St Andrew’s Academy

Department of Mathematics

Advanced Higher

Course Textbook

UNIT 3

Unit 3

N7 # Properties of Functions AAC 1.4

We wish to sketch the graphs of rational functions. A rational function is of the form $\frac{P(x)}{Q(x)}$. We wish to find stationary points and their nature and where the graph crosses the x and y axes.

Asymptotes

An asymptote is a straight line to which the curve approaches more and more closely as x becomes very large or very negative, or approaches a certain value. Asymptotes can be vertical, horizontal or slant.

Vertical Asymptotes
These are of the form *x*=*k*. They are found from the zeros of the denominator. We must find the asymptotes and how the graph approaches them from the left and the right.

Example: Find the vertical asymptotes of $f(x) = \frac{2x+3}{x^2+5x+4}$.
Vertical asymptotes occur when $x^2 + 5x + 4 = 0$
$(x + 4)(x + 1) = 0$
$x = -4$ and $x = -1$ are asymptotes.

$$f(x) = \frac{2x+3}{(x+4)(x+1)}$$

For $x = -4$
First choose a number the left of -4
$x = -4.1 \quad f(-4.1) = \frac{-}{(-)(-)} = -$
So as $x \to -4^-, y \to -\infty$.
"As x tends to -4 from the left, y tends to negative infinity."
Now choose a number to the right of -4, say -3.9.
$x = -3.9 \quad f(-3.9) = \frac{-}{(+)(-)} = +$

So as $x \to -4^{+}, y \to +\infty$.
"As x tends to -4 from the right, y tends to positive infinity."
For x=-1

$x = -1.1 \;\; f(-1.1) = \frac{+}{(+)(-)} = -$ So as $x \to -1^{-}, y \to -\infty$.
$x = -0.9 \;\; f(-0.9) = \frac{+}{(+)(+)} = +$ So as $x \to -1^{+}, y \to +\infty$.

Exercise 1: Find all the vertical asymptotes and their approaches of the following rational functions.

a) $f(x) = \frac{4}{x-2}$ b) $f(x) = \frac{3x-1}{x^2+2x-3}$ c) $f(x) = \frac{12}{x^2-2x-3}$

d) $f(x) = \frac{x+4}{x-2}$ e) $f(x) = \frac{x^2}{4-x^2}$ f) $f(x) = \frac{x(x-1)}{(x-1)(x+2)}$

g) $f(x) = \frac{(x-1)(x-4)}{x}$ h) $f(x) = \frac{x^2+3}{x-1}$ i) $f(x) = \frac{x^2}{x^2+3}$

j) $f(x) = \frac{x}{x^2+4}$ k) $f(x) = \frac{x^2}{x-1}$ l) $f(x) = \frac{2x^2}{x^2-1}$

Non-Vertical Asymptotes

Non-vertical asymptotes are horizontal or slant with equations of the form $y = c$ or $y = mx + c$. The x-axis can be an asymptote with equation $y = 0$. We will see that:

- If the degree of the numerator < the degree of the denominator then the x-axis is the asymptote.
- If the degree of the numerator = the degree of the denominator then, after dividing the numerator by the denominator, we will find that the asymptote is of the form $y = c$.
- If the degree of the numerator is one degree more than the degree of the denominator then, after dividing the numerator by the denominator, we will find that the asymptote is of the form $y = mx + c$.

If the degree of the numerator is two or more degrees greater than the denominator then there are no linear non-vertical asymptotes. The curve may approach $y = ax^2$ or $y = ax^3$ etc but that is not of interest here.

Consider $f(x) = \frac{3x+5}{x^2-4x+1}$. What happens to $f(x)$ as $x \to \pm\infty$?
Dividing numerator and denominator by the highest power, x^2, we get

$$f(x) = \frac{\frac{3}{x}+\frac{5}{x^2}}{1-\frac{4}{x}+\frac{1}{x^2}}$$

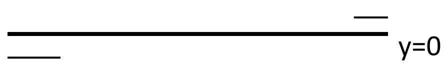
as $x \to \pm\infty$, $\frac{3}{x}$, $\frac{5}{x^2}$, $\frac{4}{x}$ and $\frac{1}{x^2}$ all tend to zero. So $f(x) \to \frac{0}{1} = 0$.
The non-vertical asymptote is $y = 0$.

When the numerator is of a lower degree than the denominator: if x becomes very large or very negative, $f(x)$ tends to zero.

Now consider the sign that $f(x) = \frac{3x+5}{x^2-4x+1}$ takes when x is very large or very negative.
For very large x the value of the polynomial is mostly determined by the highest power.

e.g. $g(x) = 3x + 5$. If x=1,000,000 adding 5 to 3,000,000 makes little difference to the value of $g(x)$.
e.g.$h(x) = x^2 - 4x + 1$. If x=1,000,000 subtracting 4 million from 1 million squared will make little difference to the value of $h(x)$.

As $x \to +\infty,\ f(x) \to \frac{+}{+} \to +$.
"As x gets larger, the curve approaches the asymptote from above."
As $x \to -\infty,\ f(x) \to \frac{-}{+} \to -$.
"As x gets more negative, the curve approaches the asymptote from below."



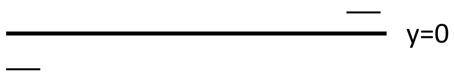
Examples: Find the non-vertical asymptotes and the approaches.

a) $f(x) = \frac{2x+3}{x^2+5x+4}$ (degree of the numerator<degree of the denominator)

$f(x) \to 0$ as $x \to \pm\infty$, so the x-axis is an asymptote.

