 + r$ is obtained.

As the turning point of $y = x^2$ is (0, 0), it follows that the new curve has a turning point at $\boldsymbol{(q\ ,\ r)}$.

A quadratic equation written as $y = p\,(x - q)^2 + r$ is said to be in the **completed square form.**



Example 2: (i) Write the following in the form $y = (x + q)^2 + r$ and find the minimum value of y.

(ii) Hence state the minimum value of y and the corresponding value of x .

a) $y = x^2 + 6x + 10$

b) $y = x^2 - 3x + 1$

Completing the Square when the x^2 Coefficient ≠ 1

Example 3: Write $y = 3x^2 + 12x + 5$ in the form $y = p(x + q)^2 + r$.

Example 4: Write $y = 5 + 12x - x^2$ in the form $y = p - (x + q)^2$.

Example 5:

a) Write $y = x^2 - 10x + 28$ in the form $y = (x + p)^2 + q$.

b) Hence find the maximum value of $\dfrac{18}{x^2 - 10x + 28}$

Solving Quadratic Equations via Completing the Square

Quadratic equations which do not easily factorise can be solved in two ways: (i) completing the square, or (ii) using the quadratic formula. In fact, both methods are essentially the same, as the quadratic formula is obtained by solving $y = ax^2 + bx + c$ via completing the square.

Example 6: State the **exact** values of the roots of the equation $2x^2 - 4x + 1 = 0$ by:

a) using the quadratic formula

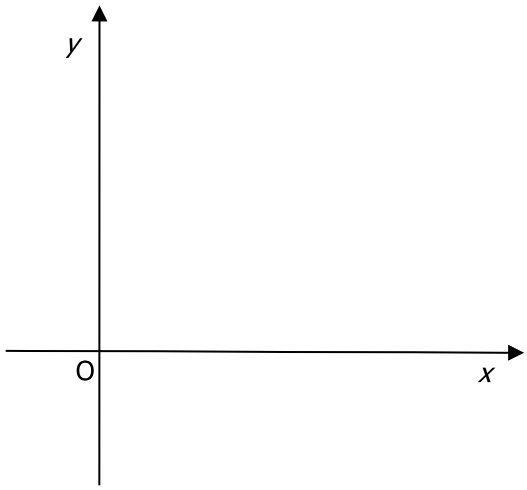
b) completing the square

Solving Quadratic Inequations

Quadratic inequations are easily solved by making a sketch of the equivalent quadratic function, and determining the regions above or below the x – axis.

Example 7: Find the values of x for which: a) $2x^2 - 7x + 6 > 0$ b) $2x^2 - 7x + 6 < 0$

First, sketch $y = 2x^2 - 7x + 6$



Roots of Quadratic Equations and The Discriminant (Revision)

For $y = ax^2 + bx + c$, $b^2 - 4ac$ is known as the **discriminant.**

- **$b^2 - 4ac > 0$ gives real, unequal roots**
- **$b^2 - 4ac = 0$ gives real, equal roots**
- **$b^2 - 4ac < 0$ gives NO real roots**

If $b^2 - 4ac$ gives a perfect square, the roots are RATIONAL
If $b^2 - 4ac$ does NOT give a perfect square, the roots are IRRATIONAL (i.e. surds)

Example 8: Determine the nature of the roots of the equation $4x(x - 3) = 9$

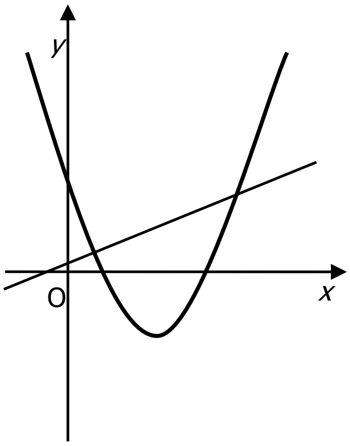
Example 9: Find the value(s) of p given that $2x^2 + 4x + p = 0$ has real roots.

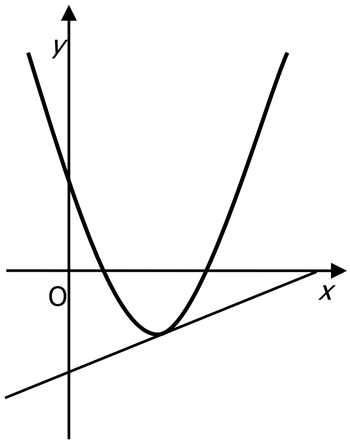
Example 10: Find the value(s) of r given that $x^2 + (r - 3)x + r = 0$ has no real roots.

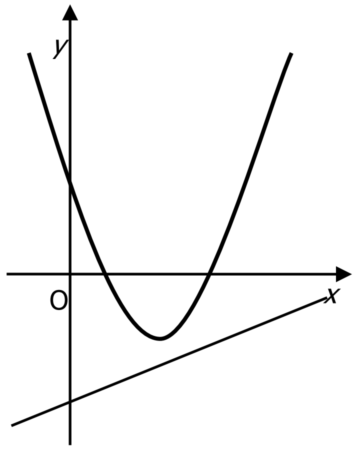
Tangents to Curves: Using the Discriminant

To find the points of contact between a line and a curve, we make the curve and line equations equal (i.e. make $y = y$) to obtain a quadratic equation, and solve to find the x-coordinates.

By finding the discriminant of this quadratic equation, we can work out **how many** points of contact there are between the line and the curve. There are 3 options:







Two points of contact	One point of contact	No points of contact
2 distinct roots	**Equal roots**	**No real roots**
$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$

The most common use for this technique is to show that a line is a tangent to a curve

Example 11: Show that the line $y = 3x - 13$ is a tangent to the curve $y = x^2 - 7x + 12$, and find the coordinates of the point of contact.

Example 12: Find **two** values of m such that $y = mx - 7$ is a tangent to $y = x^2 + 2x - 3$

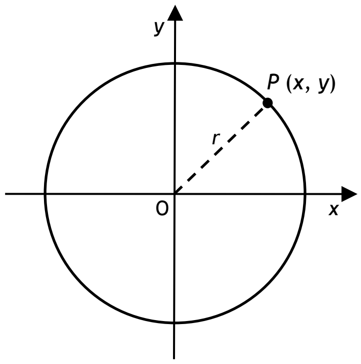
Past Paper Example 1: Express $2x^2 + 12x + 1$ in the form $a(x + b)^2 + c$.

Past Paper Example 2: Given that $2x^2 + px + p + 6 = 0$ has no real roots, find the range of values for p.

Past Paper Example 3: Show that the roots of $(k - 2)x^2 - 3kx + 2k = -2x$ are **always** real.

The Circle

If we draw, suitable to relative axes, a circle, radius *r*, centred on the origin, then the distance from the centre of any point *P* (*x*, *y*) could be determined to be $d=\sqrt{x^2+y^2}$.



As the shape is a circle, then this distance is equal to the radius. It therefore follows that:

Since $r=\sqrt{x^2+y^2}$, then $r^2=x^2+y^2$

Therefore,

The equation $x^2 + y^2 = r^2$ describes a circle with centre (0, 0) and radius r

Example 1: Write down the centre and radius of each circle.

a) $x^2 + y^2 = 64$

b) $x^2 + y^2 = 361$

c) $x^2 + y^2 = \frac{3}{25}$

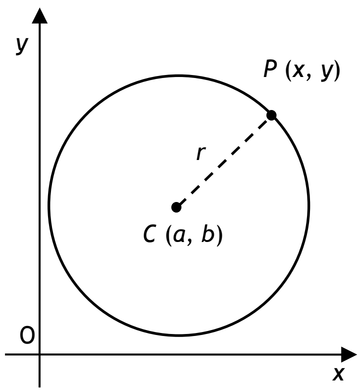
Example 2: State where the points (-2, 7), (6, -8) and (5, 9) lie in relation to the circle $x^2 + y^2 = 100$.

Circles with Centres *Not* at the Origin

The radius in the above circle is the distance between (x , y) and the origin, i.e. $r=\sqrt{(x-0)^2+(y-0)^2}$. If we move the centre to the point (a , b), then $r=\sqrt{(x-a)^2+(y-b)^2}$.

Squaring both sides, we can now also say that:

The equation $(x - a)^2 + (y - b)^2 = r^2$ describes a circle with centre (a, b) and radius r



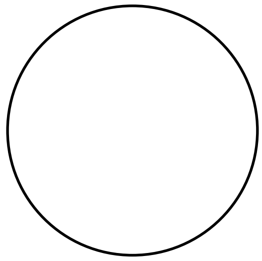
Example 3: Write down the centre and radius of each circle.

a) $(x - 1)^2 + (y + 3)^2 = 4$

b) $(x + 9)^2 + (y - 2)^2 = 20$

c) $(x - 5)^2 + y^2 = 400$

Example 4: A is the point (4, 9) and B is the point (-2, 1).
Find the equation of the circle for which AB is the diameter.



Example 5: Points P, Q and R have coordinates (-10, 2), (5, 7) and (6, 4) respectively.

a) Show that triangle PQR is right angled at Q.

b) Hence find the equation of the circle passing through points P, Q and R.

The General Equation of a Circle

For the circle described in Example 3a, we could expand the brackets and simplify to obtain the equation $x^2 + y^2 - 2x + 6y + 6 = 0$, which would **also** describe a circle with centre (1, -3) and radius 2.

For $x^2 + y^2 + 2gx + 2fy + c = 0$,

$$(x^2 + 2gx) + (y^2 + 2fy) = -c$$

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

$$(x + g)^2 + (y + f)^2 = (g^2 + f^2 - c)$$

Therefore, the circle described by

$$\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0}$$

has centre ($-g$, $-f$) and $r = \sqrt{g^2 + f^2 - c}$

Example 6: Find the centre and radius of the circle with equation $x^2 + y^2 - 4x + 8y - 5 = 0$

Example 7: State why the equation $x^2 + y^2 - 4x - 4y + 15 = 0$ does **not** represent a circle.

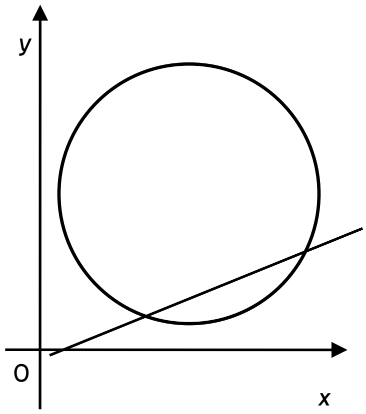
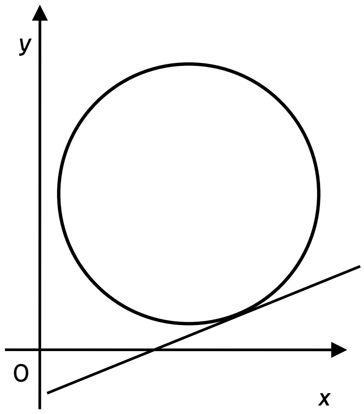
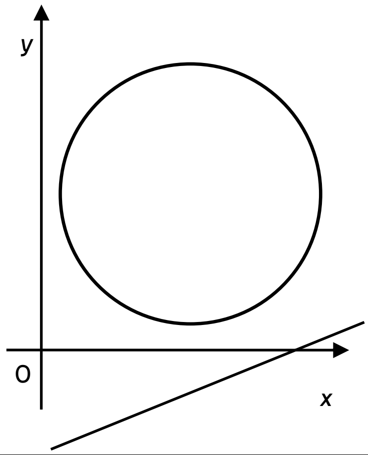
Example 8: State the range of values of c such that the equation $x^2 + y^2 - 4x + 6y + c = 0$ describes a circle.

Example 9: Find the equation of the circle concentric with $x^2 + y^2 + 6x - 2y - 54 = 0$ but with radius half its size.

Intersection of Lines and Circles

As with parabolas, there are **three** possibilities when a line and a circle come into contact, and we can examine the roots of a rearranged quadratic equation to determine which has occurred. However:

We CANNOT make the circle and line equations equal to each other: the line equation must be substituted INTO the circle equation to obtain our quadratic equation!

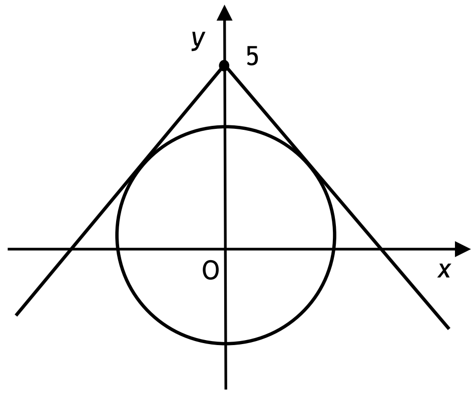
		
Two points of contact **2 distinct roots** $b^2 - 4ac > 0$	**One point of contact** **Equal roots** $b^2 - 4ac = 0$	**No points of contact** **No real roots** $b^2 - 4ac < 0$

As with parabolas, the most common use of this technique is to show tangency.

Example 10: Find the coordinates of the points of intersection of the line $y = 2x - 1$ and the circle $x^2 + y^2 - 2x - 12y + 27 = 0$.

Example 11: Show that the line $y = 3x + 10$ is a tangent to the circle $x^2 + y^2 - 8x - 4y - 20 = 0$ and establish the coordinates of the point of contact.

Example 12: Find the equations of the tangents to the circle $x^2 + y^2 = 9$ from the point (0, 5).

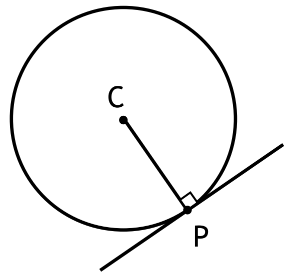


Tangents to Circles at Given Points

Remember: at the point of contact, the radius and tangent meet at 90° (i.e., they are perpendicular).

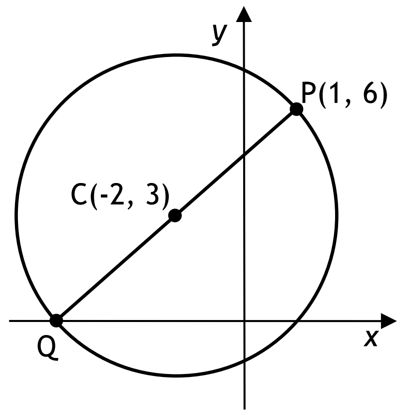
To find a tangent at a given point:

- **Find the centre of the circle**
- **Find the gradient of the radius (joining C and the given point)**
- **Find the gradient of the tangent (flip and make negative)**
- **Sub the gradient and the original point into $y - b = m(x - a)$**



Example 13: Find the equation of the tangent to $x^2 + y^2 - 14x + 6y - 87 = 0$ at the point (-2, 5).

Past Paper Example 1: A circle has centre C (-2, 3) and passes through point P (1, 6).



a) Find the equation of the circle.

b) PQ is a diameter of the circle. Find the equation of the tangent to this circle at Q.

Past Paper Example 2:

a) Show that the line with equation $y = 3 - x$ is a tangent to the circle with equation

$$x^2 + y^2 + 14x + 4y - 19 = 0$$

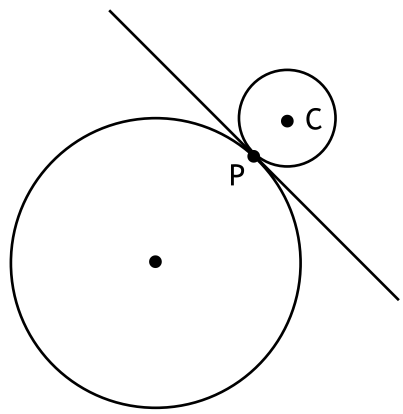
and state the coordinates of P, the point of contact.

b) Relative to a suitable set of coordinate axes, the diagram opposite shows the circle from a) and a second smaller circle with centre C.

The line $y = 3 - x$ is a common tangent at the point P.

The radius of the larger circle is three times the radius of the smaller circle.

Find the equation of the smaller circle.



Past Paper Example 3: Given that the equation

$$x^2 + y^2 - 2px - 4py + 3p + 2 = 0$$

represents a circle, determine the range of values of *p*.

Past Paper Example 4: Circle P has equation $x^2 + y^2 - 8x - 10y + 9 = 0$. Circle Q has centre (-2, -1) and radius $2\sqrt{2}$.

a) i) Show that the radius of circle P is $4\sqrt{2}$.
   ii) Hence show that circles P and Q touch.

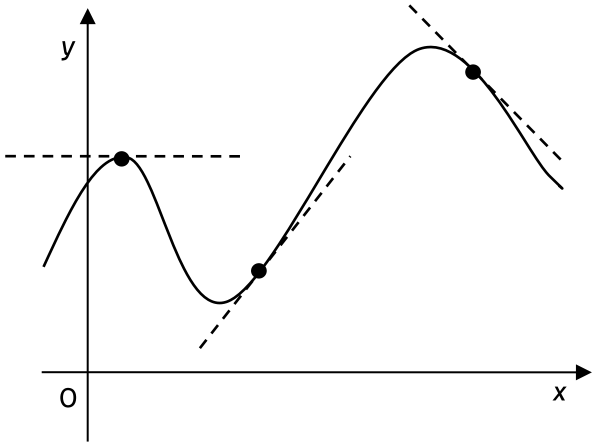
b) Find the equation of the tangent to circle Q at the point (-4, 1)

Calculus 1: Differentiation

In the chapter on straight lines, we saw that the gradient of a line is a measure of how quickly it increases (or decreases) at a constant rate.

This is easy to see for linear functions, but what about quadratic, cubic and higher functions? As these functions produce curved graphs, they do **not** increase or decrease at a constant rate.

For a function f(x), the rate of change at any point on the function can be found by measuring the gradient of a **tangent** to the curve at that point.



The rate of change at any point of a function is called the **derived function** or the **derivative.**

Finding the rate of change of a function at a given point is part of a branch of maths known as **calculus.**

For function f(x) or the graph y = f(x), the derivative is written as:

f'(x) ("f dash x")

OR

$\dfrac{dy}{dx}$ **("dy by dx")**

Derivative = Rate of Change of the Function = Gradient of the Tangent to the Curve

The Derivative of $f(x) = ax^n$

Example 1: Find the derivative of $f(x) = x^2$

To find the derivative of a function:

1. **Make sure it's written in the form $y = ax^n$**
2. **Multiply by the power**
3. **Decrease the power by one**

Example 2: $f(x) = 2x^3$. Find $f'(x)$.

This means:

At $x = 1$, the gradient of the tangent to $2x^3$ =

At $x = -2$, the gradient of the tangent to $2x^3$ =

If $f(x) = ax^n$, then $f'(x) = nax^{n-1}$

The *DE* rivative *DE* creases the power!

To find the derivative of $f(x)$:

- $f(x)$ MUST be written in the form $f(x) = ax^n$
- Rewrite to eliminate fractions by using negative indices
- Rewrite to eliminate roots by using fractional indices

Revision from National 5

Example 3: Write with negative indices:

a) $\dfrac{2}{x^2}$ b) $\dfrac{1}{4x^5}$ c) $\dfrac{3}{5x}$

Example 4: Write in index form:

a) $\sqrt{x}$ b) $\sqrt[3]{x^2}$ c) $\dfrac{2}{3\sqrt{x^7}}$

Example 5: For each function, find the derivative.

a) $f(x) = x^{35}$

b) $g(x) = -x^{-3}$ $(x \neq 0)$

c) $p(x) = \dfrac{1}{\sqrt{x}}$ $(x > 0)$

d) $y = 12x^5 + 3x^2 - 2x + 9$

e) $y = \dfrac{1}{3\sqrt{x}}$ $(x > 0)$

f) $y = \left(\sqrt{x} - 2\right)^2$ $(x \geq 0)$

Example 6: Find the rate of change of each function:

a) $f(x) = \dfrac{x^5 - 6x^3}{x^2}$

b) $y = \dfrac{(x+3)^2}{x^{2/3}}$

c) $f(x) = \dfrac{x^5 - 3x}{2x^3}$

Points to note:	• **Number terms disappear (e.g. if f(x) = 5, f'(x) = 0)** • ***x* - terms leave their coefficient (e.g. if f(x) = 135x , f'(x) = 135)** • **Give your answer back in the same form as the question**

Equation of a Tangent to a Curve

Example 7: Find the equation of the tangent to the curve $y = x^2 - 2x - 15$ when $x = 4$.

To find the equation of a tangent to a curve:

- **Find the point of contact (sub the value of *x* into the equation to find *y*)**
- **Find $\frac{dy}{dx}$**
- **Find *m* by substituting *x* into $\frac{dy}{dx}$**
- **Use $y - b = m(x - a)$**

Example 8:

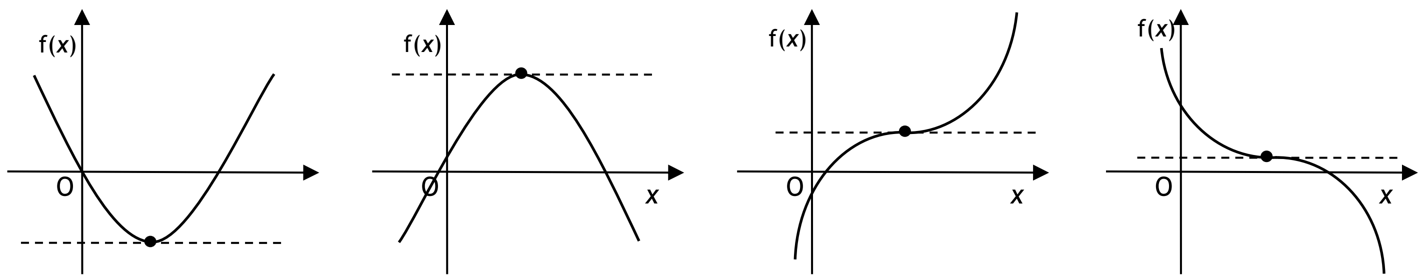
a) Find the gradient of the tangent to the curve $y = x^3 - 2x^2$ at the point where $x = \frac{7}{3}$.

b) Find the other point on the curve where the tangent has the same gradient.

Example 9: Find the point of contact of the tangent to the curve with equation $y = x^2 + 7x + 3$ when the gradient of the tangent is 9.

Stationary Points and their Nature

Any point on a curve where the tangent is horizontal (i.e. the gradient or $\frac{dy}{dx} = 0$) is commonly known as a **stationary point.** There are four types of stationary point:



Minimum Turning Point | **Maximum Turning Point** | **Rising Point of Inflection** | **Falling Point of Inflection**

To locate the position of stationary points, we find the derivative, make it equal zero, and solve for x. To determine their type (or **nature**), we must use a **nature table.**

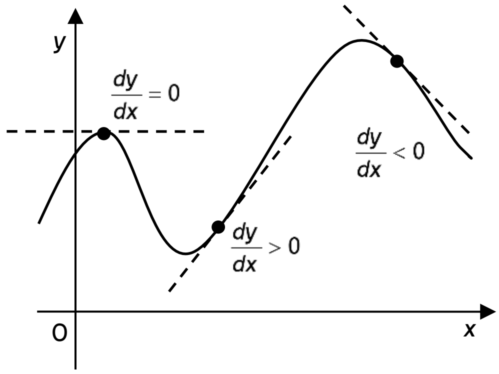
Example 10: Find the stationary points of the curve $y = 2x^3 - 12x^2 + 18x$ and determine their nature.

For any curve,

if $\frac{dy}{dx} > 0$, then y is **increasing**

if $\frac{dy}{dx} < 0$, then y is **decreasing**

if $\frac{dy}{dx} = 0$, then y is **stationary**



If a function is **always** increasing (or decreasing), it is said to be **strictly** increasing (or decreasing).

Example 11: State whether the function $f(x) = x^3 - x^2 - 5x + 2$ is **increasing**, **decreasing** or **stationary** when:

a) $x = 0$

b) $x = 1$

c) $x = 2$

Example 12: Show algebraically that the function $f(x) = x^3 - 6x^2 + 12x - 5$ is never decreasing.

Example 13: Find the intervals in which the function $f(x) = 2x^3 - 6x^2 + 5$ is increasing and decreasing.

Curve Sketching

To accurately sketch and annotate the curve obtained from a function, we must consider:

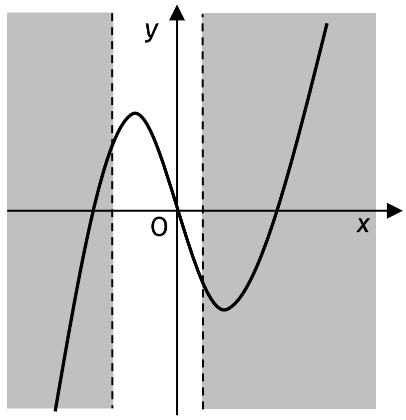
1. **x – and y – intercepts**
2. **Stationary points and their nature**

Example 14: Sketch and annotate fully y = $x^3(4 - x)$

Closed Intervals

Sometimes, we may only be interested in a small section of the curve of a function. To find the maximum and minimum values of a function in a given interval, we find stationary points as normal, but we also need to consider the value of the function at the ends of the interval.

Example 15: Find the greatest and least values of y = $x^3 - 12x$ on the interval $-3 \leq x \leq 1$.



Note: In a closed interval. The maximum and minimum values of a function occur either at a Stationary Point within the interval or at the end point of the interval.

Differentiation in Context: Optimisation

Differentiation can be used to find the maximum or minimum values of things which happen in real life. Finding the maximum or minimum value of a system is called **optimisation.**

Example 16: A carton is in the shape of a cuboid with a rectangular base and a volume of 3888cm^3.

The surface area of the carton can be represented by the formula $A(x) = 4x^2 + \dfrac{5832}{x}$.

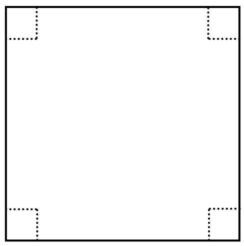
Find the value of x such that the surface area is a minimum.

In exams, optimisation questions almost always consist of two parts: part one asks you to **show** that a situation can be described using an algebraic formula or equation, whilst part two asks you to use the given formula to find a maximum or minimum value by differentiation.

Leave part 1 of an optimisation question until the end of the exam (if you have time), as they are almost always (i) more difficult than finding the stationary point and (ii) worth fewer marks.

Remember that part 2 is just a well-disguised "find the minimum/maximum turning point of this function" question!

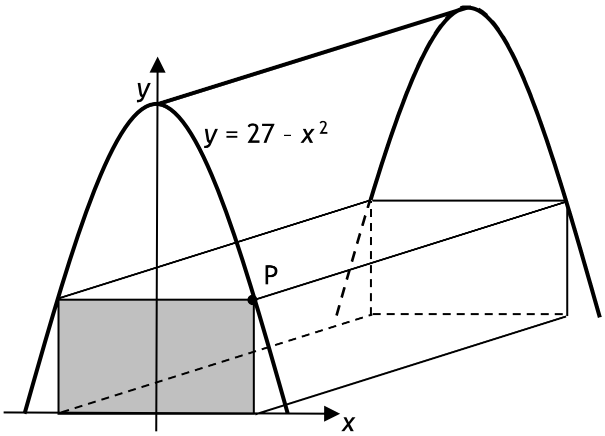
Example 17: A square piece of card of side 30cm has a square of side x cm cut from each corner.
An open box is formed by turning up the sides.



a) Show that the volume, V , of the box may be expressed as $900x - 120x^2 + 4x^3$

b) Find the maximum volume of the box.

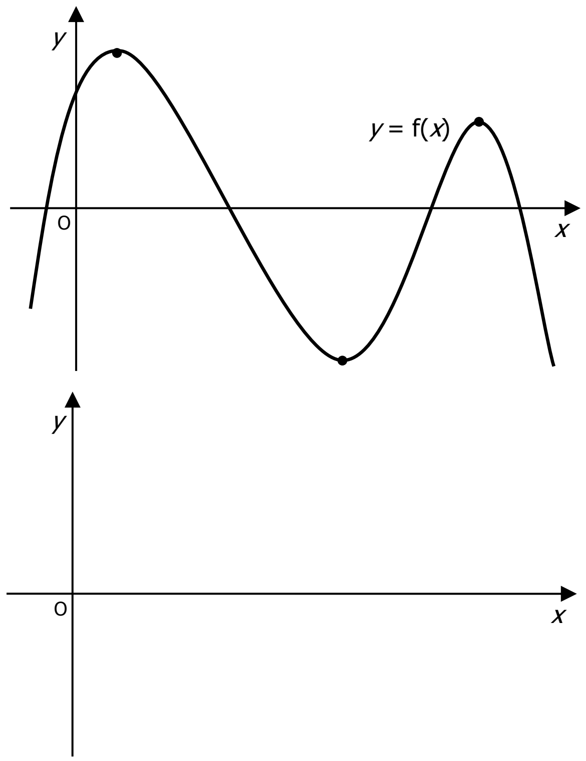
Example 18: An architect has designed a new open-plan office building using two identical parabolic support beams spaced 25m apart as shown below. The front beam, relative to suitable axes, has the equation $y = 27 - x^2$. The inhabited part of the building is to take the shape of a cuboid.



a) By considering the point P in the corner of the front face of the building, show that the area of this face is given by $\boldsymbol{A(x) = 54x - 2x^3}$.

b) Find the maximum volume of the inhabited section of the building.

Graph of the Derived Function



From the graph of y = f(x), we can obtain the graph of y = f'(x) by considering its stationary points. On the graph of y = f'(x), the y-coordinate comes from the derivative of y = f(x).

1. Draw a set of axes directly under a copy of y = f(x).

2. Locate the stationary points.

3. At SP's, f'(x) = 0, so the y coordinate of f'(x) = 0 on the new graph.

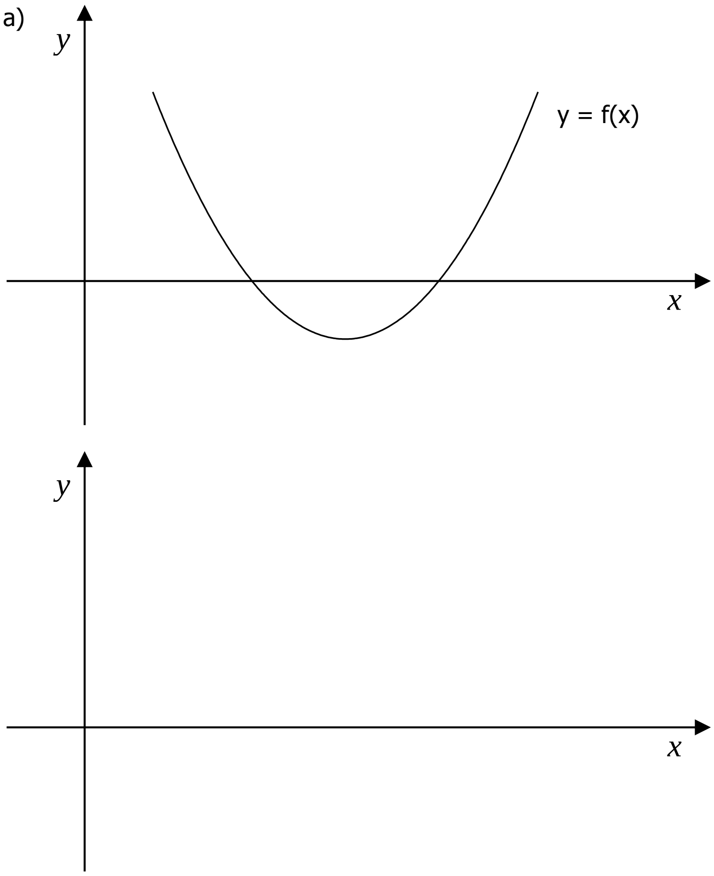
4. Where f(x) is increasing, f'(x) is **above** the x – axis.

5. Where f(x) is decreasing, f'(x) is **below** the x – axis.

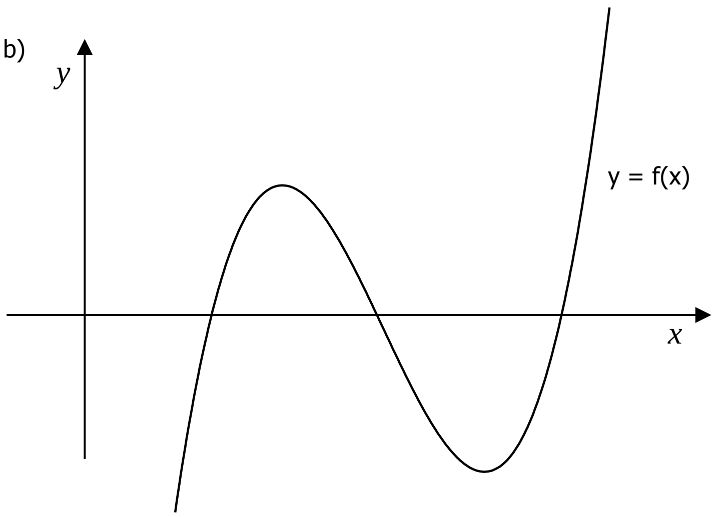
6. Draw a smooth curve which fits this information.

Example 19: For the graphs below. Sketch the corresponding derived graphs of y = f'(x)

a)



b)

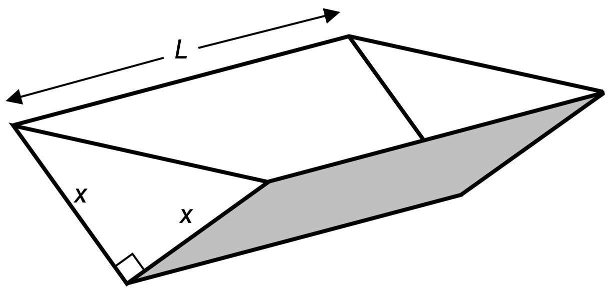


Past Paper Example 1: A curve has equation $y = x^4 - 4x^3 + 3$. Find the position and nature of its stationary points.

Past Paper Example 2: Find the equation of the two tangents to the curve $y = 2x^3 - 3x^2 - 12x + 20$ which are parallel to the line $48x - 2y = 5$.

Past Paper Example 3: An open water tank, in the shape of a triangular prism, has a capacity of 108 litres. The tank is to be lined on the inside in order to make it watertight.

The triangular cross-section of the tank is right-angled and isosceles, with equal sides of length x cm. The tank has a length of L cm.



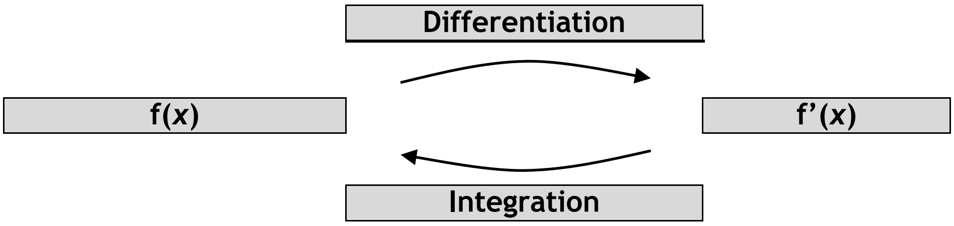
a) Show that the surface area to be lined, A cm^2, is given by $A(x) = x^2 + \dfrac{432000}{x}$

b) Find the minimum surface area of the tank.

Past Paper Example 4: A function is defined on the domain $0 \leq x \leq 3$ by $f(x) = x^3 - 2x^2 - 4x + 6$. Determine the maximum and minimum values of f.

Calculus 2: Integration

The reverse process to differentiation is known as **integration.**



As it is the opposite of finding the derivative, the function obtained by integration is sometimes called the **anti-derivative,** but is more commonly known as the **integral**, and is given the sign $\int$.

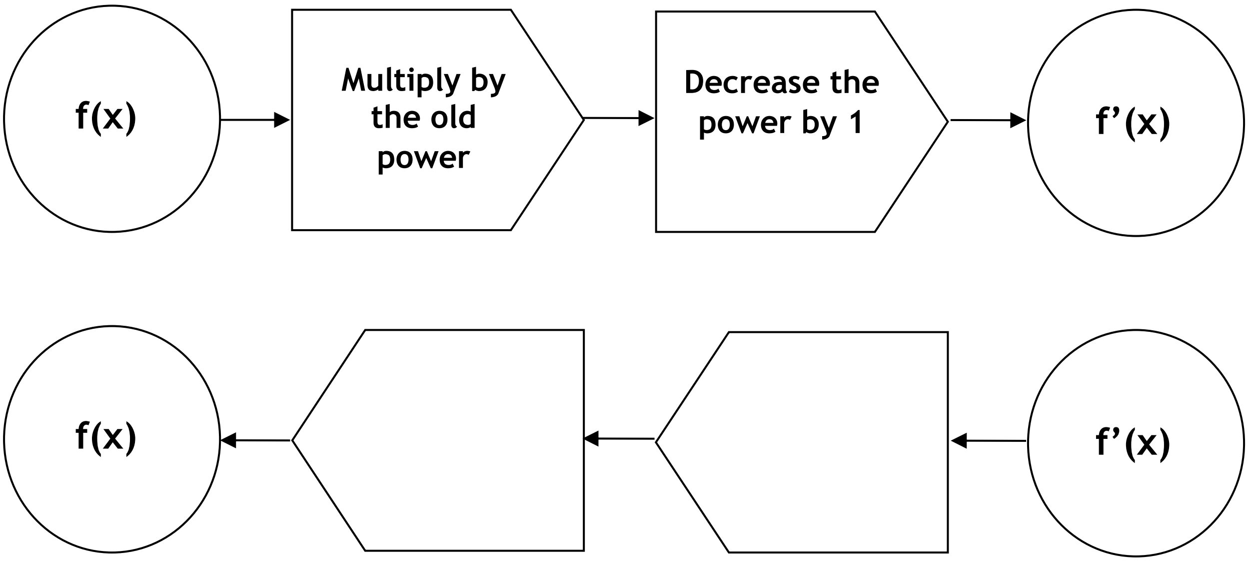
If f(x) = x^n, then $\int x^n dx$ is "the integral of x^n with respect to x "

Indefinite Integrals and the Constant of Integration

Consider the three functions a(x) = $3x^2 + 2x + 5$, b(x) = $3x^2 + 2x - 8$ and c(x) = $3x^2 + 2x - \frac{13}{4}$.

In each case, the derivative of the function is the same, i.e. $6x + 2$. This means that $\int (6x+2)dx$ has more than one answer. Because there is more than one answer, we say that this is an **indefinite integral**, and we must include in the answer a constant value C, to represent the 5, -8, $-\frac{13}{4}$ etc which we would need to distinguish a(x) from b(x) from c (x) etc.