As $x \to +\infty$, $f(x) \to \frac{+}{+} \to +$

As $x \to -\infty$, $f(x) \to \frac{-}{+} \to -$



b) $f(x) = \frac{x^2+2x+1}{x^2+5x+4}$ (degree of the numerator=degree of the denominator)

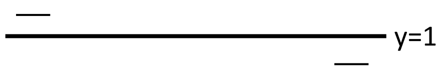
First divide the numerator by the denominator, this gives

$f(x) = 1 - \frac{3x+3}{x^2+5x+4}$

The fraction part will tend to zero for large values of x (as above) and f(x) will get closer to the value of 1.

As $x \to +\infty$, $f(x) \to 1 - \frac{+}{+} \to less\ than\ 1$

As $x \to -\infty$, $f(x) \to 1 - \frac{-}{+} \to more\ than\ 1$



c) $f(x) = \frac{x^2+4x+3}{x+2}$ (degree of the numerator>degree of the denominator)

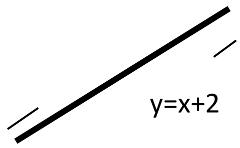
First divide the numerator by the denominator, this gives

$f(x) = x + 2 - \frac{1}{x+2}$

The fraction part will tend to zero as x becomes very large so the asymptote is $y = x + 2$.

As $x \to +\infty$, $f(x) \to x + 2 - \frac{1}{+} \to less\ than\ x + 2$

As $x \to -\infty$, $f(x) \to x + 2 - \frac{1}{-} \to more\ than\ x + 2$



Exercise 2: Find all the non-vertical asymptotes and their approaches of the following rational functions.

a) $f(x) = \frac{4}{x-2}$ b) $f(x) = \frac{3x-1}{x^2+2x-3}$ c) $f(x) = \frac{12}{x^2-2x-3}$

d) $f(x) = \frac{x+4}{x-2}$ e) $f(x) = \frac{x^2}{4-x^2}$ f) $f(x) = \frac{x(x-1)}{(x-1)(x+2)}$

g) $f(x) = \frac{(x-1)(x-4)}{x-2}$ h) $f(x) = \frac{x^2+3}{x-1}$ i) $f(x) = \frac{x^2}{x^2+3}$

j) $f(x) = \frac{x}{x^2+4}$ k) $f(x) = \frac{x^2}{x-1}$ l) $f(x) = \frac{2x^2}{x^2-1}$

Curve Sketching

Steps in curve sketching for rational functions:

1. Find all the asymptotes and investigate the approaches
2. Find all the stationary points and their nature
3. Find where the graph crosses the y-axis and x-axis

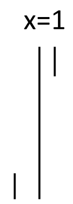
Remember that you now have many methods of differentiation and for finding the nature stationary points. The methods used here are just one of the options.

Examples: Sketch the graph of

a) $f(x) = \frac{2x^2+x-1}{x-1} = \frac{(x+1)(2x-1)}{x-1}$

Asymptotes

Vertical: $x - 1 = 0 \implies x = 1$ is an asymptote

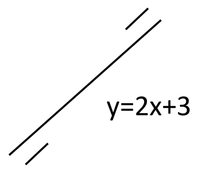


As $x \to 1^-,\ y \to \frac{(+)(+)}{-} \to -\infty$

As $x \to 1^+,\ y \to \frac{(+)(+)}{+} \to +\infty$

Non-Vertical: By division, $f(x) = 2x + 3 + \frac{2}{x+1}$

So $y = 2x + 3$ is an asymptote.



As $x \to +\infty, f(x) \to 2x + 3 + \frac{2}{+} \to more\ than\ 2x + 3$

As $x \to +\infty, f(x) \to 2x + 3 + \frac{2}{-} \to less\ than\ 2x + 3$

Stationary Points

$f(x) = 2x + 3 + 2(x + 1)^{-1}$

$f'(x) = 2 - 2(x - 1)^{-2}$

Set f'(x)=0 for stationary points

$2 - \frac{2}{(x-1)^2} = 0$

$2 = \frac{2}{(x-1)^2}$

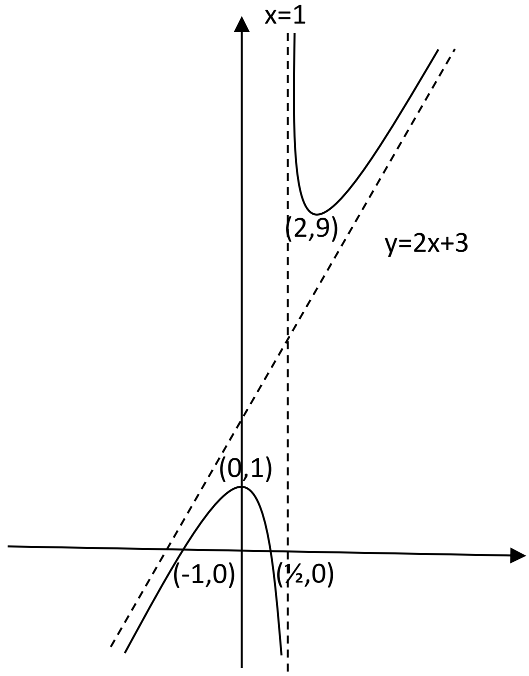
$(x - 1)^2 = 1$

$x = 2\ \ or\ \ x = 0$

$y = 9 \qquad y = 1$

$f''(x) = 4(x - 1)^{-3}$

$f''(0) = - \implies (0,1)$ is a max t.p.
$f''(2) = + \implies (2,9)$ is a min t.p.