To find the integral of a function, we do the **opposite** of what we would do to find the derivative:



In general:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

IN **tegration** ***IN*** **creases the power!**
1. **Write as ax^n**
2. **Increase the power by 1**
3. **Divide by the new power**

Example 1: Find **(remember "+C"):**

a) $\int 2x\,dx$ **b)** $\int 4t^2\,dt$ **c)** $\int (3x^5 - 4)dx$

d) $\int \frac{3}{g^4}\,dg$ $(g \neq 0)$

e) $\int 6\sqrt[5]{p^3}\,dp$

f) $\int \frac{4y-3}{y^{2/3}}\,dy$ $(y \neq 0)$

The Definite Integral

A **definite integral** of a function is the difference between the integrals of f(x) at two values of x. Suppose we integrate f(x) and get F(x). Then the integral of f(x) when x = a would be F(a), and the integral when x = b would be F(b).

The definite integral of f(x), with respect to x, between a and b, is written as:

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{(where b > a)}$$

For example, the integral of f(x) = $2x^2 - 4$ between the values x = -3 and x = 5 is written as $\int_{-3}^{5}(2x^2-4)\,dx$ and reads "the integral from -3 to 5 of $2x^2 - 4$ with respect to x".

Note: definite integrals do NOT include the constant of integration!

$$\int_a^b f(x) = [F(b)+C] - [F(a)+C] = F(b) - F(a)$$

Example 2: Evaluate $\int_{-1}^{3}(2x-1)\,dx$

To find a definite integral:

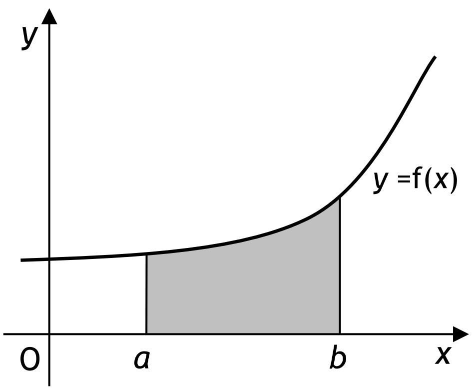
- **prepare the function for integration**
- **integrate as normal, but write inside square brackets with the limits to the right**
- **sub each limit into the integral, and subtract the integral with the lower limit from the one with the higher limit**

Example 3: Evaluate $\int_{0}^{2}(p+1)(p-1)\,dp$

Example 4: Evaluate $\int_{1}^{\sqrt{3}}(x^2 - 2x)\,dx$

Example 5: Find the value of g such that $\int_{-2}^{g}(6x+5)\,dx = 6$.

Area Between a Curve and the *x* - axis.



In the diagram opposite, the area of the shaded section can be obtained by finding the area under the graph from O to b, and subtracting the area from O to a.

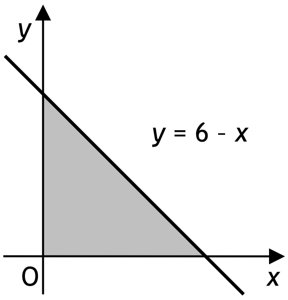
The value of each of these areas can be determined by integrating the function and substituting b or a respectively.

The area enclosed by the curve $y = f(x)$, the lines $y = a$, $y = b$ and the x - axis is equal to the definite integral of $f(x)$ between a and b

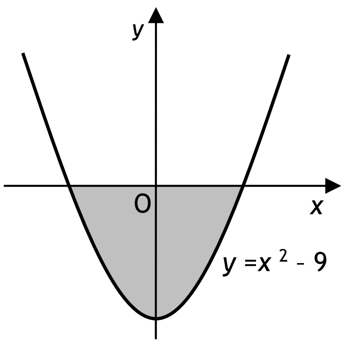
i.e. Area = $\int_{a}^{b} f(x)\,dx$

Example 6: For each graph below,
(i) write down the integrals which describe the shaded regions
(ii) calculate the area of the shaded region

a)



b)

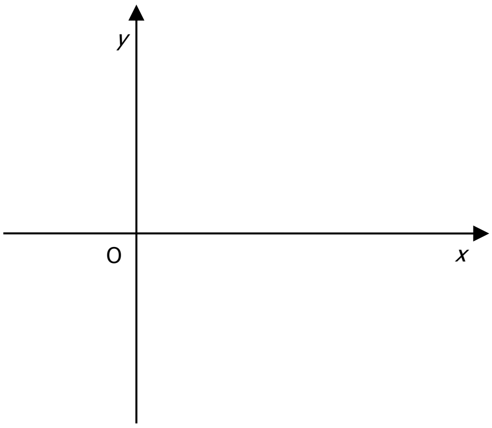


NOTE: Example 6b shows that areas UNDER the *x* - axis give NEGATIVE values!

Example 7:

a) Evaluate $\int_{-1}^{7}(2x-6)\,dx$

b) (i) Sketch below the area described by the integral $\int_{-1}^{7}(2x-6)\,dx$.

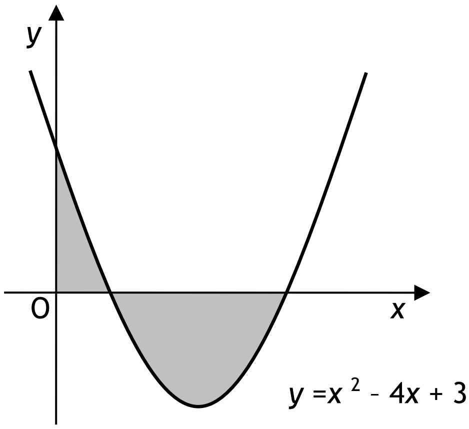


The answers for 5a and 5b do not match! This is because the area below the axis and the area above cancel each other out (as in 4b, areas below the *x* – axis give negative values).

To find the area between a curve and the x-axis:

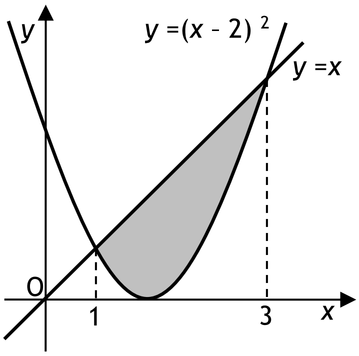
1. **Determine the limits which describe the sections above and below the axis**
2. **Calculate areas separately**
3. **Find the total, *IGNORING THE NEGATIVE VALUE OF THE SECTION BELOW THE AXIS*.**

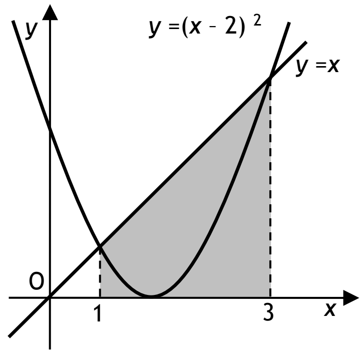
Example 8: Determine the area of the regions bounded by the curve $y = x^2 - 4x + 3$ and the *x* - and *y* - axes.

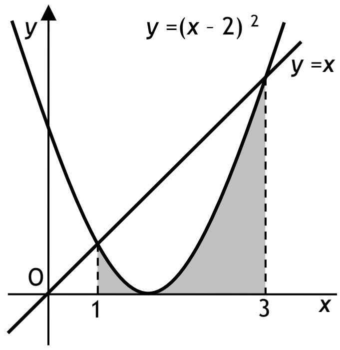


Area Between Two Curves

Consider the area bounded by the curves $y = (x - 2)^2$ and $y = x$.

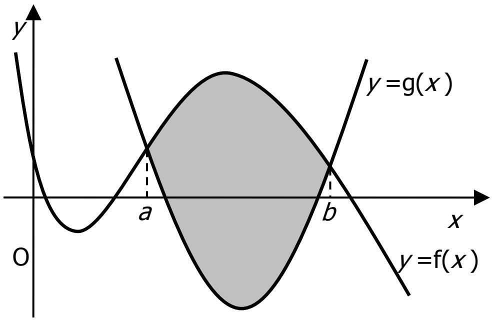






Area $= \int_1^3 x\,dx - \int_1^3 (x-2)^2\,dx$

The diagrams above show that the area between the curves is equal to the area between the top function (x) and the x – axis MINUS the area between the bottom curve ($(x - 2)^2$) and the x – axis.



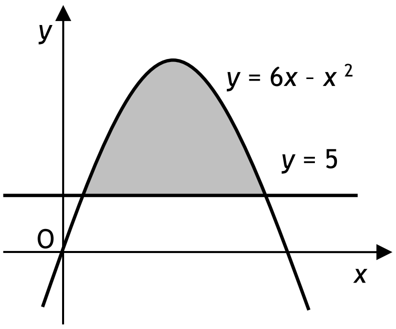
The area between the curves $y = f(x)$ and $y = g(x)$ (which meet at the points where $x = a$ and $x = b$) is given by:

$$\mathbf{A = \int_a^b (f(x) - g(x))dx}$$

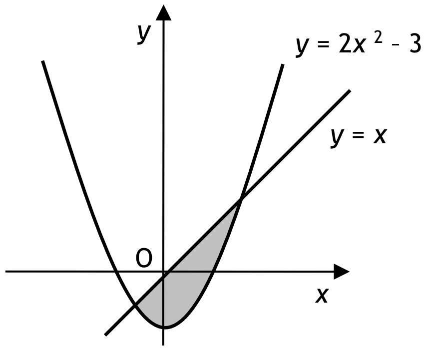
where:
- $f(x)$ is the TOP function and $g(x)$ is the BOTTOM
- $b > a$

Example 9: Write down the integrals used to determine the areas shown below:

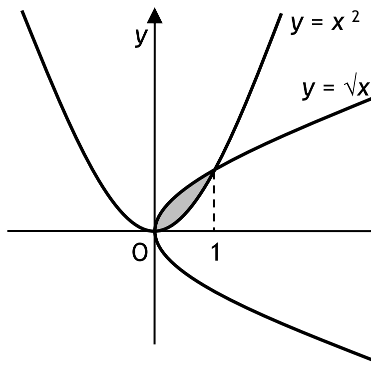
a)



b)



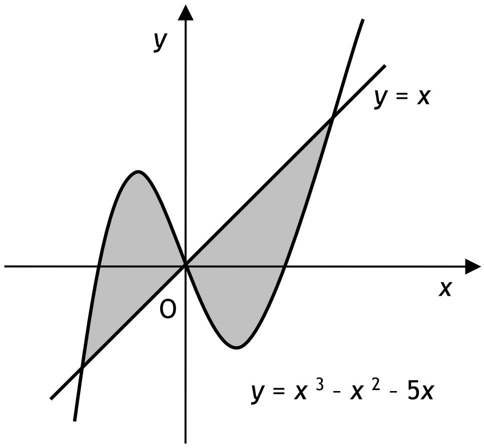
c)



To find the area between two curves:

1. **Make a sketch (if one has not been given)**
2. **Find points of intersection (make $y = y$ and solve)**
3. **Subtract the bottom function from the top function,** ***PUTTING THE BOTTOM FUNCTION IN BRACKETS!***
4. **Integrate**

Example 10: Find the area enclosed between the curve $y = x^3 - x^2 - 5x$ and the line $y = x$



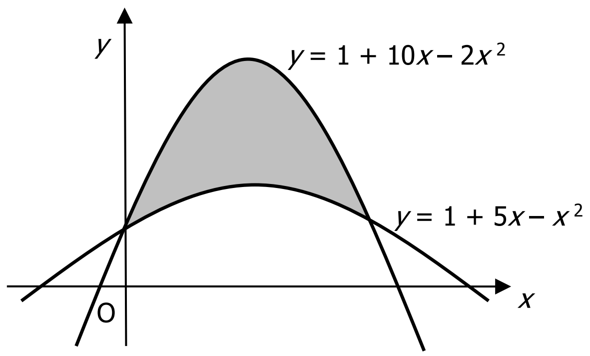
Differential Equations

If we know the derivative of a function (e.g. $f'(x) = 6x^2 - 3$), we can obtain a formula for the original function by integration. This is called a **differential equation**, and gives us the function in terms of x and C (which we can then evaluate if we have a point on the graph of the function).

Example 11: The gradient of a tangent to the curve of $y = f(x)$ is $24x^2 + 10x$, Express y in terms of x, given than the graph of $y = f(x)$ passes through the point (-1, -10).

Past Paper Example 1: Evaluate $\int_1^9 \frac{x+1}{\sqrt{x}}\,dx$

Past Paper Example 2: Find area enclosed between the curves $y = 1 + 10x - 2x^2$ and $y = 1 + 5x - x^2$.

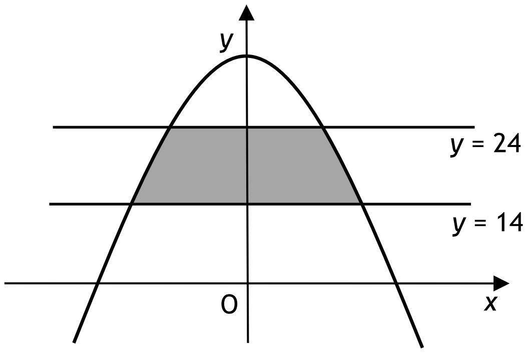


Past Paper Example 3: The parabola shown in the diagram has equation

$$y = 32 - 2x^{2}.$$

The shaded area lies between the lines $y = 14$ and $y = 24$

Calculate the shaded area.



Applications Unit Topic Checklist (Unit Assessment Topics in Bold)			
Topic		Questions	Done?
Straight Line	**Gradients (inc. m = tanθ)**	Exercise 1A, Q 8 – 10; Exercise 1B, p 4, Q 4, 5	Y/N
	Perpendicular Gradients	Exercise 1D, Q 1 – 4, 7	Y/N
	Equations of straight lines	Exercise 1E, Q 1, 3, 7, 8 ($y = mx + c$)	Y/N
		Exercise 1F, Q 1, 2 ($Ax + By + C = 0$)	Y/N
		Exercise 1G, Q 2, 3 ($y - b = m(x - a)$)	Y/N
	Collinearity	Exercise 1B, Q 1 – 3, 9	Y/N
	Perpendicular bisectors	Exercise 1I, Q 1, 2; Exercise 1N, Q 5	Y/N
	Altitudes	Exercise 1K, Q 1, 5; Exercise 1N, Q 1 – 3	Y/N
	Medians	Exercise 1M, Q 1, 3; Exercise 1N, Q 4	Y/N
	Distance Formula	Exercise 12B, Q 1	Y/N
Recurrence Relations	**Finding terms**	Exercise 5D, Q 1 – 3	Y/N
	Creating & using formulae	Exercise 5C, Q 5 – 11	Y/N
	Finding a limit	Exercise 5H, Q 1 – 3	Y/N
		Exercise 5H, Q 4 – 10; Exercise 5L, p 83, Q 2, 4	Y/N
	Solving to find *a* and *b*	Exercise 5I, Q 1, 2	Y/N
		Exercise 5I, Q 3, 4	Y/N
The Circle	**Circles centred on O**	Exercise 12D, Q 1 – 3	Y/N
	$(x - a)^2 + (y - b)^2 = r^2$	Exercise 12F, Q 1 – 3, 10	Y/N
	General equation	Exercise 12H, Q 1, 4, 12 – 15; Exercise 12M, Q 1, 7	Y/N
	Intersection of lines & circles	Exercise 12J, Q 3	Y/N
	Tangency	Exercise 12K, Q 2, 6; Exercise 12M, Q 4, 8	Y/N
	Equations of tangents	Exercise 12L, Q 1 – 4	Y/N
Calculus	**Optimisation**	Exercise 6Q, Q 1, 2, 4	Y/N
		Exercise 6R, Q 1, 5; Exercise 6S, Q 19	Y/N
	Area under a curve	Exercise 9K, Q 1; Exercise 9N, Q 1, 3, 4	Y/N
	Area between two curves	Exercise 9P, Q 1, 2, 4; Exercise 9R, Q 7, 11	Y/N
	Differential Equations	Exercise 9Q, Q 2, 3; Exercise 9R, Q 14, 15	Y/N

Polynomials

A **polynomial** is an expression with terms of the form ax^n, where n is a whole number.
For example, $5p^4 - 3p^3$ is a polynomial, but $3p^{-1}$ or $\sqrt[3]{p^2}$ are not.

The **degree** of a polynomial is its highest power, e.g. the polynomial above has a degree of 4.

The number part of each term is called its **coefficient**, e.g. the coefficients of p^4, p^3 and p above are 5, -3 and 0 (as there is no p term!) respectively (note that $5p^4$ would also be a polynomial on its own, with coefficients of zero for all other powers of p).

Evaluating Polynomials

An easy way to find out the value of a polynomial function is by using a **nested table.**

Example 1: Evaluate f(4) for $f(x) = 2x^4 - 3x^3 - 10x^2 - 5x + 7$.

4	2	-3	-10	-5	7	←	Line up coefficients
	↓						
	2						

Example 2: Evaluate f(-1) for $f(x) = 3x^5 - 2x^3 + 4$.

-1	3	0	-2	0	0	4
	↓					
	3					

Missing powers have coefficients of zero!

Synthetic Division

Dividing 67 by 9 gives an answer of "7 remainder 4". We can write this in two ways:

67 ÷ 9 = 7 remainder 4 **OR** **9 x 7 + 4 = 67**

For this problem, 9 is the **divisor**, 7 is the **quotient**, and 4 is the **remainder** (note that if we were dividing 63 by 9, the remainder would be zero, since 9 is a **factor** of 63).

Example 3:

a) Remove brackets and simplify:

$(x - 3)(x^2 + 9x - 12) + 11$

b) Evaluate f(3) for $f(x) = x^3 + 6x^2 - 39x + 47$

This shows that: $x^3 + 6x^2 - 39x + 47 = (x - 3)(x^2 + 9x - 12) + 11$

OR

$(x^3 + 6x^2 - 39x + 47) \div (x - 3) = (x^2 + 9x - 12)$ remainder 11

Compare the numbers on the **bottom row** of the nested table in part b) with the coefficients in part a). This shows that we can use nested tables to divide polynomial expressions to give both the quotient and remainder (if one exists). This process is known as **synthetic division.**

Example 4: Find the remainder on dividing $x^3 - x^2 - x + 5$ by $(x + 5)$.

Example 5: Write $4p^4 + 2p^3 - 6p^2 + 3 \div (2p - 1)$ in the form $(ap - b)\,Q(p) + R$

Remainder Theorem and Factor Theorem

Considered together, these two theorems allow us to factorise algebraic functions (remember that a factor is a number or term which divides **exactly** into another, leaving **no remainder**).

If polynomial $f(x)$ is divided by $(x - h)$, then the remainder is $f(h)$

On division of polynomial $f(x)$ by $(x - h)$, if $f(h) = 0$, then $(x - h)$ is a factor of $f(x)$

In other words, if the result of synthetic division on a polynomial by h is zero, then h is a **root** of the polynomial, and $(x - h)$ is a **factor** of it.

Example 6: $f(x) = 2x^3 - 9x^2 + x + 12$.

a) Show that $(x - 4)$ if a factor of $f(x)$.

b) Hence factorise $f(x)$ fully.

Example 7: Factorise fully $3x^3 + 2x^2 - 12x - 8$.

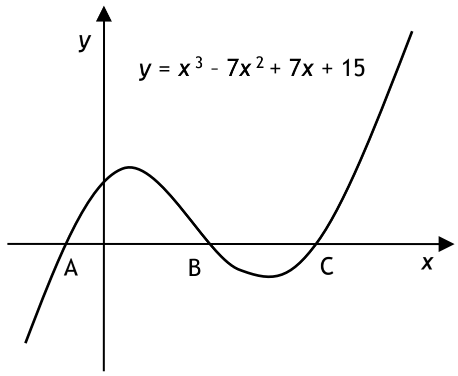
Example 8: Find the value of k for which $(x + 3)$ is a factor of $x^3 - 3x^2 + kx + 6$

Example 9: Find the values of a and b if $(x - 3)$ and $(x + 5)$ are both factors of $x^3 + ax^2 + bx - 15$

Solving Polynomial Equations

Polynomial equations are solved in exactly the same way as we solve quadratic equations: **make the right hand side equal to zero, factorise, and solve to find the roots.**

Example 10: The graph of the function $y = x^3 - 7x^2 + 7x + 15$ is shown.
Find the coordinates of points A, B and C.

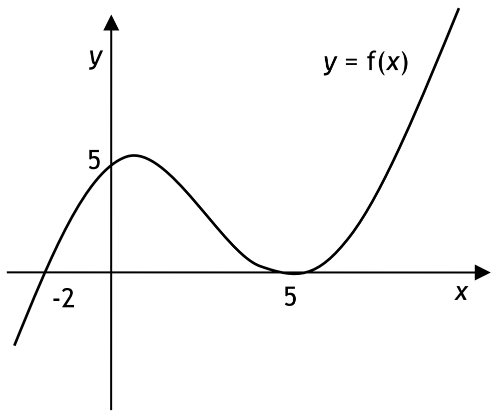


Finding a Function from its Graph

This uses exactly the same system as that for quadratic graphs, but with more brackets (see page 19).

Remember: tangents to the x – axis have repeated roots!

Example 11: Find an expression for cubic function $f(x)$.



Sketching Polynomial Functions

Example 12: a) Find the x – and y – intercepts of the graph of $y = x^4 - 6x^3 + 13x^2 - 12x + 4$.

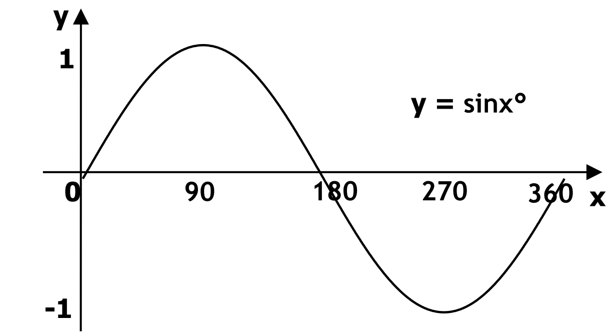
b) Find the position and nature of the stationary points of $y = x^4 - 6x^3 + 13x^2 - 12x + 4$.

c) Hence, sketch and annotate the graph of $y = x^4 - 6x^3 + 13x^2 - 12x + 4$.

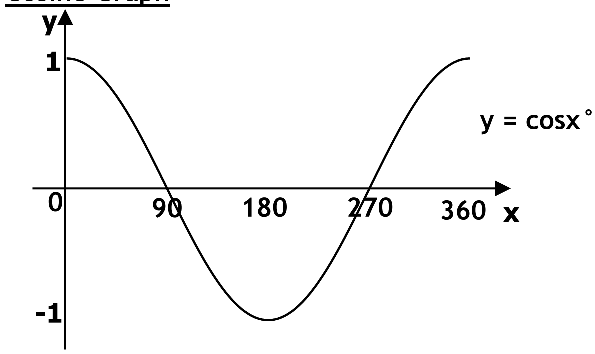
Trigonometry: Addition Formulae and Equations

What you must know from National 5!!!

Sine Graph



Cosine Graph

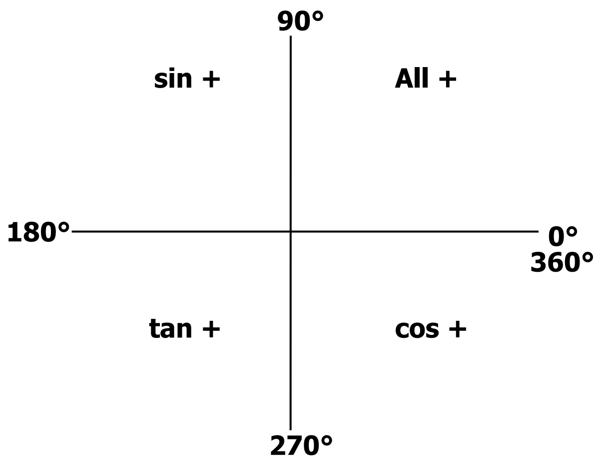


We can use the above graphs to find the values of:

$\sin 0° = 0$	$\cos 0° = 1$
$\sin 90° = 1$	$\cos 90° = 0$
$\sin 180° = 0$	$\cos 180° = -1$
$\sin 270° = -1$	$\cos 270° = 0$
$\sin 360° = 0$	$\cos 360° = 1$

We can use these graphs to solve the following:

$\sin x° = 0$ $(0 \leq x \leq 360)$ $x = 0°, 180°, 360°$	$\sin x° = -1$ $(0 \leq x \leq 360)$ $x = 270°$	$\sin x° = 1$ $(0 \leq x \leq 360)$ $x = 90°$
$\cos x° = 0$ $(0 \leq x \leq 360)$ $x = 90°, 270°$	$\cos x° = -1$ $(0 \leq x \leq 360)$ $x = 270°$	$\cos x° = -1$ $(0 \leq x \leq 360)$ $x = 0°, 360°$



Remember , this means that:

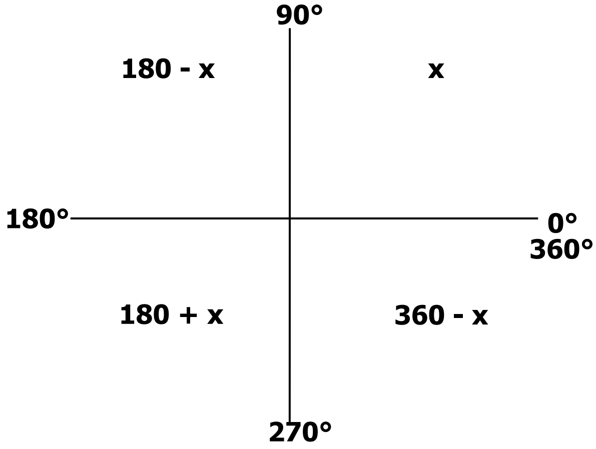
sin 160° would be +

cos 200° would be -

tan 200° would be +

sin 320° would be -
and so on...

Related Angles



This diagram can be used to find families of related angles.

For example, for x = 30°.
The family of related angles would be:
30°, 150°, 210°, 330°

These angles are related since:
$\sin 30° = 0.5$
$\sin 150° = 0.5$
$\sin 210° = -0.5$
$\sin 330° = -0.5$

Note: The sine of these angles have the same numerical value.

Equations

Example A: $\sin x° = 0.423 \quad (0 \le x \le 360)$ $x = \sin^{-1}(0.423)$ $x = 25°$ (R.A) $x = (0 + 25)°, (180 - 25)°$ $x = 25°, 155°$	**Step 1: Consider 0.423** **Step 2: We know that we can find the other 3 angles in the family 155°, 205°, 335°** **Step 3: We only want the angles which will give +ve answers for sin.**
Example B: $\cos x° = -0.584 \quad (0 \le x \le 360)$ $x = \cos^{-1}(0.584)$ $x = 54.3°$ (R.A) $x = (180 - 54.3)°, (180 + 54.3)°$ $x = 125.7°, 234.3°$	**Step 1: Consider 0.584 (ignore -ve)** **Step 2: We know that we can find the other 3 angles in the family 125.7°, 234.3°, 305.7°** **Step 3: We only want the angles which will give -ve answers for cos.**

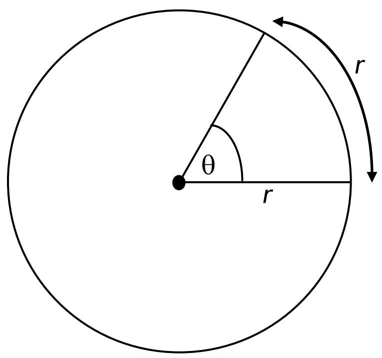
Radians

If we draw a circle and make a sector with an arc of exactly one radius long, then the angle at the centre of the sector is called a **radian.**

Remember that Circumference = $\pi D = 2\pi r$. This means that there are 2π radians in a full circle.

360° = 2π radians

180° = π radians



Example 1: Convert:

a) 90° to radians

b) 60° to radians

c) 225° to radians

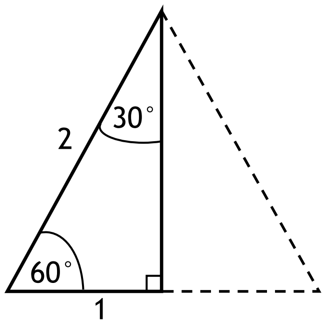
d) $\frac{\pi}{4}$ radians to degrees

e) $\frac{4\pi}{3}$ radians to degrees

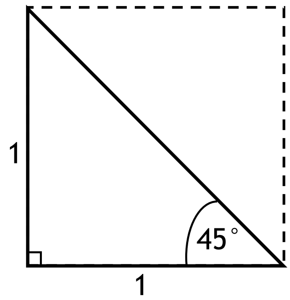
f) $\frac{11\pi}{6}$ radians to degrees

Exact Values

Consider the following triangles:



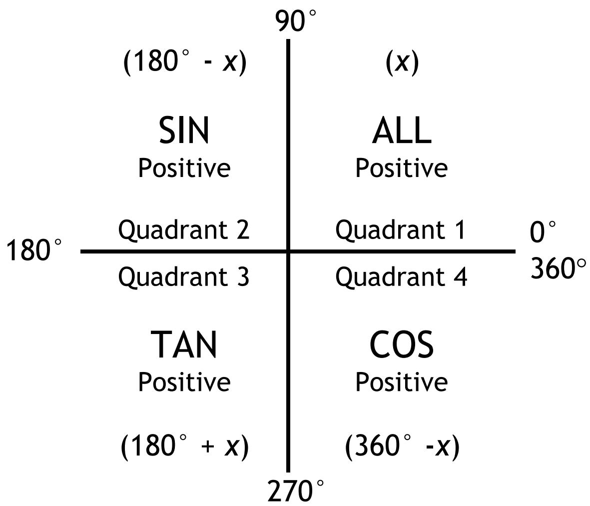
A right-angled triangle made by halving an equilateral triangle of side 2 units



A right-angled triangle made by halving an square of side 1 unit

Once we have found the lengths of the missing sides (by Pythagoras' Theorem), the following table of values can be constructed:

	0°	30°	45°	60°	90°
	0	$\left(\frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}\right)$
Sin					
Cos					
Tan					



Example 2: State the exact values of:

a) $\sin 150^\circ$

b) $\tan 315^\circ$

c) $\cos\frac{7\pi}{6}$

Addition Formulae

Finding the value of a compound angle is not quite as simple as adding together the values of the component angles, e.g. sin90° ≠ sin60° + sin30°. The following formulae must be used:

sin(A + B) = sinAcosB + cosAsinB	**sin(A - B) = sinAcosB - cosAsinB**

Example 3: Expand each of the following:

a) sin(X + Y)

b) sin(Q + 3P)

Example 4: Find the exact value of sin75°.

Example 5: A and B are acute angles where $\tan A = \frac{12}{5}$ and $\tan B = \frac{3}{4}$. Find the value of sin(A + B).

Example 6: Expand each of the following:

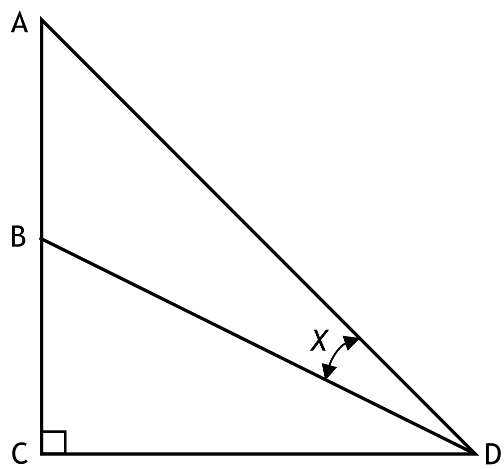
a) sin(α - β)

b) $\sin\left(2B - \frac{2\pi}{3}\right)$

Example 7: In the diagram opposite:

AC = CD = 2 units, and AB = BC = 1 unit.

Show that sin *X* is exactly $\frac{1}{\sqrt{10}}$.



cos(A + B) and cos (A - B)

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

Example 8: Expand the following:

a) $\cos(X - Y)$

b) $\cos(X + 315)°$

Example 9:

a) Show that $\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

b) Hence find the exact value of $\cos\frac{\pi}{12}$

To summarise:

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

Trigonometric Identities

NOTE: these are important formulae which are **not provided** in the exam paper formula sheets!

$\dfrac{\sin x^\circ}{\cos x^\circ} = \tan x^\circ$	$\sin^2 x^\circ + \cos^2 x^\circ = 1$

Note that due to the second formula, we can also say that:

$\cos^2 x^\circ = 1 - \sin^2 x^\circ$	AND	$\sin^2 x^\circ = 1 - \cos^2 x^\circ$

To prove that an identity is true, we need to show that the expression on the left hand side of the equals sign can be changed into the expression on the right hand side.

Example 10: Prove that:

a) $\cos^4\alpha - \sin^4\alpha = \cos^2\alpha - \sin^2\alpha$

b) $\tan 3\theta + \tan\theta = \dfrac{\sin 4\theta}{\cos\theta\cos 3\theta}$

c) $\tan x - \dfrac{1}{\tan x} = \dfrac{2\sin^2 x - 1}{\sin x\cos x}$

Double Angle Formulae

sin2A = sin(A + A)

=

cos2A = cos(A + A)

=

Since $\cos^2x° = 1 - \sin^2x°$ and $\sin^2x° = 1 - \cos^2x°$, we can further expand the formula for cos2A:

cos2A = $\cos^2A - \sin^2A$

=

cos2A = $\cos^2A - \sin^2A$

=

To summarise:

sin2A = 2sinAcosA

	= $\cos^2A - \sin^2A$
cos2A	**= $2\cos^2A - 1$**
	= $1 - 2\sin^2A$

Example 11: Express the following using double angle formulae:

a) sin2X

b) sin6Y

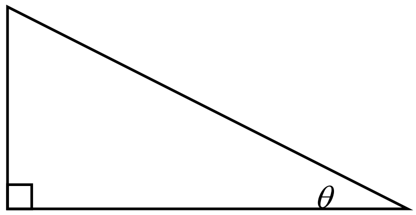
c) cos2X (sine version)

d) cos8H (cosine version)

e) sin5Q

f) $\cos\theta$ (cos and sin version)

Example 12: $\sin\theta = \frac{2}{\sqrt{13}}$, where θ is an acute angle. Find the exact values of:



a) sin(2θ)

b) cos(2θ)

Example 13: Prove that $\frac{\sin 2x}{1+\cos 2x} = \tan x$

Solving Complex Trig Equations

Trig equations can also often involve (i) powers of sin, cos or tan, and (ii) multiple and/or compound angles.

Example 14: Solve $4\cos^2 x - 3 = 0$ for $0 \leq x \leq 2\pi$

Trig equations can also be written in forms which resemble quadratic equations: to solve these, treat them as such, and solve by factorisation.

Example 15: Solve $6\sin^2 x° - \sin x° - 2 = 0$ for $0 \leq x \leq 360°$

If the equation contains a multiple angle term, solve as normal (paying close attention to the range of values of x).

Example 16: Solve $\sqrt{3}\tan(2x - 135)° = 1$ for $0 \leq x \leq 360°$

To solve trig equations with combinations of double- and single-angle angle terms:

- **Rewrite the double angle term using the formulae on Page 59**
- **Factorise**
- **Solve each factor for *x***

When the term is $\cos 2X$, the version of the double angle formula we use depends on the other terms in the equation: use $2\cos^2 x - 1$ if the other term is $\cos x$; $1 - 2\sin^2 x$ if the other term is $\sin x$.

Example 17: Solve $\sin 2x° - 2\sin x° = 0$, $0 \leq x \leq 360°$

Example 18: Solve $2\cos 2x - 7\cos x = 0, \quad 0 \le x \le 2\pi$

Formulae for $\cos^2 x$ and $\sin^2 x$

Rearranging the formulae for cos2x allows us to obtain the following formulae for $\cos^2 x$ and $\sin^2 x$

$$\cos^2 x = \frac{1}{2}\left(1 + \cos 2x\right)$$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right)$$

Example 19: Express each of the following without a squared term:

a) $\cos^2\theta$

b) $\sin^2 3X$

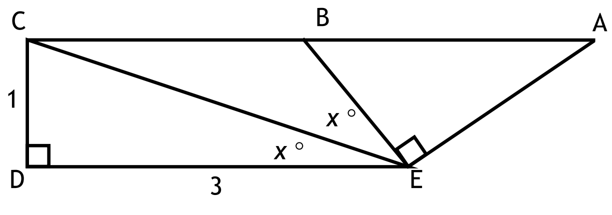
c) $\sin^2(\frac{x}{2})$

Past Paper Example 1: In the diagram,

$\angle DEC = \angle CEB = x°$, and $\angle CDE = \angle BEA = 90°$.

CD = 1 unit; DE = 3 units.

By writing $\angle DEA$ in terms of x, find the exact value of $\cos(D\hat{E}A)$.



Past Paper Example 2: Find the points of intersection of the graphs of $y = 3\cos 2x° + 2$ and $y = 1 - \cos x°$ in the interval $0 \leq x \leq 360°$.

Past Paper Example 3: Solve algebraically the equation

$$\sin 2x = 2\cos^2 x \qquad \text{for } 0 \leq x \leq 2\pi$$

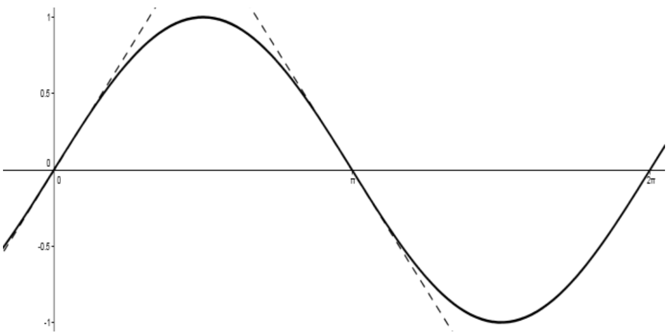
Calculus 3: Further Calculus

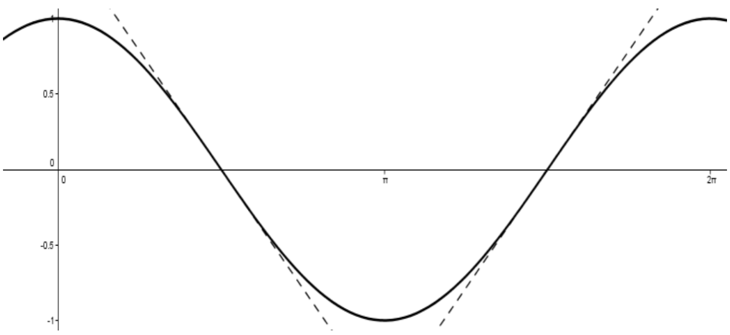
Let $f(x)$ = sin$x$ and $g(x)$ = cos$x$. The graphs of $y = f(x)$ and $y = g(x)$ are shown below, where the x-axis is measured in radians. Tangents to each curve have been drawn at the following points:

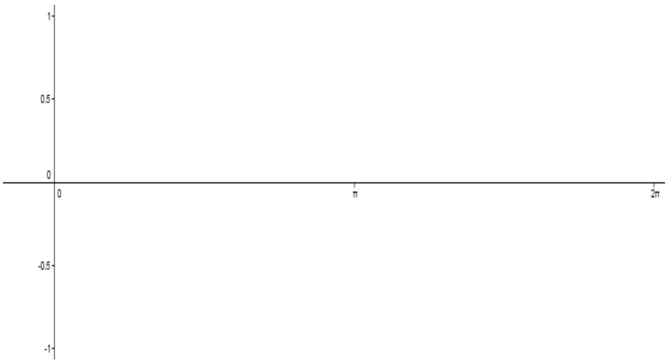
On y = sin$x$, the tangent at $x = 0$ has $m = 1$, and the tangent at $x = \pi$ has $m = -1$.