<u>Axes Crossings</u>

x-axis: $y = 0 \ i.e. \ \frac{(x+1)(2x-1)}{x-1} = 0$

$(x+1)(2x-1) = 0$

$x = -1 \ \ or \ \ x = \frac{1}{2}$

$(-1,0) \qquad \left(\frac{1}{2}, 0\right)$

y-axis: $x = 0 \ \Rightarrow \ y = 1$

$(0,1)$

b) $f(x) = \frac{2x^2+4x+3}{x^2-1} = \frac{2x^2+4x+3}{(x+1)(x-1)}$

<u>Asymptotes</u>

Vertical: $(x+1)(x-1) = 0$

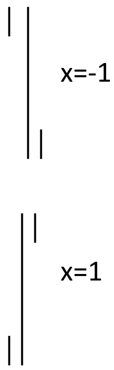
$x = -1$ and $x = 1$ are asymptotes

As $x \to -1^-$, $y \longrightarrow \frac{+}{(-)(-)} \longrightarrow +\infty$

As $x \to -1^+$, $y \longrightarrow \frac{+}{(-)(-)} \longrightarrow +\infty$

As $x \to 1^-$, $y \longrightarrow \frac{+}{(+)(-)} \longrightarrow -\infty$

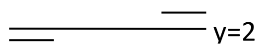
As $x \to 1^+$, $y \longrightarrow \frac{+}{(+)(+)} \longrightarrow +\infty$



Non-Vertical: $f(x) = 2 + \frac{4x+5}{x^2-1}$, so $y = 2$ is an asymptote

As $x \to +\infty, y \to 2 + \frac{+}{+} \to more \ than \ 2$

As $x \to -\infty, y \to 2 + \frac{-}{+} \to less \ than \ 2$



<u>Stationary Points</u>

$f(x) = 2 + \frac{4x+5}{x^2-1}$

$f'(x) = \frac{4(x^2-1)-(4x+5)2x}{(x^2-1)^2}$

$= \frac{-2(x+2)(2x+1)}{(x^2-1)^2}$

Set f’(x)=0 for stationary points

$-2(x+2)(2x+1) = 0$

$x = -2 \;\; or \;\; x = -\frac{1}{2}$

$y = 1 \qquad\quad y = -2$

x	→-2→
-2	- - -
x+2	- 0 +
2x+1	- - -
$(x^2-1)^2$	+ + +
f'(x)	- 0 +
tangent	__/

x	→-½→
-2	- - -
x+2	+ + +
2x+1	- 0 +
$(x^2-1)^2$	+ + +
f'(x)	+ 0 -
tangent	/‾‾\\

Min t.p. at (-2,1)

Max t.p. at (-½,-2)

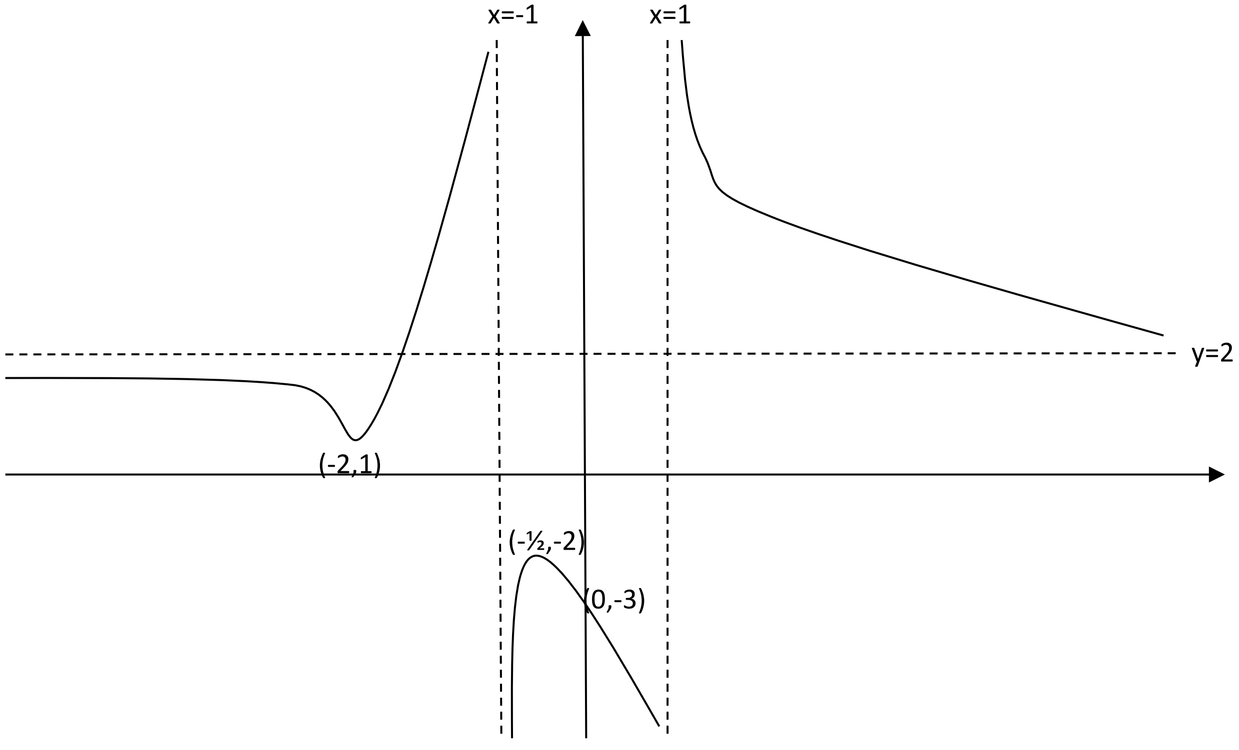
Axes Crossings

x-axis: $y = 0 \; i.e. \; \frac{2x^2+4x+3}{(x+1)(x-1)} = 0$

$2x^2 + 4x + 3 = 0$, however $b^2-4ac<0$ so there are no real roots and f(x) does not cross the x-axis.

y-axis: $x = 0 \;\; \Rightarrow \;\; y = -3$

$(0, -3)$



Exercise 3: Sketch the following curves

a) $f(x) = \frac{4}{x-2}$

b) $f(x) = \frac{x-2}{x-1}$

c) $f(x) = \frac{x^2+x-2}{x^2+x-6}$

d) $f(x) = \frac{x^2+2x+5}{x+1}$

e) $f(x) = \frac{x+1}{x^2+2x+5}$

f) $f(x) = \frac{2x^2}{x^2-1}$

g) $f(x) = \frac{x^2-10x+9}{x^2+10x+9}$

h) $f(x) = \frac{2x^2-3x-3}{x^2-3x+2}$

Concavity

The graph of $y=f(x)$ is <u>concave downward</u> in an interval when $f''(x)<0$. Concave downward can be pictured as a ∩ shaped graph.

The graph of $y=f(x)$ is <u>concave upward</u> in an interval when $f''(x)>0$. Concave upward can be pictured as a ∪ shaped graph.

A curve has a point of inflexion if the concavity changes at the point i.e. (a,b) is a point of inflexion if $f''(x)$ changes sign at x=a.

Examples:

1 Determine the concavity of the function $f(x)=x^3-3x+4$.

$f'(x)=3x^2-3$

$f''(x)=6x$

There will be a possible point of inflexion when $f''(x)=0$

$6x=0$

$x=0$

Is there a change in the sign of $f''(x)$ at x=0?

x	→ 0 →
f''(x)	- 0 +

So (0,4) is a point of inflexion and the curve is concave downward to the left of zero and concave upward to the right of zero. Note that (0,4) is not a horizontal point of inflexion as $f'(x)\neq 0$ at x=0.

2 Find the points of inflexion on the curve $y=(x-1)^4-32x$

$f'(x)=4(x-1)^3-32$

$f''(x)=12(x-1)^2$

Possible points of inflexion when $12(x-1)^2=0$

$x=1$

But $12(x-1)^2$ is always greater than or equal to zero so there is no point of inflexion at x=1 and the curve is concave up.

Exercise 4:

1. Use the second derivative to show that the graph of f(x)=lnx is always concave down.

2. Find the points of inflexion on the curve $y=2x^3-3x^2$.

3. Describe the concavity of the function $f(x)=(x+2)^3+4$ and identify the point of inflexion.

4. Describe the concavity of the function $f(x) = x^2 + \frac{16}{x}$ and identify the point of inflexion.

New Terms

A <u>critical point</u> is any point on a curve where the gradient of the tangent is zero or where f'(x) is undefined.

A <u>local maximum point</u> occurs when a function has a greater value at that point than at any points close to it. It may not be the greatest value of the function. There can be more than one local maximum turning point.

A <u>local minimum point</u> is defined in a similar way.

A <u>global maximum point</u> occurs when f(x) has its greatest value over the whole domain, at a point i.e. a function has a global maximum at a is f(a)≥f(x) for all x in the domain of f.

A <u>global minimum point</u> is defined in a similar way.

<u>Continuous Function</u>

A function is said to be continuous if there is no break in the graph of the function. If a graph can be drawn without lifting the pencil from the paper, the function is continuous.

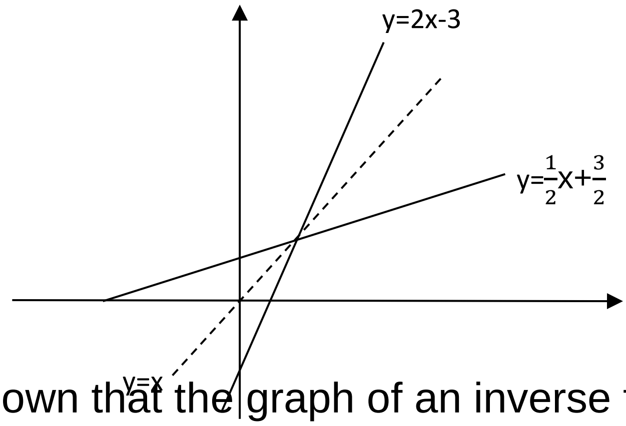
The Graph of Inverse Functions

The graph of the inverse of simple functions has already been met at Higher level

Examples:

a) $f(x)=2x-3$ has an inverse of $f^{-1}(x)=\frac{1}{2}(x+3)=\frac{1}{2}x+\frac{3}{2}$

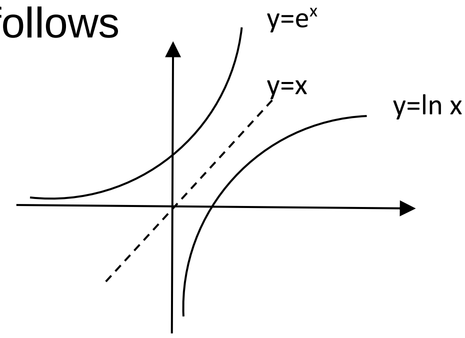
The graphs are as follows



It has already been shown that the graph of an inverse function can be found by reflecting the graph of the function in the line y=x.