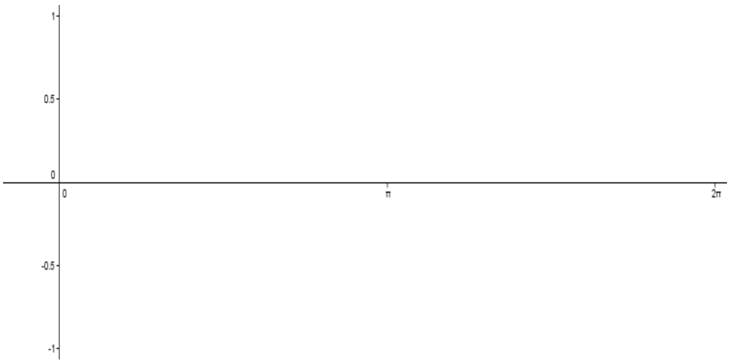
On y = cos$x$, the tangent at $x = \frac{\pi}{2}$ has $m = -1$, and the tangent at $x = \frac{3\pi}{2}$ has $m = 1$.

Draw the graphs of $y = f'(x)$ and $y = g'(x)$ below.









The graphs of the derived functions therefore show that:

If y = sin$x$, $\frac{dy}{dx}$ =

If y = cos$x$, $\frac{dy}{dx}$ =

Example 1: Find the derivative in each case:

a) $y = 4\sin x$

b) $f(x) = 2\cos x$

c) $g(x) = -\frac{1}{2}\cos x$

d) $h = -5\sin k$

As integration is the opposite of differentiation, we can also say that:

$\int \cos x\,dx =$

$\int \sin x\,dx =$

Example 2: Find:

a) $\int 24\cos x\,dx$

b) $\int -3\sin s\,ds$

c) $\int (3x - \cos x)\,dx$

IMPORTANT!

- **Definite Integrals of sin and cos functions MUST be done in radians!**
- **NEVER ignore any brackets where the limit is zero!**

Example 3: Evaluate:

a) $\int_0^{\pi/2} \sin x \, dx$

b) $\int_0^{\pi/4} (\sin x - \cos x) \, dx$

c) $\int_0^{3} 2\cos x \, dx$

The Chain Rule

Example 4: By first expanding the brackets, find the derivatives of the following functions:

a) $y = (3x + 1)^2$

$\therefore \frac{dy}{dx} = \_\_\_(3x + 1) \times \_\_\_\_$

b) $y = (2x^2 - 1)^2$

$\therefore \frac{dy}{dx} = \_\_\_(2x^2 - 1) \times \_\_\_\_$

c) $y = (2x + 1)^3$

$\therefore \frac{dy}{dx} = \_\_\_(2x + 1)^2 \times \_\_\_\_$

In each case, we can factorise the answer to give us back the original function, which has been differentiated as if it was just an x^2 or x^3 term (multiply by the old power, drop the power by one), and then **multiplied by the derivative of the function in the bracket**.

This is known as the **Chain Rule,** and can be written generally for brackets with powers as:

For $f(x) = a\,(\ldots\ldots\ldots)^n$, $f'(x) = an\,(\ldots\ldots\ldots)^{n-1} \times$ (DOB)

where DOB = the Derivative Of the Bracket

Example 5: Use the chain rule to differentiate:

a) $f(x) = (4x - 2)^4$

b) $g(x) = \dfrac{1}{\sqrt{2x^2 + x}}$ $(x < -\frac{1}{2},\ x > 0)$

c) $y = \sin^2 x$

The Chain Rule can also be applied to sine and cosine functions with double or compound angles, or to more complicated composite functions containing sine and cosine.

For functions including sine and cosine components:

For f(x) = sin(……),	For f(x) = cos(……),
f'(x) = cos(……) x DOB	**f'(x) = - sin(……) x DOB**

Example 6: Differentiate:

a) $y = \sin(3x)$

b) $f(x) = \cos\left(\dfrac{\pi}{4} - 2x\right)$

c) $y = \sin\left(x^2\right)$

Example 7: Find the equation of the tangent to $y = \sin\left(2x + \dfrac{\pi}{3}\right)$ when $x = \dfrac{\pi}{6}$.

Further Integration

We have seen that integration is **anti-differentiation**, i.e. the opposite of differentiating.

As finding the derivative of a function with a bracket included multiplying by DOB, then integrating must also include **dividing** by DOB.

To integrate: $$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)\times a} + C$$

Important Point: Integration is more complicated than differentiation!

This method only works for **linear** functions inside the bracket, i.e. the highest power = 1. To find, e.g., $\int \left(g^3 - 7\right)^2 dg$, we would have to multiply out the bracket and integrate each term separately.

Example 8: Evaluate:

a) $\int (x+3)^3 dx$

b) $\int (4x - 7)^9 dx$

c) $\int \frac{dt}{\left(4t+9\right)^2} \quad \left(t \neq -\frac{9}{4}\right)$

d) $\int_1^2 \left(2t+5\right)^3 dt$

e) $\int_0^6 \frac{dx}{\sqrt{4x+1}} \quad \left(x > -\frac{1}{4}\right)$

For functions including sine and cosine components:

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+C$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+C$$

Example 9: Evaluate:

a) $\int \sin 4x\, dx$

b) $\int 3\cos 2x\, dx$

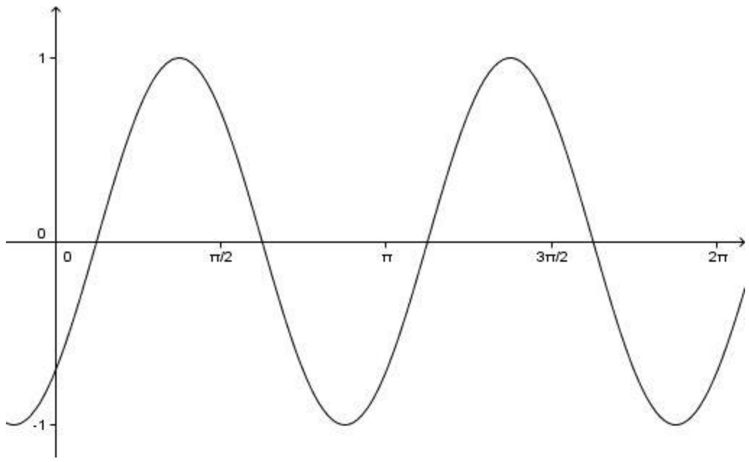
c) $\int \sin(1-2x)\, dx$

d) (i) Write $\cos^2 x$ in terms of $\cos 2x$

(ii) Hence find $\int 4\cos^2 x\, dx$

e) Evaluate $\int_0^{2\pi} \sin\left(\frac{1}{2}x\right)dx$

Example 10: Find the area enclosed by $y = \sin\left(2x - \frac{\pi}{4}\right)$, the x - axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.



In summary, for trig functions:

Differentiation	
f(*x*)	f'(*x*)
sin*ax*	*a* cos*ax*
cos*ax*	-*a* sin*ax*

Integration	
f(*x*)	$\int f(x)\,dx$
sin*ax*	$-\frac{1}{a}\cos ax$
cos*ax*	$\frac{1}{a}\sin ax$

Uses of Calculus in Real Life Situations

In the same way that geometry is the study of shape, calculus is the study of how functions change. This means that wherever a system can be described mathematically using a function, calculus can be used to find the ideal conditions (as we have seen using optimisation) or to use the rate of change at a given time to find the total change (using integration).

As a result, calculus is used throughout the sciences: in Physics (Newton's Laws of Motion, Einstein's Theory of Relativity), Chemistry (reaction rates, radioactive decay), Biology (modelling changes in population), Medicine (using the decay of drugs in the bloodstream to determine dosages), Economics (finding the maximum profit), Engineering (maximising the strength of a building whilst using the minimum of material, working out the curved path of a rocket in space) and more.

Example 11: In Physics, the formulae for kinetic energy (E_k) and momentum (p) are respectively.

$$E_k = \frac{1}{2}mv^2 \qquad \text{and} \qquad p = mv$$

a) How could the formula for momentum be obtained from the formula for kinetic energy?

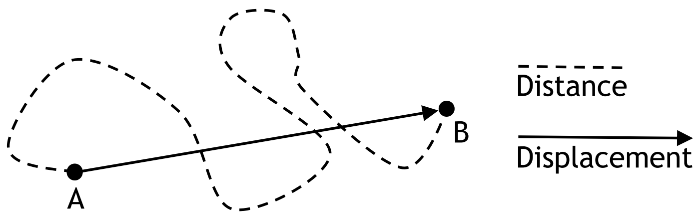
b) How could the formula for kinetic energy be obtained from the formula for momentum?

Displacement, Velocity and Acceleration

The most common use of this approach considers the link between displacement, velocity and acceleration.

When an object moves on a journey, we normally think of the total distance travelled.

Displacement is the straight line distance between the start and end points of a journey (so the displacement is not necessarily the same as the distance travelled!)



As displacement is a "straight-line" measurement, it involves direction and therefore is a **vector** quantity: another name for displacement is the **position**.

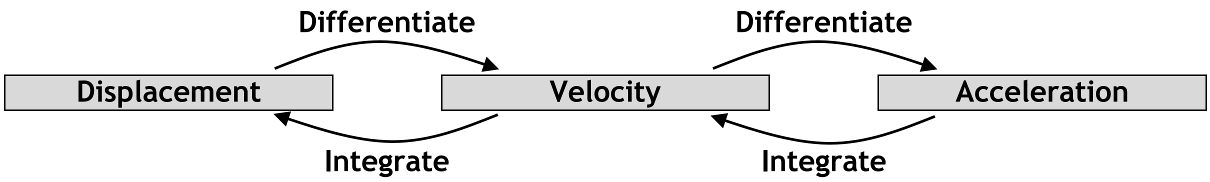
Velocity is the vector equivalent of speed, i.e. if speed is a measure of the distance travelled in a given time, then velocity is a measure of the change in displacement which occurs in a given time.

Velocity is defined as the **rate of change of displacement with respect to time.**

Acceleration measures the change in velocity of an object in a given time: if two race cars have the same top speed, then the one which can get to that top speed first would win a race.

Acceleration is defined as the **rate of change of velocity with respect to time.**

If one of either displacement, velocity or acceleration can be described using a function, then the other two can be obtained using either differentiation or integration, i.e.:



Example 12: The displacement s cm at a time t seconds of a particle moving in a straight line is given by the formula $s = t^3 - 2t^2 + 3t$.

a) Find its velocity v cm/s after 3 seconds.

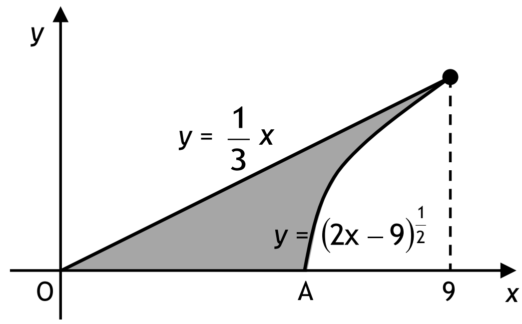
b) The time at which its acceleration a is equal to 26cm/s^2.

Example 13: The velocity of an electron is given by the formula $v(t) = 5\sin\left(2t - \frac{\pi}{4}\right)$.

a) Find the first time when its acceleration is at its maximum.

b) Find a formula for the displacement of the electron, given that $s = 0$ when $t = 0$.

Past Paper Example 1: A curve has equation $y = (2x - 9)^{\frac{1}{2}}$.
Part of the curve is shown in the diagram opposite.



a) Show that the tangent to the curve at the point where $x = 9$ has equation $y = \frac{1}{3}x$.

b) Find the coordinates of A, and hence find the shaded area.

Past Paper Example 2: A curve for which $\frac{dy}{dx} = 3\sin 2x$ passes through the point $\left(\frac{5\pi}{12}, \sqrt{3}\right)$.
Find y in terms of x .

Past Paper Example 3: Find the values of x for which the function $f(x) = 2x + 3 + \frac{18}{x-4}$, $x \neq 4$, is increasing.

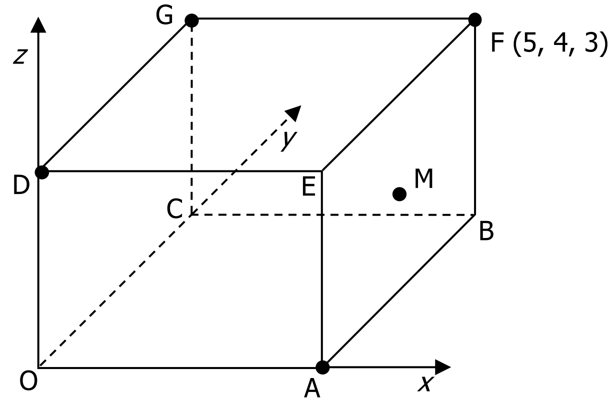
Relationships & Calculus Unit Topic Checklist: Unit Assessment Topics in Bold			
Topic		Questions	Done?
Polynomials	**Synthetic Division**	Exercise 7C, Q 2, 4	Y/N
	Factorising polynomials	Exercise 7E, Q 1 – 7	Y/N
	Solving polynomial equations	Exercise 7G, Q 2, 4, 6	Y/N
	Finding coefficients	Exercise 7F, Q 1, 2	Y/N
	Functions from graphs	Exercise 7H, Q 1 – 15	Y/N
	Roots using b^2 - $4ac$	Exercise 8H, Q 1, 2; Exercise 8I, Q 1, 2, 5, 6, 8	Y/N
		Exercise 8K, Q 10, 12	Y/N
	Solving Trig Equations (including use of double angle formulae)	Exercise 4H, Q 1, 2, 5; Exercise 4I, Q 1 – 3	Y/N
		Exercise 11H, Q 1, 2	Y/N
Differentiation	**Finding derivatives of functions**	Exercise 6F, (all); Exercise 6G, (all)	Y/N
		Exercise 6H, Q 2, 4, 5, 7, 9; Exercise 6I, Q 1, 2, 4	Y/N
	Equations of tangents to curves	Exercise 6J, Q 1, 2; Exercise 6S, Q 13	Y/N
	Increasing & decreasing functions	Exercise 6L, Q 1 - 7	Y/N
	Stationary points	Exercise 6M, (all); Exercise 6S, Q 14	Y/N
	Curve Sketching	Exercise 6N, Q 1 – 3	Y/N
	Closed Intervals	Exercise 6O, Q 2	Y/N
Integration	**Finding indefinite integrals**	Exercise 9H, (all); Exercise 9I, Q 1 (a – n)	Y/N
	Definite Integrals	Exercise 9L, Q 1 – 3	Y/N
	Differentiating and integrating sin$x$ and cos$x$	Exercise 14C, Q 1, 2, 5, 6	Y/N
	The Chain Rule	Exercise 14H, Q 3, 4, 5; Exercise 14I, Q 1, 3, 4, 5	Y/N
	Integrating a (…..)^n	Exercise 14J, Q 1, 4, 5, 8	Y/N
	Integrating sin(ax + b) and cos(ax + b)	Exercise 14K, Q 1, 2, 5, 6, 8	Y/N
		Exercise 14L, Q 10, 12, 13	Y/N

Vectors

Revision from National 5

A measurement which only describes the magnitude (i.e. size) of the object is called a **scalar** quantity, e.g. Glasgow is 11 miles from Airdrie. A **vector** is a quantity with **magnitude and direction**, e.g. Glasgow is 11 miles from Airdrie on a bearing of 270°.

The position of a point in 3-D space can be described if we add a third coordinate to indicate height.



Example 1: OABC DEFG is a cuboid, where F is the point (5, 4, 3). Write down the coordinates of the points:

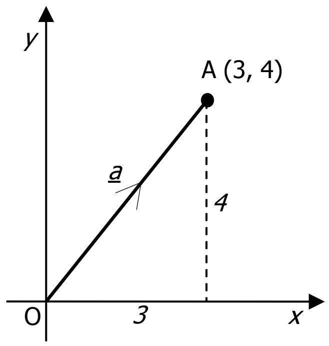
a) A

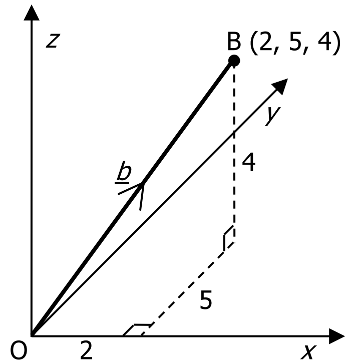
b) D

c) G

d) M, the centre of face ABFE

The rules of vectors can be used in either 2 or 3 dimensions:





Directed line segment $\overrightarrow{OA}$	Directed line segment $\overrightarrow{OB}$
Position vector $\underline{a}$	**Position vector $\underline{b}$**
Components $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$	**Components** $\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$

The **magnitude** of a vector is its length, which can be determined by Pythagoras' Theorem. The magnitude of $\underline{a}$ is written as $|\underline{a}|$.

If $\underline{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, **then** $|\underline{u}| = \sqrt{a^2 + b^2}$

If $\underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, **then** $|\underline{u}| = \sqrt{a^2 + b^2 + c^2}$

Example 2: Determine $|\underline{a}|$ and $|\underline{b}|$ in the examples above.

Addition of Vectors

Two (or more) vectors can be added together to produce a **resultant vector.**

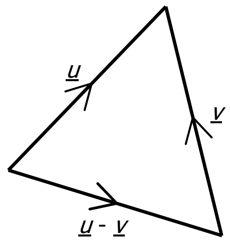
In general: $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ and **If $\underline{u}$ = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underline{v}$ = $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$, then $\underline{u}$ + $\underline{v}$ = $\begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$**

Example 3: Find $\underline{p}$ + $\underline{q}$ when $\underline{p}$ = $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\underline{q}$ = $\begin{pmatrix} 7 \\ -7 \\ 4 \end{pmatrix}$.

Example 4: Find values of x and y such that

$$\begin{pmatrix} x \\ 4 \end{pmatrix}+\begin{pmatrix} 12 \\ y \end{pmatrix}=\begin{pmatrix} 9 \\ -2 \end{pmatrix}$$

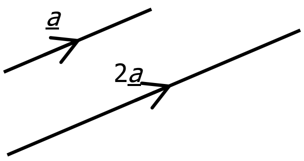
Subtraction of Vectors



Subtraction of vectors can be considered as going along the second vector in the **wrong direction**.

If $\underline{u}$ = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underline{v}$ = $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$, then $\underline{u}$ - $\underline{v}$ = $\begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$

Multiplication by a Scalar Quantity



If we go along $\underline{a}$ twice, the resultant vector is $\underline{a}$ + $\underline{a}$ = 2$\underline{a}$. As we have not changed direction, it follows that 2$\underline{a}$ must be **parallel** to $\underline{a}$.

If $\underline{u}$ = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $k\underline{u}$ = $\begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$

If $\underline{v}$ = $k\underline{u}$, then $\underline{u}$ and $\underline{v}$ are parallel

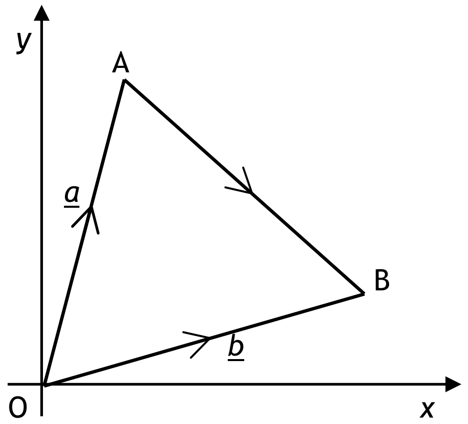
Example 5: If $\underline{b}$ = $\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$ and $\underline{c}$ = $\begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix}$, find:

a) 3$\underline{b}$

b) 2$\underline{b}$ + $\underline{c}$

c) $\underline{c}$ - $\frac{1}{2}\underline{b}$

Position Vectors



Consider the vector $\overrightarrow{AB}$ in the diagram opposite. $\overrightarrow{AB}$ is the resultant vector of going along $\underline{a}$ in the opposite direction, followed by $\underline{b}$ in the correct direction.

So, $\overrightarrow{AB} = -\underline{a} + \underline{b}$, i.e.:

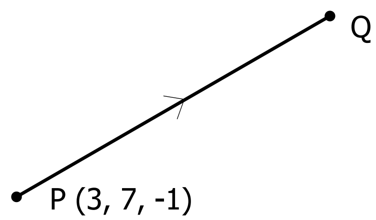
$$\overrightarrow{AB} = \underline{b} - \underline{a}$$

Example 6: L is the point (4, -7, 2), M is the point (-5, -3, -1).

Find the components of $\overrightarrow{LM}$.

Example 7: P is the point (3, 7, -1). $\overrightarrow{PQ}$ has components $\begin{pmatrix} -4 \\ 9 \\ -3 \end{pmatrix}$.

Find the coordinates of Q.



Unit Vectors

A **unit vector** is a vector with magnitude = 1.

Example 8: Find the components of the unit vector parallel to $\underline{h} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

To find the components of a unit vector:

- **Find the magnitude of the given vector**
- **Divide components by the magnitude**

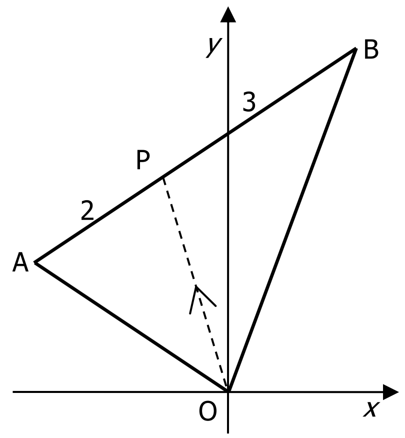
Collinearity

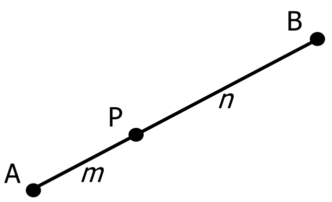
Example 9: Points F, G and H have coordinates (6, 1, 5), G (4, 4, 4), and (-2, 13, 1) respectively. Show that F, G and H are collinear, and find the ratio in which G divides $\overrightarrow{FH}$.

The Section Formula

P divides $\overrightarrow{AB}$ in the ratio 2:3. By examining the diagram, we can find a formula for $\underline{p}$ (i.e. $\overrightarrow{OP}$).

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$





In general, if P divides AB in the ratio m:n, then:

$$\underline{p} = \frac{1}{n+m}(n\underline{a} + m\underline{b})$$

Example 10: A is the point (3, -1, 2) and B is the point (7, -5, 14). Find the coordinates of P such that P divides AB in the ratio 1:3.

Vectors in 3D can also be described in terms of the three unit vectors $\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, which are parallel to the x, y, and z axes respectively.

Example 11: $\underline{u} = 3\underline{i} + 2\underline{j} - 6\underline{k}$, $\underline{v} = -\underline{i} + 5\underline{j}$.

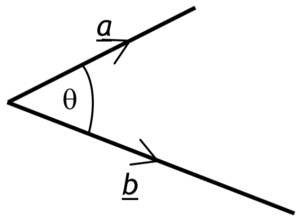
a) Express $\underline{u} + \underline{v}$ in component form

b) Find $|\underline{u} + \underline{v}|$

The Scalar Product (Angle Form)

The **scalar product** is the result of a type of multiplication of two vectors to give a scalar quantity. (i.e. a number with no directional component)

For vectors $\underline{a}$ and $\underline{b}$, the scalar product (or **dot** product) is given as:

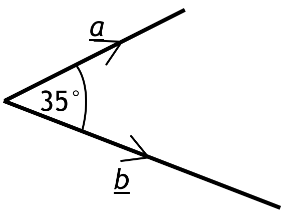


$$\underline{\boldsymbol{a}}.\underline{\boldsymbol{b}} = |\underline{\boldsymbol{a}}||\underline{\boldsymbol{b}}|\cos\theta$$

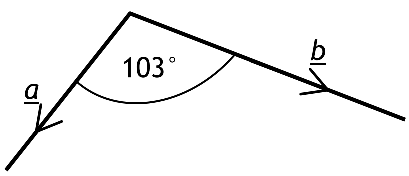
Note:
- $\underline{a}$ and $\underline{b}$ point **away** from the vertex
- $0 \le \theta \le 180^\circ$

Example 12: Find the scalar product in each case below, where $|\underline{a}| = 6$ and $|\underline{b}| = 10$.

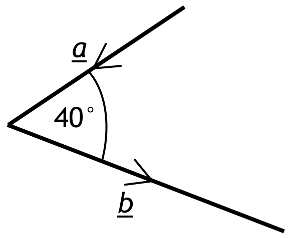
a)



b)



c)



The Scalar Product (Component Form)

We can use the formula below to find the scalar product when we have been given the component forms of the two vectors but not the angle in between them.

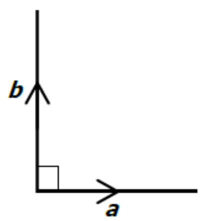
$$\text{If } \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{, then} \qquad \underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Example 13: $\underline{a} = \underline{i} + 2\underline{j} + 2\underline{k}$, and $\underline{b} = 2\underline{i} + 3\underline{j} - 6\underline{k}$. Evaluate $\underline{a} \cdot \underline{b}$

Example 14: A is the point (1, 2, 3), B is the point (6, 5, 4) and C is the point (-1, -2, -6).
Evaluate $\overrightarrow{AB}.\overrightarrow{BC}$

Perpendicular Vectors

A special case of the scalar product occurs when we have perpendicular vectors i.e. when $\theta = 90°$:



$$\begin{aligned} \boldsymbol{a}.\boldsymbol{b} &= |\boldsymbol{a}||\boldsymbol{b}| \cos 90° \\ &= |\boldsymbol{a}||\boldsymbol{b}| \times 0 \\ &= 0 \end{aligned}$$

If $\boldsymbol{a}.\boldsymbol{b} = 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular

Example 15: P, Q and R are the points (1, 1, 2), (-1, -1, 0) and (3, -4, -1) respectively. Find the components of $\overrightarrow{QP}$ and $\overrightarrow{QR}$, and hence show that the vectors are perpendicular.

The Angle Between Two Vectors

We can rearrange the angle form of the scalar product to give $\cos\theta = \dfrac{\underline{a}.\underline{b}}{|\underline{a}|\,|\underline{b}|}$.

More specifically: $\cos ABC = \dfrac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$

If the question gives you three points, you MUST find the components of the vectors pointing AWAY from the vertex first!

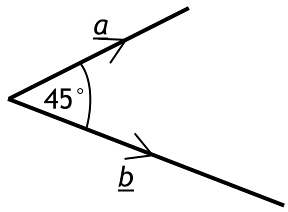
Example 16: $\underline{a} = \underline{i} + 2\underline{j} + 2\underline{k}$ and $\underline{b} = 2\underline{i} + 3\underline{j} - 6\underline{k}$. Find the angle between $\underline{a}$ and $\underline{b}$.

Example 17: A is the point (1, 2, 3), B (6, 5, 4), and C (-1, -2, -6). Calculate $\angle$ABC.

For vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$:

$\underline{a} . \underline{b} = \underline{b} . \underline{a}$

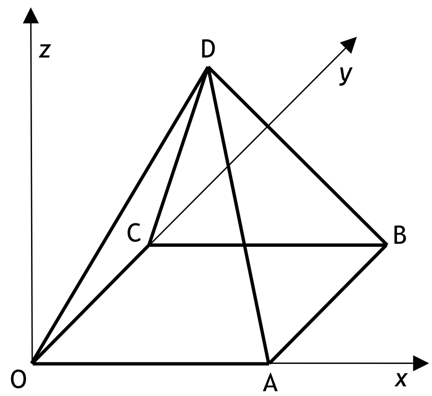
$\underline{a} .(\underline{b} + \underline{c}) = \underline{a} . \underline{b} + \underline{a} . \underline{c}$



Example 18: $|\underline{a}| = 5$ and $|\underline{b}| = 8$. Find $\underline{a} .(\underline{a} + \underline{b})$

Past Paper Example 1: The diagram shows a square-based pyramid of height 8 units. Square OABC has a side length of 6 units. The coordinates of A and D are (6, 0, 0) and (3, 3, 8). C lies on the *y* – axis.

a) Write down the coordinates of B.



b) Determine the components of $\overrightarrow{DA}$ and $\overrightarrow{DB}$.

c) Calculate the size of $\angle$ADB.

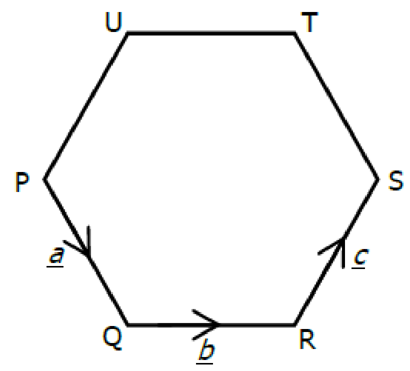
Past Paper Example 2:

a) Show that the points A (-7, -8, 1), T (3, 2, 5) and B (18, 17, 11) are collinear and state the ratio in which T divides AB.

b) The point C lies on the x-axis.

If TB and TC are perpendicular, find the coordinates of C.

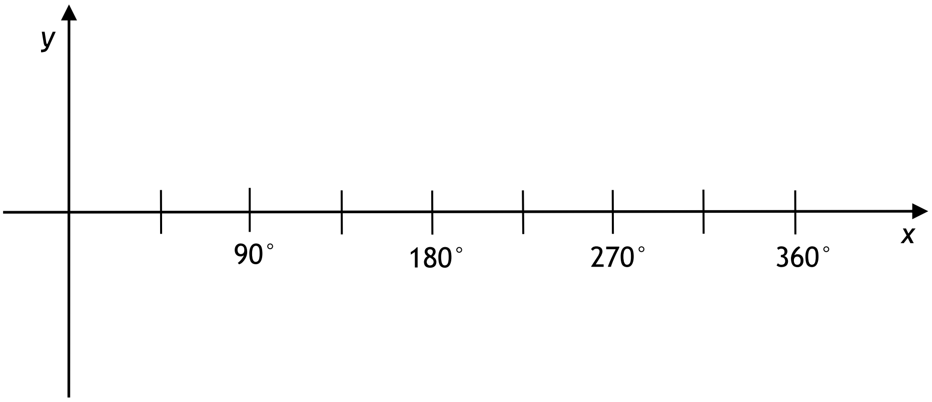
Past Paper Example 3: PQRSTU is a regular hexagon of side 2 units. $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$ represent the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively. Find the value of $\underline{a} \bullet (\underline{b} + \underline{c})$



Trigonometry: The Wave Function

It is possible to model the behaviour of waves in real-life situations (e.g. the interaction of sound waves or the tides where two bodies of water meet) using trigonometry. Consider the result of combining the waves represented by the functions $y = \sin x°$ and $y = \cos x°$. To find what the resultant graph would look like, complete the table of values (accurate to 1 d.p.) and plot on the axes below.

	0°	45°	90°	135°	180°	225°	270°	315°	360°
$\sin x°$									
$\cos x°$									
$\sin x° + \cos x°$									



Max = __________

Min = __________

Max when x = __________

Min when x = __________

$\therefore$ **y =**

____

Looking at the graph of $y = \sin x° + \cos x°$ above, we can compare it to cosine graph shifted 45° to the right (i.e. $y = \cos(x - \alpha)°$), and stretched vertically by a factor of roughly 1.4 (i.e. $y = k\cos x°$).

It is important to note, however, that the graph could also be described as a cosine graph shifted to the *left*, and also as a sine graph! Therefore, $y = \sin x° + \cos x°$ could **also** be written as:

y = 1.4cos(x + ________) OR **y = 1.4sin(x - ________)** OR **y = 1.4sin(x + ________)**

Rather than drawing an approximate graph, it is more useful if we use an algebraic method.

NOTE: you will only be asked to use one specific form to describe a function, not all four!

Example 1: Write $\sin x° + \cos x°$ in the form $k\cos(x - \alpha)°$, where $0 \le \alpha \le 360$.

This technique can also include the difference between waves and to include double (or higher) angles, **but only when the angles of both the sin and cos term are the same (**i.e. $2\cos 2x + 5\sin 2x$ can be written as a wave function, but $2\cos 2x + 5\sin 3x$ could not).

Example 2: Write $\sin x - \sqrt{3}\cos x$ in the form $k\cos(x - \alpha)$, where $0 \leq \alpha \leq 2\pi$

Example 3: Write $12\cos x° - 5\sin x°$ in the form $k\sin(x - \alpha)°$, where $0 \leq \alpha \leq 360$

Example 4: Write $2\sin 2\theta - \cos 2\theta$ in the form $k\sin(2\theta + \alpha)$, where $0 \leq \alpha \leq 2\pi$

Solving Trig Equations Using the Wave Function

In **almost all** cases, questions like these will be split into two parts, with a) being a "write ………. in the form $y = k\cos(x - \alpha)$" followed by b) asking "hence or otherwise solve………..".

Use the wave function from part a) to solve the equation!

Example 5:

a) Write $2\cos x° - \sin x°$ in the form $k\cos(x - \alpha)°$ where $0 \le \alpha \le 360$

b) Hence solve $2\cos x° - \sin x° = -1$ where $0 \le x \le 360$

Maximum and Minimum Values and Sketching Wave Function Graphs

Look back at the graph you drew of $\sin x° + \cos x°$. The maximum value of the graph is $\sqrt{2}$ at the point where $x = 45°$, and the minimum value is $-\sqrt{2}$ at the point where $x = 225°$. Compare these to the maximum and minimum of $y = \cos x°$, i.e. a maximum of 1 where $x = 0°$ or $360°$ and a minimum of -1 where $x = 180°$.

Since $\sin x° + \cos x° = \sqrt{2}\cos(x - 45)°$, we can see that the maximum and minimum values change from ± 1 to $\pm k$.

The maximum value occurs where $\sqrt{2}\cos(x - 45)° = \sqrt{2}$, i.e. $\cos(x - 45)° = 1$. Similarly, the minimum value occurs where $\sqrt{2}\cos(x - 45)° = -\sqrt{2}$, i.e. $\cos(x - 45)° = -1$

For $a\sin x + b\cos x = k\cos(x - \alpha)$, $k > 0$:

Maximum = k
when $\cos(x - \alpha) = 1$

Minimum = $-k$
when $\cos(x - \alpha) = -1$

Example 6:

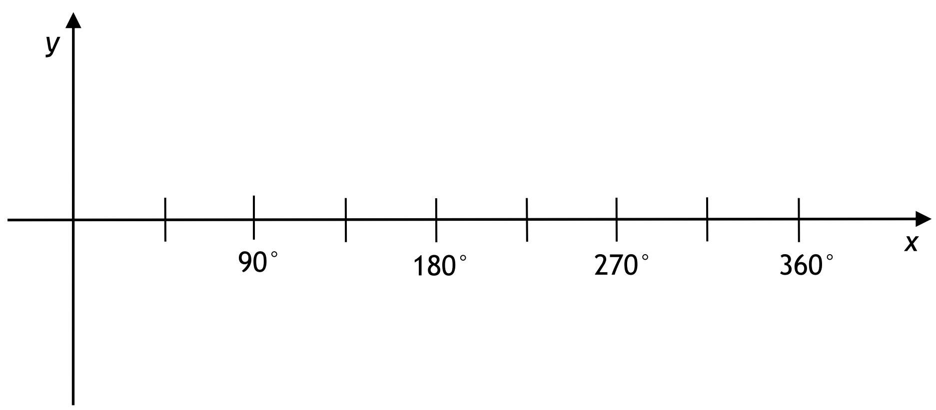
a) Write $\sqrt{3}\sin x + \cos x$ in the form $k\cos(x - \alpha)^\circ$, where $0 \leq \alpha \leq 360^\circ$

b) Find algebraically for $0 \leq x \leq 360^\circ$:

(i) The maximum and minimum turning points of $y = \sqrt{3}\sin x^\circ + \cos x^\circ$.

(ii) The points of intersection of $y = \sqrt{3}\sin x^\circ + \cos x^\circ$ with the coordinate axes.

c) Sketch and annotate the graph of $y = \sqrt{3}\sin x^\circ + \cos x^\circ$ for $0 \leq x \leq 360^\circ$.



Recognising Trig Equations

The trig equations we can be asked to solve at Higher level can be split into three types based on the **angle** (i.e. $x°$, $2x°$, $3x°$ etc) and the **function(s)** (i.e. sin, cos, tan, sin & cos).

Type One: **One Function** **One Angle**	e.g.: $2 \sin 4x + 1 = 0$ $\tan^2 x = 3$ $3\sin^2 x - 4\sin x + 1 = 0$	**1. Factorise (if necessary)** **2. Rearrange to sin(…) = (…) [or cos, or tan]** **3. Inverse sin/cos/tan to solve**
Type Two: **Two Functions** **One Angle**	e.g.: $\sin x + \cos x = 1$ $3\cos(2x) + 4\sin(2x) = 0$ $\cos(4\theta) - \sqrt{3}\sin(4\theta) = -1$	**1. Rewrite as a WAVE FUNCTION** **(choose $k\cos(x - \alpha)$ unless told differently)** **2. Solve as Type One**
Type Three: **Two Angles**	e.g.: $5\cos(2\theta) = \cos\theta - 2$ $2\sin(2x) + \sin(x) = -0.5$ $2\cos 2x - \sin x + 5 = 0$	**1. Rewrite the double angle and factorise** **(change cos2x to the SINGLE ANGLE function)** **2. Solve as Type One**

Past Paper Example:

a) The expression $\sqrt{3}\sin x° - \cos x°$ can be written in the form $k\sin(x - \alpha)°$, where $k > 0$ and $0 \leq \alpha < 360$.

Calculate the values of k and α.

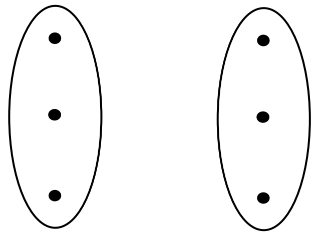
b) Determine the maximum value of $4 + 5\cos x° - 5\sqrt{3}\sin x°$, where $0 \leq \alpha < 360$, and state the value of x for which it occurs.

Inverse Functions

If a function links every number in the domain to **only** one number in the range, the function is called a **one to one correspondence**.