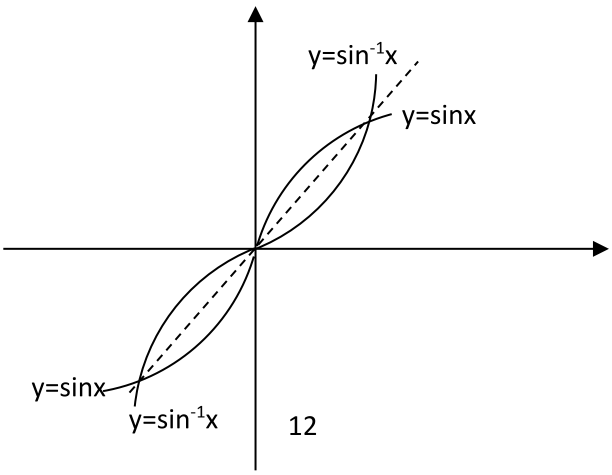
b) $f(x)=e^x$ has an inverse of $f^{-1}(x)=\ln x$

The graphs are as follows



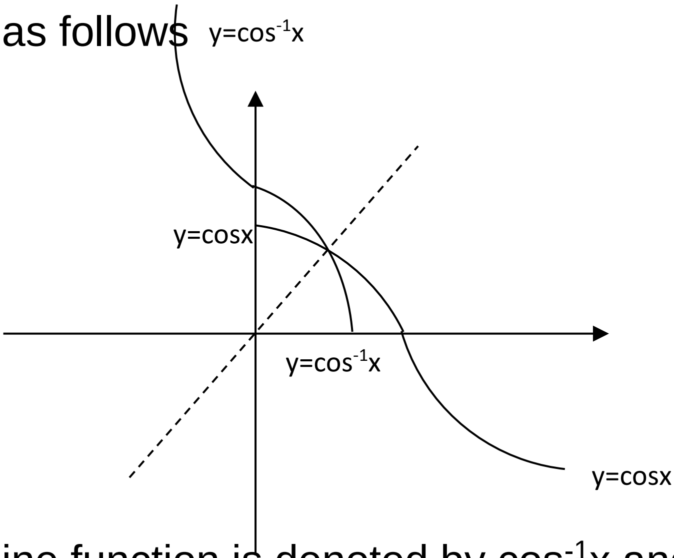
c) $f(x)=\sin x$ has an inverse of $f^{-1}(x)=\sin^{-1}x$

The graphs are as follows



The inverse sine function is denoted by $\sin^{-1}x$ and read as sin inverse of x or arc sine x.
The inverse sine function is defined as the angle whose sine is x.
If we consider the graph of y=sinx, we see that there is an infinite number of angles whose sine could be x.
Consequently, in order that the inverse sine should be a function, we must restrict the angle concerned.
We chose the simplest possible restriction for the angle – the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. A revised definition is therefore that the inverse sine function, $\sin^{-1}x$, is the angle in the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, whose sine is x.

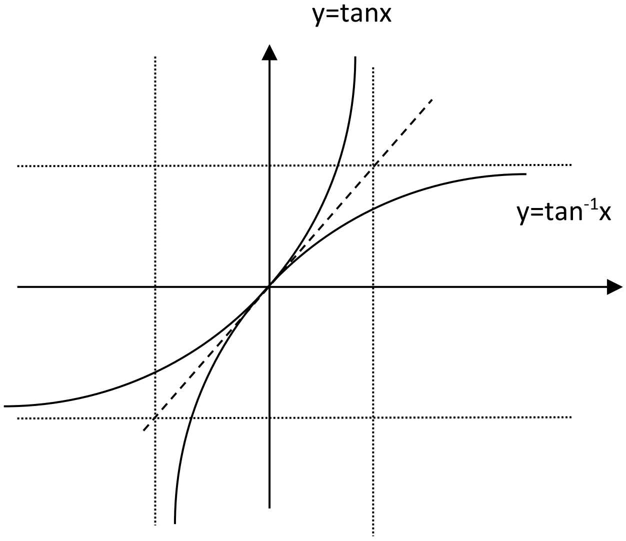
d) $f(x)=\cos x$ has an inverse of $f^{-1}(x)=\cos^{-1}x$
The graphs are as follows



The inverse cosine function is denoted by $\cos^{-1}x$ and is read as cos inverse or arc cos x.
In this case we restrict the angle to the closed interval $[0, \pi]$.
The inverse cosine function is defined as the angle, in the closed interval $[0, \pi]$, whose cosine is x.

e) $f(x)=\tan x$ has an inverse of $f^{-1}(x)=\tan^{-1}x$

The graphs are as follows

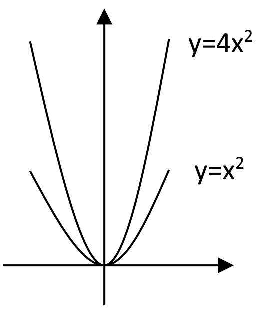


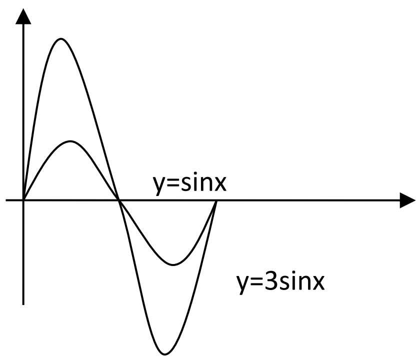
The inverse tangent function is denoted by $\tan^{-1}x$ and is read as tan inverse x or arc tan x. In this case we restrict the angle to the open interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

The inverse tangent function is defined as the angle in the open interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x.