When function f(x) is a one to one correspondence from A to B, the function which maps from B back to A is called the **inverse function**, written $\mathbf{f^{-1}(x)}$.

For example, if f(x) = 2x, the inverse would be the function which "cancels out" multiplication by 2, i.e. $f^{-1}(x) = \frac{1}{2}x$



A B

Finding the Formula of an Inverse Function

We can find the formula for the inverse of a function through a process very similar to changing the subject of a formula.

Example 1: For each function shown find a formula for the inverse function.

a) $f(x) = 2x + 5$

b) $g(x) = \frac{1}{2}(x - 9)$

c) $p(x) = 3x^3 - 4$

d) $h(x) = \dfrac{3x + 17}{x - 4}$

If $f(g(x)) = x$, then $f(x)$ and $g(x)$ are inverse functions, so that

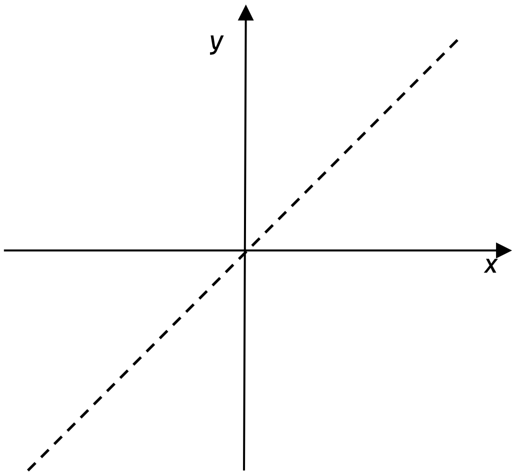
$f(x) = g^{-1}(x)$ AND $g(x) = f^{-1}(x)$

Example 2: $f(x) = 2x + 5$ and $g(x) = \dfrac{x - 5}{2}$. Show that $g(x) = f^{-1}(x)$

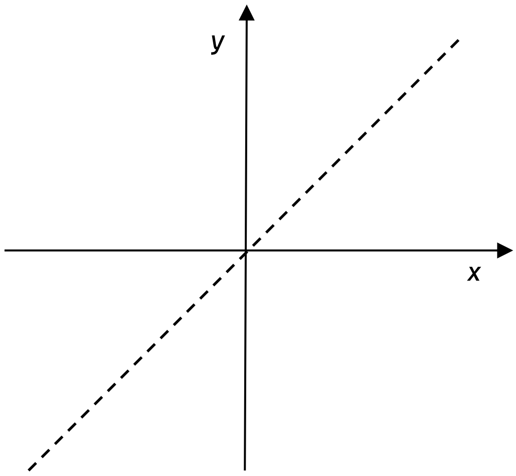
Graphs of Inverse Functions

Example 3: Sketch on the same graphs below:

a) $y = f(x)$ and $y = f^{-1}(x)$ where $f(x) = 2x + 5$



b) $y = f(x)$ and $y = f^{-1}(x)$ where $f(x) = \frac{x}{2} - 9$

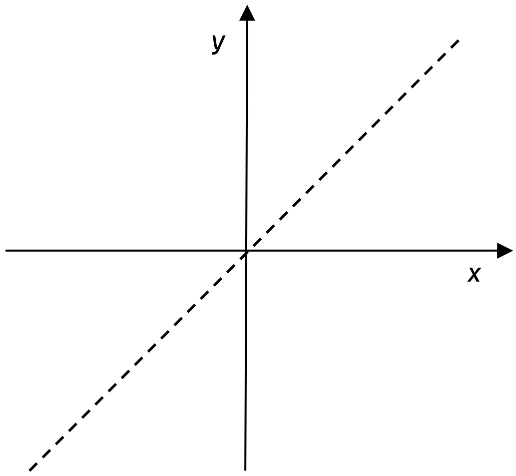


The dotted lines on each diagram are the line y = x. In each case, the graph of an inverse function can be obtained from the graph of the original function by **reflecting in the line y = x.**

Example 4: $g(x) = x^3 + 6$

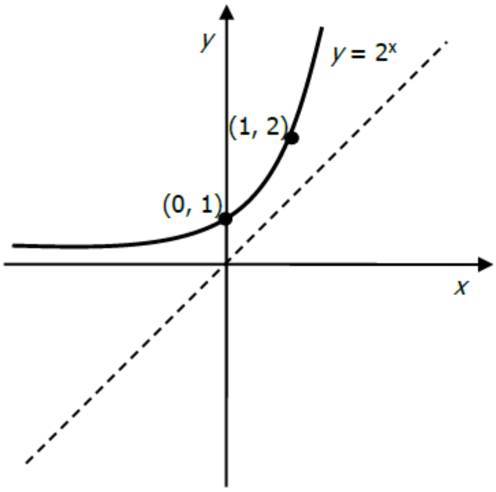
a) Sketch the graph of $y = g(x)$.

b) Show that $g^{-1}(x) = \sqrt[3]{x-6}$



c) Hence sketch the graph of $y = \sqrt[3]{x-6}$

Exponential functions have the formula $f(x) = a^x$, $x \in \mathbf{R}$, where a is called the **base.**
The graph of $y = 2^x$ is shown below.



The graph of $y = 2^x$ passes through the points (0, 1) and (1, 2). As reflection in the line y = x will produce the inverse of $y = 2^x$, then the inverse of $f(x) = 2^x$ must pass through the points (1, 0) and (2, 1)

The inverse of an exponential function is known as a **logarithmic** function.

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$
("log to the base a of x")

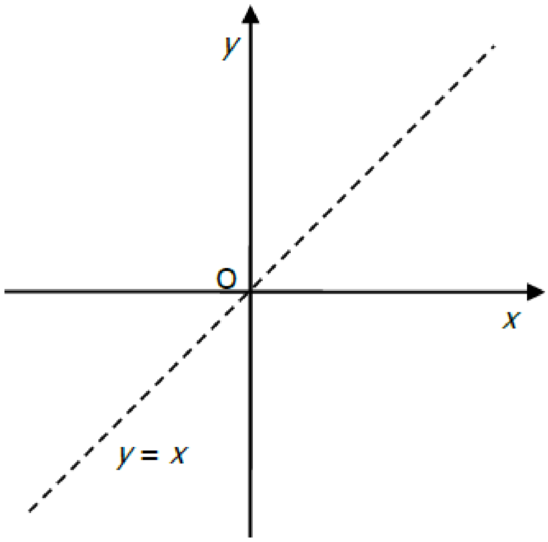
Example 5: Add the graph of $y = \log_2 x$ to the graph opposite.

Note that:

$y = a^x$ passes through (0, 1) and (1, a)
$y = \log_a x$ passes through (1, 0) and (a, 1)

For logarithms:

If $y = a^x$
then
$\log_a y = x$



Example 6: On the graph above, sketch and annotate the graphs of:

a) $y = 5^x$

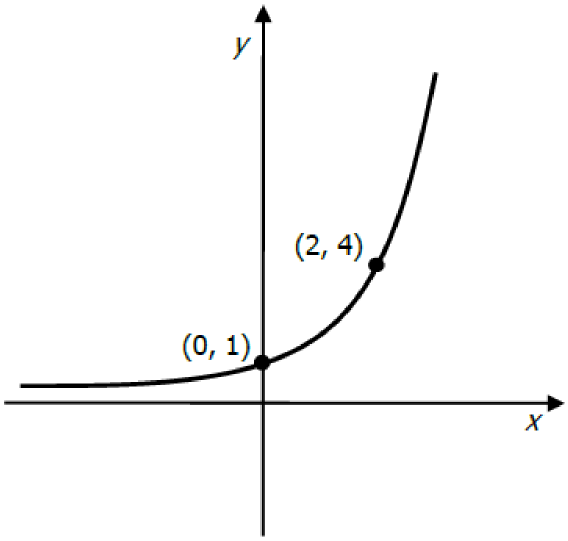
b) $y = \log_5 x$

Example 7: Write as logarithms:

a) $y = 3^x$

b) $q = 13^r$

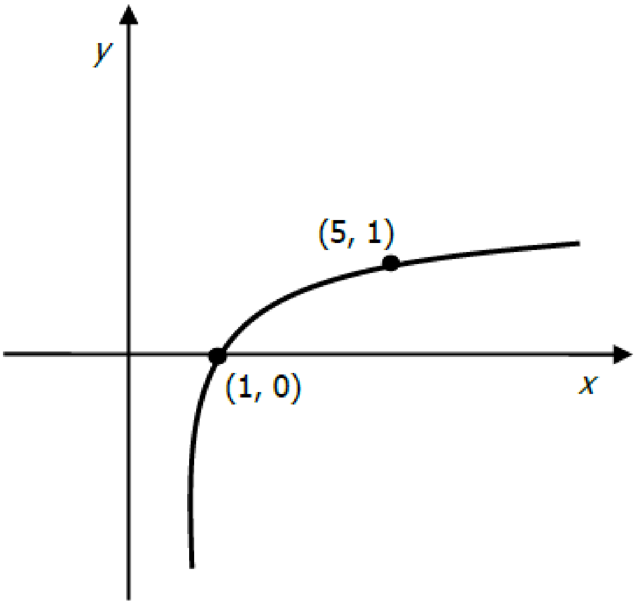
$y = a^x$ means "a multiplied by itself x times gives y"
$y = \log_a x$ means "y is the number of times I multiply a by itself to get x"



Example 8: Shown is the graph of the function $f(x) = 2^x$. To the diagram opposite, add the annotated graphs of the functions:

a) $y = 2^x - 3$

b) $y = 2^{(x - 2)}$



Example 9: Shown is the graph of the function $y = \log_5 x$. To the diagram opposite, add the annotated graphs of the functions:

a) $y = 2\log_5 x$

b) $y = \log_5(x + 1)$

Past Paper Example:

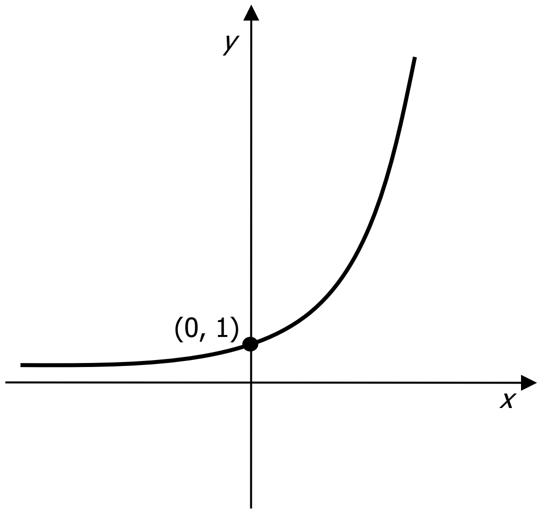
Functions f and g are defined on the set of real numbers. The inverse functions f^{-1} and g^{-1} both exist.

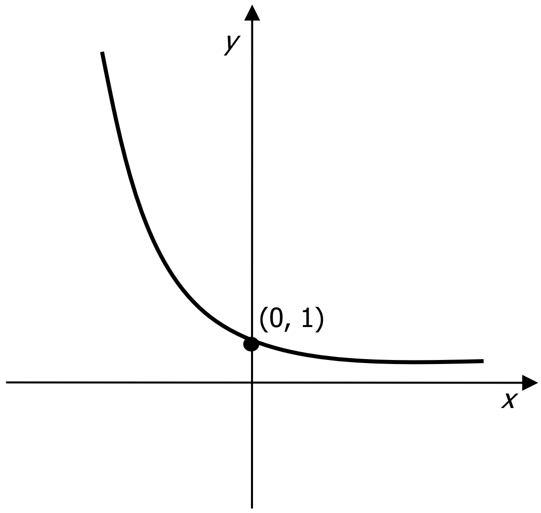
a) Given $f(x) = 3x + 5$, find $f^{-1}(x)$

b) If $g(2) = 7$, write down the value of $g^{-1}(7)$.

Exponential and Logarithmic Functions

Exponential functions are those with variable powers, e.g. $y = a^x$. Their graphs take two forms:





When a > 1, the graph:
- is always increasing
- is always positive
- never cuts the *x* – axis
- passes through (0, 1)
- shows **exponential growth**

When 0 < a < 1, the graph
- is always decreasing
- is always positive
- never cuts the *x* – axis
- passes through (0, 1)
- shows **exponential decay**

Exponential Functions as Models

Example 11: Ulanda's population in 2016 was 100 million and it was growing at 6% per annum.

a) Find a formula P_n for the population in millions, n years later.

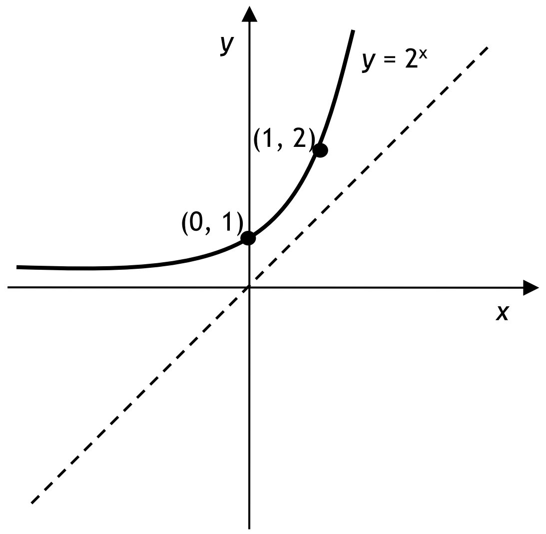
b) Estimate the population in the year 2026

Example 12: 8000 gallons of oil are lost in an oil spill in Blue Sky Bay. At the beginning of each week a filter plant removes 67% of the oil present.

a) Find a formula G_n for the amount of oil left in the bay after n weeks.

b) After how many **complete** weeks will there be less than 10 gallons left?

Logarithmic Functions



The inverse of an exponential function is known as a **logarithmic** function.

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$

("log to the base a of x")

We have seen that the graph of the inverse of a function can be obtained by reflection in the line $y = x$.

Since the graph of $y = 2^x$ passes through the points (0, 1) and (1, 2), then the inverse of $f(x) = 2^x$ must pass through the points (1, 0) and (2, 1).

Example 13: Add the graph of $y = \log_2 x$ to the graph opposite.

Note that:

$y = a^x$ passes through (0, 1) and (1, a)

$y = \log_a x$ passes through (1, 0) and (a, 1)

$y = a^x$ means "a multiplied by itself x times gives y"

$y = \log_a x$ means "y is the number of times I multiply a by itself to get x"

Since the graph does not cross the y-axis, we can only take the logarithm of a positive number

The expression "$\log_a x$" can be read as "a to the power of what is equal to x ?", e.g. $\log_2 8$ means "2 to the power of what equals 8?", so $\log_2 8 = 3$.

Example 14: Write in logarithmic form:

a) $5^2 = 25$

b) $12^1 = 12$

c) $8^{1/3} = 2$

d) $8^x = y$

e) $1 = q^0$

f) $(x - 3)^4 = k$

Example 15: Write in exponential form:

a) $3 = \log_5 125$

b) $\log_7 49 = 2$

c) $\log_4 4096 = 6$

d) $\log_2\left(\frac{1}{4}\right) = -2$

e) $\log_b g = 5h$

f) $1 = \log_7 7$

Example 16: Evaluate:

a) $\log_8 64$

b) $\log_2 32$

c) $\log_{3.5} 3.5$

d) $\log_{25} 5$

e) $\log_4\left(\frac{1}{2}\right)$

Since $a^1 = a$, then **$\log_a a = 1$**

Since $a^0 = 1$, then **$\log_a 1 = 0.$**

Laws of Logarithms

$\log_a xy = \log_a x + \log_a y$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_a x^n = n \log_a x$

Example 17:

a) $\log_2 4 + \log_2 8 - \log_2\frac{1}{2}$

b) $2\log_5 10 - \log_5 4$

c) Simplify $\frac{1}{4}(\log_3 810 - \log_3 10)$

Solving Logarithmic Equations

You **MUST** memorise the laws of logarithms to solve log equations! As we can only take logs of **positive** numbers, we must remember to discard any answers which violate this rule!

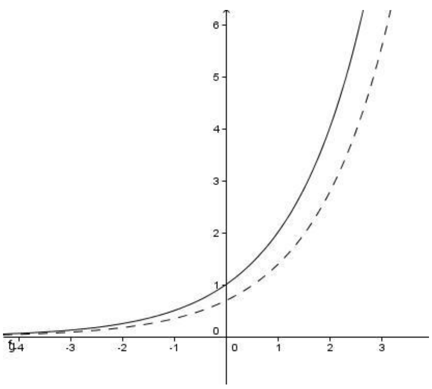
Example 18: Solve:

a) $\log_4 (3x - 2) - \log_4 (x + 1) = \frac{1}{2}$ $\left(x > \frac{2}{3}\right)$

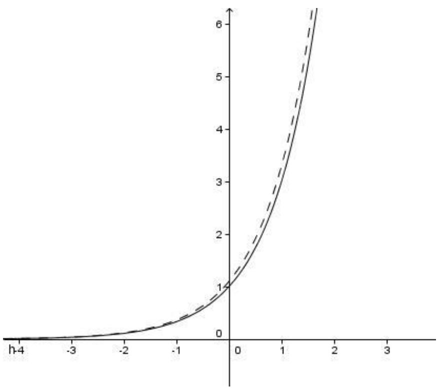
b) $\log_6 x + \log_6 (2x - 1) = 2$ $\left(x > \frac{1}{2}\right)$

The Exponential Function and Natural Logarithms

The graph of the derived function of $y = a^x$ can be plotted and compared with the original function. The new graphs are also exponential functions. Below are the graphs of $y = 2^x$ and $y = 3^x$ (solid lines) and their derived functions (dotted).



$f(x) = 2^x$



$f(x) = 3^x$

The derived function of $y = 2^x$ lies **under** the original graph, but the derived function of $y = 3^x$ lies **above** it.

This means that there must be a value of a between 2 and 3 where the derived function lies **on** the original.

i.e. where $f(x) = f'(x)$

The value of the base of this function is known as e, and is approximately 2.71828.

The function $y = e^x$ is known as The Exponential Function.

The function $y = \log_e x$ is known as the Natural Logarithm of x, and is also written as $\ln x$.

Example 19: Evaluate:

a) e^3

b) $\log_e 120$

Example 20: Solve:

a) $\ln x = 5$

b) $5^{x-1} = 16$

Example 21: Atmospheric pressure P_t at various heights above sea level can be determined by using the formula $P_t = P_0 e^{rt}$, where P_0 is the pressure at sea level, t is the height above sea level in thousands of feet, and r is a constant.

a) At 20 000 feet, the air pressure is half that at sea level. Find r accurate to 3 significant figures.

b) Find the height at which P is 10% of that at sea level.

Example 22: A radioactive element decays according to the law $A_t = A_0 e^{kt}$, where A_t is the number of radioactive nuclei present at time t years and A_0 is the initial amount of radioactive nuclei.

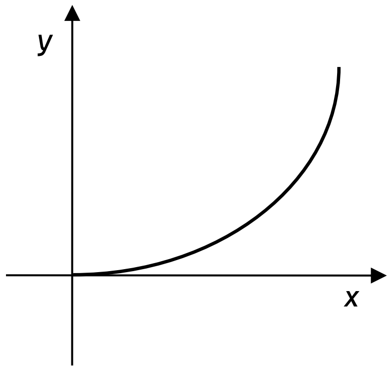
a) After 150 years, 240g of this material had decayed to 200g.
Find the value of k accurate to 3 s.f.

b) The half-life of the element is the time taken half the mass to decay. Find the half-life of the material.

Using Logs to Analyse Data, Type 1: $y = kx^n \Leftrightarrow \log y = n \log x + \log k$

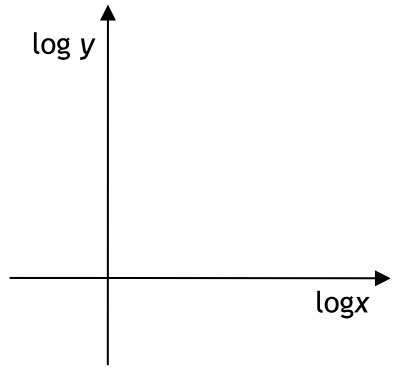
When the data obtained from an experiment results in an exponential graph of the form $y = kx^n$ as shown below, we can use the laws of logarithms to find the values of k and n.

To begin, take logs of both sides of the exponential equation.



$y = kx^n$

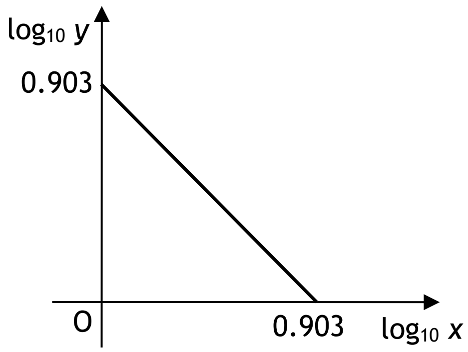
$y = kx^n$



This gives a straight line graph!

$\log y = n \log x + \log k$

Note: the base is not important, as long as the **same** base is used on both sides.



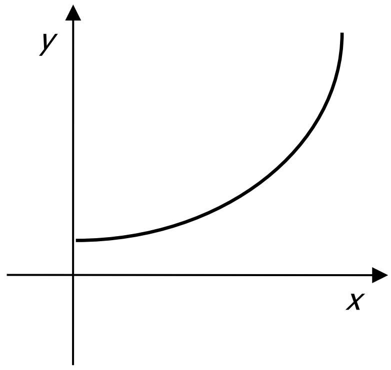
Example 23: Data are recorded from an experiment and the graph opposite is produced.

a) Find the equation of the line in terms of $\log_{10} x$ and $\log_{10} y$.

b) Hence express y in terms of x.

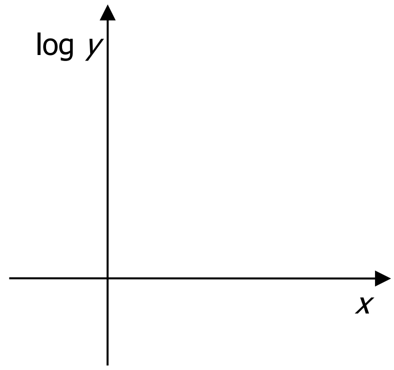
Using Logs to Analyse Data, Type 2: $y = kn^x \Leftrightarrow \log y = \log n\,(x) + \log k$

A similar technique can be used when the graph is of the form $y = kn^x$ (i.e. x is the index, not the base as before).



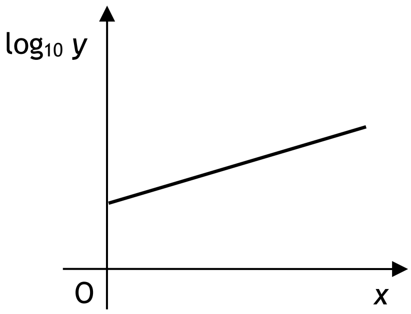
$y = kn^x$

$y = kn^x$



$\log y = (\log n)\, x + \log k$

Example 24: The data below are plotted and the graph shown is obtained.



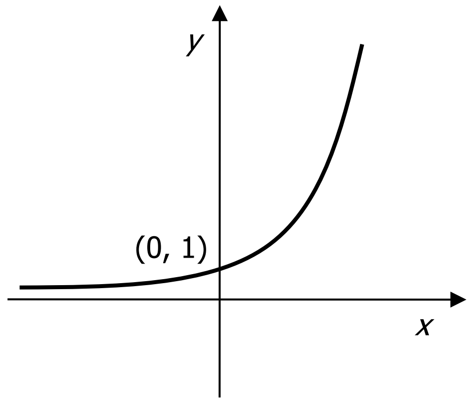
x	0	1	2	3	4
$\log_{10} y$	0.602	1.079	1.556	2.033	2.510

a) Express $\log_{10} y$ in terms of x.

b) Hence express y in terms of x.

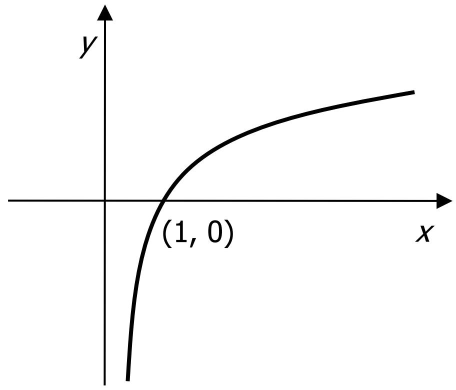
Related Graphs of Exponentials and Logs

$y = e^x + a$



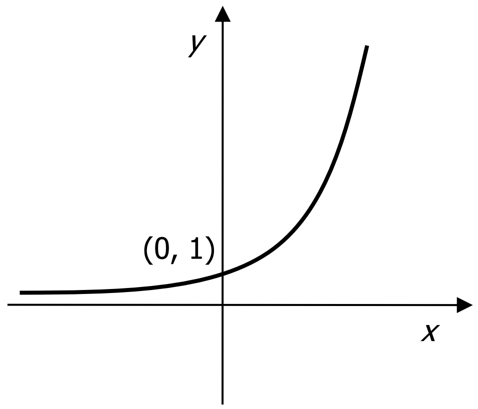
$y = e^x + a$ is obtained by **sliding** $y = e^x$:
Vertically upwards if a>0
Vertically downwards if a<0

$y = \ln(x + a)$



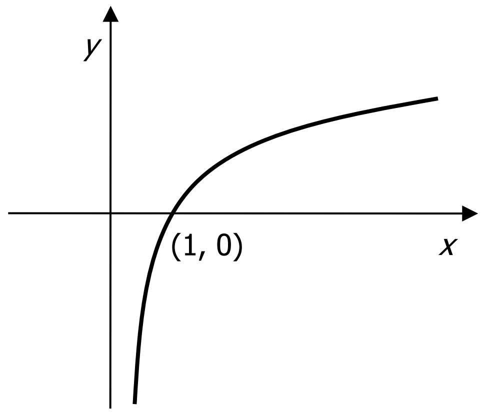
$y = \ln(x + a)$ is obtained by **sliding** $y = \ln x$
Horizontally left if a>0
Horizontally right if a<0

$y = e^{(x + a)}$



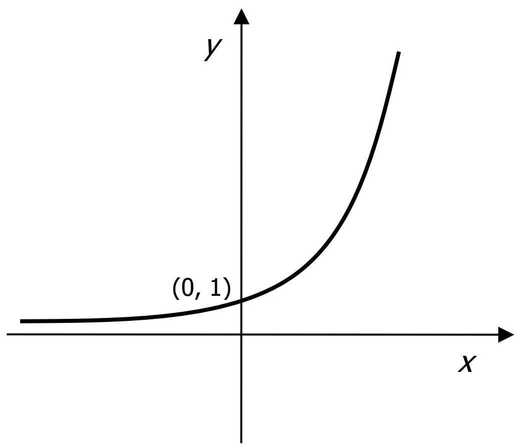
$y = e^{(x + a)}$ is obtained by **sliding** $y = e^x$
Horizontally left if a>0
Horizontally right if a<0

$y = k \ln x$



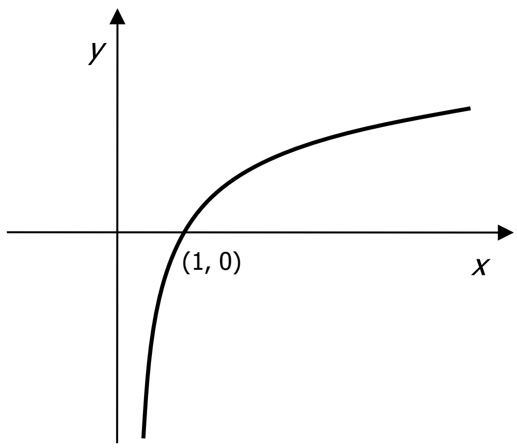
$y = k \ln x$ is obtained by **vertically:**
stretching $y = \ln x$ if $k > 1$
compressing $y = \ln x$ if $0 < k < 1$

$y = e^{-x}$

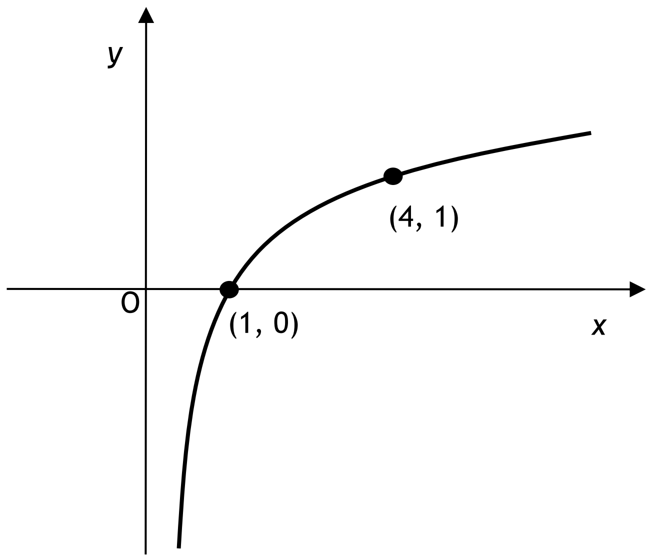


$y = e^{-x}$ is obtained by **reflecting** $y = e^x$**:**
in the y -axis

$y = - \ln x$



$y = - \ln x$ is obtained by **reflecting** $y = \ln x$:
in the x - axis



Example 25: The graph of $y = \log_4 x$ is shown. On the same diagram, sketch:

a) $y = \log_4 4x$

b) $y = \log_4\left(\frac{1}{4x}\right)$

Past Paper Example 1:

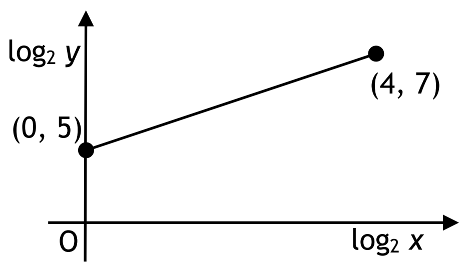
a) Show that $x = 1$ is a root of $x^3 + 8x^2 + 11x - 20 = 0$, and hence factorise $x^3 + 8x^2 + 11x - 20$ fully

b) Solve $\log_2(x + 3) + \log_2(x^2 + 5x - 4) = 3$

Past Paper Example 2: Variables x and y are related by the equation $y = kx^n$.

The graph of $\log_2 y$ against $\log_2 x$ is a straight line through the points (0, 5) and (4, 7), as shown in the diagram.

Find the values of k and n.



Past Paper Example 3: The concentration of the pesticide *Xpesto* in soil is modelled by the equation:

$$P_t = P_0 e^{-kt}$$

where:
P_0 is the initial concentration
P_t is the concentration at time t
t is the time, in days, after the application of the pesticide.

a) Once in the soil, the half-life of a pesticide is the time taken for its concentration to be reduced to one half of its initial value.

If the half-life of *Xpesto* is 25 days, find the value of k to 2 significant figures.

b) Eighty days after the initial application, what is the percentage decrease in *Xpesto*?

Past Paper Example 4: Simplify the expression $3\log_e 2e - 2\log_e 3e$ giving your answer in the form $A + \log_e B - \log_e C$, where A, B and C are whole numbers.

Past Paper Example 5: Two variables x and y satisfy the equation $y = 3\left(4^x\right)$.

A graph is drawn of $\log_{10} y$ against x. Show that its equation will be of the form $\log_{10} y = Px + Q$, and state the gradient and y-intercept of this line.

Expressions & Functions Unit Topic Checklist: Unit Assessment Topics in Bold			
Topic		Questions	Done?
Logs & Exponentials	**Logarithms**	Exercise 15E, (all)	Y/N
	Laws of logarithms	Exercise 15F, Q 1	Y/N
	Log equations	Exercise 15G, Q 1, 2, 3	Y/N
	e^x and Natural logarithms	Exercise 15D, Q 1 – 5	Y/N
	Exponential growth/decay	Exercise 15H, Q 4 - 7	Y/N
	Data and straight line graphs	Exercise 15J, Q 2; Exercise 15K, Q 2	Y/N
	Related log & exponential graphs	Exercise 15K, Q 1 - 7	Y/N
Trigonometry	**Radians**	Exercise 4C, Q 1 – 3	Y/N
	Exact Values	Exercise 4E, Q 1, 3	Y/N
	Trig Identities	Exercise 17, Q 2; Exercise 11J, Q 20	Y/N
	Compound and double angle formulae	Exercise 11D, Q 6 – 8; Exercise 11F, Q 1 – 4, 7, 9	Y/N
	k cos(x - α)	Exercise 16C, Q 1 – 5; Exercise 16D, Q 2	Y/N
	k cos(x + α)	Exercise 16E, Q 1	Y/N
	k sin($x \pm \alpha$)	Exercise 16E, Q 2, 3; Exercise 16E, Q 4, 5	Y/N
	Wave Fn Maxima and minima	Exercise 16G, Q 1, 3, 4, 5, 7	Y/N
	Solving Wave Fn equations	Exercise 16H, Q 1 - 4	Y/N
Sets, Functions & Graphs	**Transforming graphs**	Exercise 3P, Q 1 – 9	Y/N
	Naming/Sketching trig graphs	Exercise 4B, (all)	Y/N
	Completing the square	Exercise 8D, Q 4, 6; Exercise 5, Q 3, 4	Y/N
	Graphs of derived functions	Exercise 6P, (all)	Y/N
	Set Notation	Exercise 2A, Q 2 & 3	Y/N
	Composite Functions	Exercise 2C, Q 5 - 10	Y/N
	Inverse Functions	Exercise 2D, Q 2; Exercise 2I, Q 1	Y/N
	Graphs of inverse functions	Exercise 2F, Q 1 & 2	Y/N
	Exponential & log graphs	Exercise 3N, Q 3, 4; Exercise 3O, p 47, Q 2, 3	Y/N
Vectors	**Resultant vectors**	Exercise 13N (all)	Y/N
	Unit Vectors (inc. i, j, k)	Exercise 13F, Q 1, 2;	Y/N
	Collinearity	Exercise 13N, Q 15 – 18, 23	Y/N
	Section Formula	Exercise 13N, Q 20 - 24	Y/N
	Scalar Product	Exercise 13O, Q 1; Exercise 13P, Q 1, 2	Y/N
	Angle between vectors	Exercise 13Q, Q 1, 2; Exercise 13S, Q 4 – 7	Y/N
	Perpendicular Vectors	Exercise 13R, Q 1 - 8	Y/N
	Properties of Scalar Product	Exercise 13U, Q 1, 2, 4, 5	Y/N

Past Paper Questions by Topic

	Topic	2009		2010		2011		2012		2013		2014		2015		2016		2017	
		P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2
Functions and Graphs	Ranges and Domains					20				13		12			2c	15b			
	Composite Functions	3		3			2a,b		1a	1			3a	5b	2a	12a		1	
	Inverse Functions													5a		6		6	
	Transforming graphs		7			3			4	11		11		13c					
	Interpreting trig functions and graphs			11b	4a			9		4				4					
	Exact Values							1				13							
Recurrence Relation	Terms of Recurrence Relations	4		7a							1	1			3a	3a		9a	8
	Creating & using RR formulae																		
	Finding a limit of an RR			7b			3c		6	8		10			3b	3b,c		9b	
	Solving RR's to find *a* and *b*						3a,b												
Straight Line	Gradients from points or equations					2				2									
	Gradients from angles with *x* - axis					8		4					1c						1b
	Equations of straight lines		1	1		21a		23b		5	2a					1	1b	11	1b
	Perpendicular bisectors					21c		23a					1a						1a
	Altitudes	1b													1a				
	Medians	1a													1b		1a	7	
	Points of intersection of lines	1c				21b		23c			2b,c		1b		1c		1c		1c
	Distance Formula							23d											
	Collinearity													9					10a
Differentiation	Finding derivatives of functions							6,8						7				8	4b
	Equations of tangents to curves		3a,10b		5a	4		2				2	2	2		2			
	Increasing & decreasing functions											21a				9b		15c	4c
	Stationary points	5		9b		22b		18				21b				9a			7a
	Curve Sketching			9c		22ac													
	Closed Intervals							12	3										7b
	Graphs of derived functions																		
	Optimisation		12		6						7				8		7		
	Velocity, Acceleration, Displacement												9						
Integration	Finding indefinite integrals					11, 16		11		7									
	Definite Integrals												5						
	Area under a curve	6		8c				21b						12			3b	15b	
	Area between two curves						4				4		7		4			10	
	Differential Equations		5		10b									15			9		

Past Paper Questions by Topic

	Topic	2009		2010		2011		2012		2013		2014		2015		2016		2017	
		P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2
Quadratics	Completing the square	8				5		3		21		17			2b	12b		15a	4a
	Quadratic Inequations					18		19		19				8					
	Roots using $b^2 - 4ac$		2			9			1b	3			3b				2	4	
	Tangency using $b^2 - 4ac$		3b	4			2c, d			22c,d	3b			11b					
Polynomials	Synthetic Division	9b, c		8b		7		21a		6	3a	22		3			3a		2
	Intersections of curves																		
	Functions from graphs				10a	17		13		17		15				15a			
	Approximate roots (Iteration)																		
Circles	$(x - a)^2 + (y - b)^2 = r^2$						7		2b			23c				4	4a		10b
	General equation of a circle	2a	4	5			7			22a			8				4a		
	Lines cutting circles								2a			23ab		14	5				3
	Tangents to circles	2b			3, 5c	6				22b		2		11a		8		2	
	Distance between centres vs radii																4b		
Trigonometry	Trig Equations	7		6	4b	10	6b			15	8		6						6
	Compound angle formulae		8b		2a	12				10						13	8b		
	Double angle formulae	7	8a,b	6	2b	23		5		9		7,18		10					
	Trigonometric identities														7b		11a		11a
Vectors	Interpreting vector diagrams											19	4a						5a
	Unit Vectors		6					15								11b			
	Position Vectors and Components			2		1	1a, b	10	5ai	12		6	4b			7	5a		5b
	Collinearity					15				24									
	Section Formula							7								11a			
	Scalar Product (angle form)																	5b	
	Scalar Product (components)	9a						17									5b	5a	
	Angle between vectors	9d			1		1c		5aii, b				4c				5b		5c
	Perpendicular Vectors											14		1					
	Properties of Scalar Product					14				14		16			6				
WF	Wave Functions		10a	11a			6a	22a		23		4,9			9		8a	14	
Further Calculus	Chain Rule (inc. trig functions)			10		13		16		18		8					10a, 11c	3	11b
	Integrating a (…..)^n							14,22b		16		5	5				10b	13	
	Integrating sin and cos		9, 10b		7		6b				6				7a,c	5			
Logs and Exponentials	Exponential growth/decay		11								9						6		
	Log equations				8			20			5	3,20		6		14			
	Exponential and log graphs				9	19			7					13a,b		10			
	Linearisation	10			11		5			20		24							9