More Related Graphs from Higher

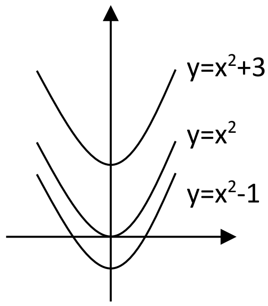
a) The graph of kf(x)

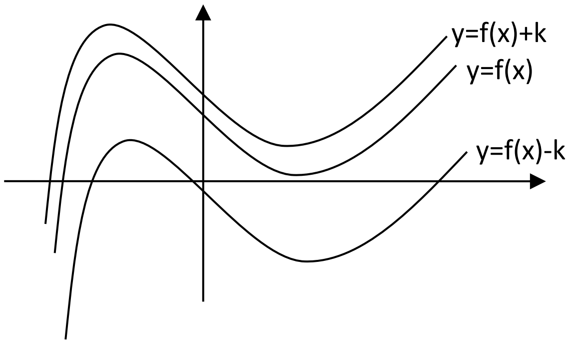




The graph of f(x) is stretched vertically if k>1 and compressed vertically if 0<k<1.

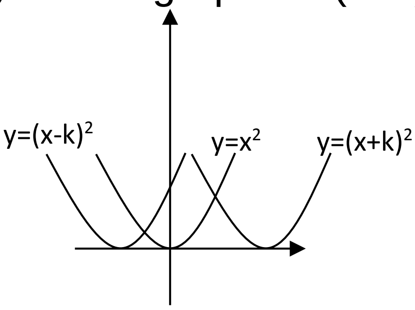
b) The graph of $f(x)+k$

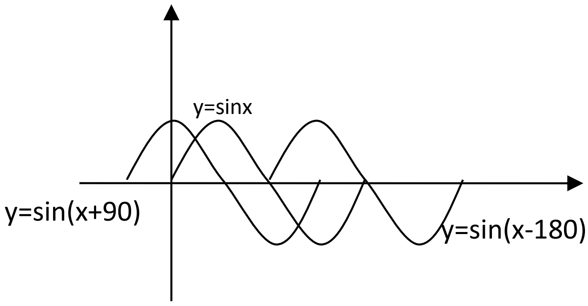




The graph of $f(x)$ is moved vertically upwards if $k>0$ and downwards if $k<0$.

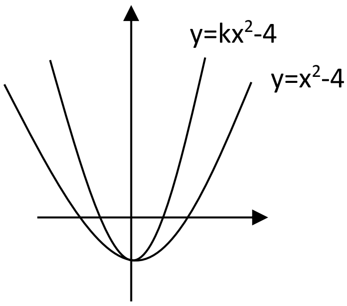
c) The graph of $f(x+k)$

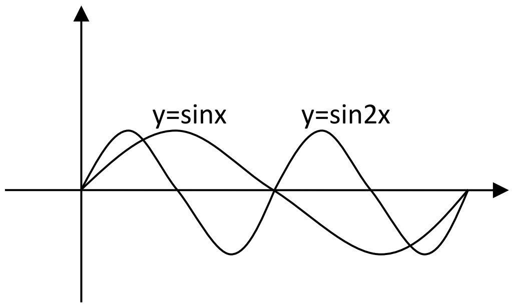




The graph of $f(x)$ is moved horizontally left if $k>0$ and right if $k<0$.

d) The graph of $f(kx)$



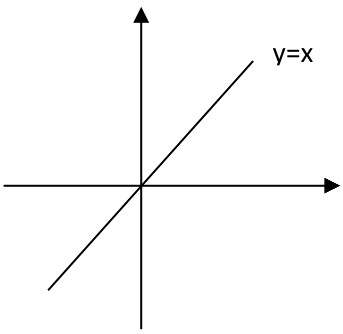


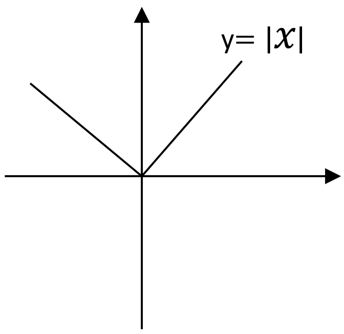
The graph of $f(x)$ is compressed if $k>1$ and stretched if $k<1$.

New Related Graphs

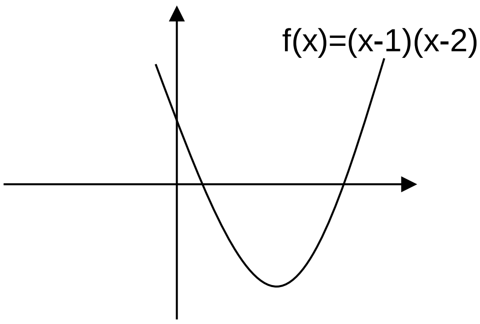
a) The graph of $|f(x)|$ - the modulus function

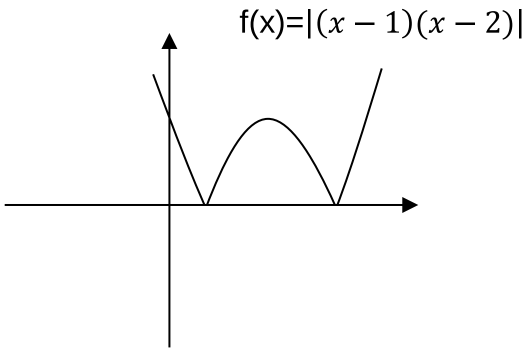
i) When f(x)=x, f(x) is negative when x is negative.
But if f(x)=$|x|$, f(x) takes the positive numerical value of x.
e.g. when x=-3, $|x|$=3 so f(x) is always positive.
The graph of f(x)=$|x|$ can be obtained from the graph of f(x)=x by simply flipping any parts of the graph of f(x), which appear below the x-axis, over the x-axis.





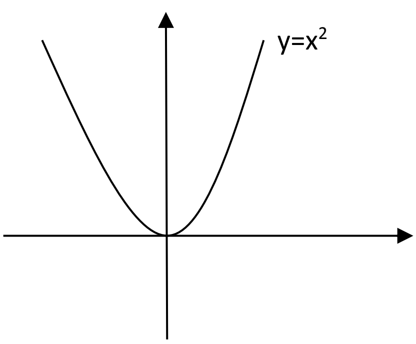
ii) To sketch f(x)=$|(x-1)(x-2)|$, we start by sketching f(x)=(x-1)(x-2). We then reflect in the x-axis the part of the curve that is below the x-axis.

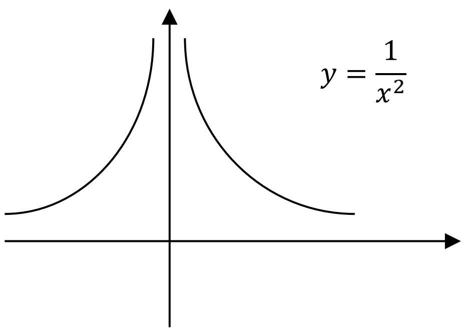




b) Even Functions

An even function is any function whose curve has the y-axis as a line of symmetry. Curves, having only even powers of x, are symmetrical on the y-axis.





An alternative method of defining an even function is the show that f(-x)=f(x) for all values of x.

Examples: Show that these functions are even

i) $f(x)=x^2$

$f(-x)=(-x)^2$

$f(-x)=x^2$

$f(-x)=f(x)$

ii) $f(x)=\frac{1}{x^2}$

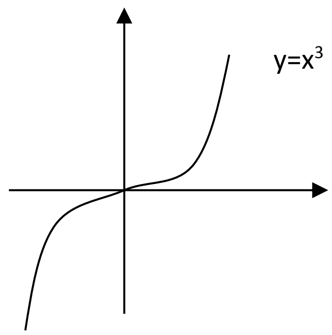
$f(-x)=\frac{1}{(-x)^2}$

$f(-x)=\frac{1}{x^2}$

$f(-x)=f(x)$

c) Odd Functions

An odd function is any function whose curve has 180^0 rotational symmetry about the origin. Curves having only odd powers of x have 180^0 rotational symmetry about the origin.



An alternative method of defining an odd function is to show that f(-x)=-f(x), for all values of x.

Examples: Show that this function is odd

$f(x)=x^3$

$f(-x)=(-x)^3$

$f(-x)=-x^3$

$f(-x)=-f(x)$

Exercise 5:

1 Write down the equations of the inverses of the following functions:

a) $f(x)=2x$ *b)* $f(x)=2-x$ *c)* $f(x)=\frac{2}{x}$

d) $f(x)=2^x$ *e)* $f(x)=1-2x$ *f)* $f(x)=ln(x-2)$

2 Evaluate:

a) $sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ b) $tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ c) $tan^{-1}(1)$

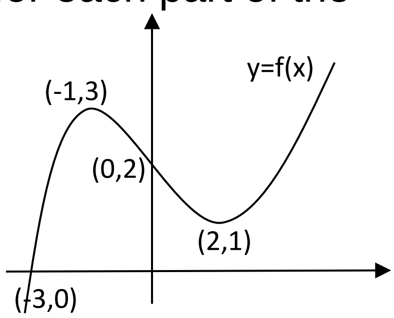
d) $sin^{-1}\left(\frac{1}{2}\right)$ e) $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ f) $tan^{-1}\left(\sqrt{3}\right)$

3 Sketch the following graph of the graph y=f(x) for each part of the following question.

On separate graphs, sketch the graphs of:

a) $f(x-3)$ *b)* $f(x+3)$ *c)* $f(x)+2$

d) $2f(x)$ *e)* $-f(x)$ *f)* $f(-x)$



4 Sketch the graphs of y=f(x) and y=$|f(x)|$:

a) $f(x)=x+2$ *b)* $f(x)=5-2x$ *c)* $f(x)=x^2-2x-3$

d) $f(x)=3x-x^2$ *e)* $f(x)=x^3+1$ *f)* $f(x)=\frac{1}{x}-2$

5 Which of the following functions are odd, even or neither?

a) $f(x)=(x+4)(x-2)$ *b)* $f(x)=3x^2+5$ *c)* $f(x)=2x-x^3$

d) $f(x)=\sin 2x$ *e)* $f(x)=x+\dfrac{1}{x}$ *f)* $f(x)=x-\dfrac{1}{x}$

g) $f(x)=\sin x\cos x$ *h)* $f(x)=e^{x^2}$ *i)* $f(x)=e^x-e^{-x}$

j) $f(x)=e^x+e^{-x}$ *k)* $f(x)=\ln x$ *l)* $f(x)=\sin x+\cos x$

N7 — Motion and Optimisation — AAC 1.5

Motion in a Straight Line

Take the x-axis to be the straight line along which the motion takes place. The displacement is defined as the distance from the origin in time *t* and is denoted as *x(t).*

Velocity is defined as the rate of change of displacement with respect to time is denoted by *v(t).*

$v(t) = \frac{d}{dt}\left(x(t)\right)$ or simply $v = \frac{dx}{dt}$

Acceleration is defined as the rate of change of velocity with respect to time and is denoted by *a(t).*

$a(t) = \frac{d}{dx}(v(t))$ or simply $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

If this has a positive value, it is called acceleration, a negative value is called deceleration.

Another notation used is

x = displacement $\dot{x}$ = velocity $\ddot{x}$ = acceleration

Examples

1 A car is travelling along a straight road. The distance, x metres, travelled in t seconds is $x = 10t - 5t^2$.

Find the velocity when t = 0.5 secs.

$x = 10t - 5t^2$

$v = 10 - 10t$

at t = 0.5, v = 10 – 5 = 5m/s

2 A car it travelling along a straight line. Its velocity, v metres per second, in t seconds, is $v=10 + 6t^2 - t^3$.

Find the acceleration when t = 3 secs.

$v=10 + 6t^2 - t^3$

$a = 12t - 3t^2$

at t = 3, a = 36 – 27 = 9m/s^2

3 A car is travelling along a straight road. Its distance, x metres, travelled in t seconds, is $x = 5 + 2t + t^3$.

Find the velocity and acceleration at t = 3.

$x = 5 + 2t + t^3$

$v = 2 + 3t^2$

at t = 3, v = 2 + 27 = 29m/s

a = 6t

at t = 3, v = 18m/s^2

Exercise 1

1 A body moves in a straight line and the motion is such that x, the number of metres from a fixed point after t seconds, is given by

$$x = 3 - 4t + t^2$$

a) How far is the body from the fixed point at the start?

b) What is the position after 4 seconds?

c) What is its velocity after 3 seconds?

d) What is its initial acceleration?

2 If $x = 4t^3 - 3t^2 - 2t - 1$, where x is in metres and t is in seconds, find

a) The velocity at the end of the 3rd and 4th seconds.

b) The acceleration at the end of the 3rd and 4th seconds.

c) The average velocity during the 4th second.

d) The acceleration during the 4th second.

3 A motor bike starts from rest and its displacement x metres after t seconds is given by $x = \frac{1}{6}t^3 + \frac{1}{4}t^2$.

Calculate the initial acceleration and the acceleration at the end to the 2nd second.

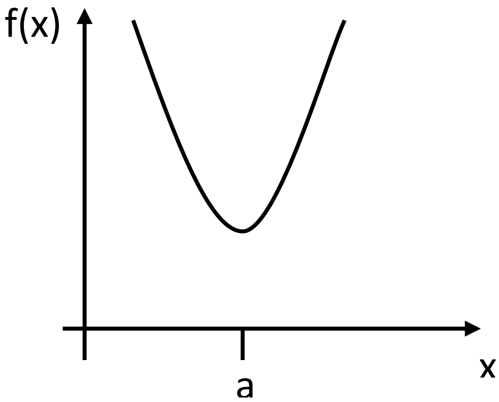
4 A body moves along a straight line so that after t seconds its displacement from a f ixed point O on the line is x metres.

If $x = 3t^2(3-t)$, find

a) the initial velocity and acceleration.

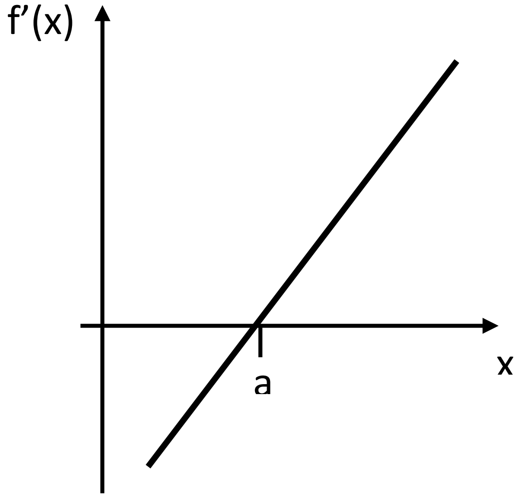
b) the velocity and acceleration after 3 seconds.

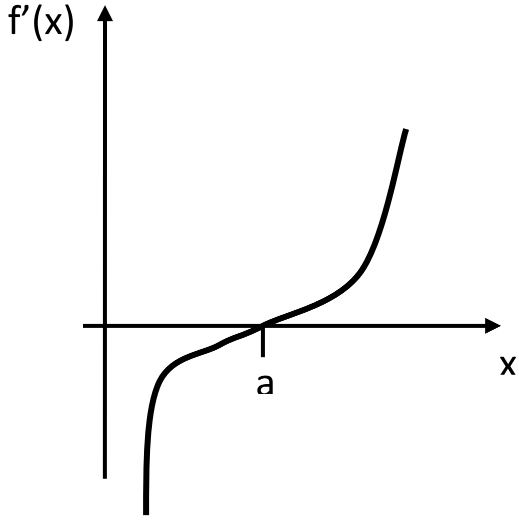
Stationary Points



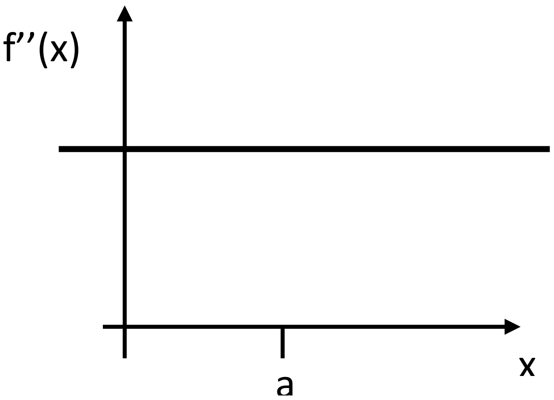
Consider a function f(x) with a minimum turning point at x=a. When we draw tangents to the left of a they slope down so f'(x) is negative. At f'(a)=0. To the right of a the tangents slope upwards so f'(x) is positive

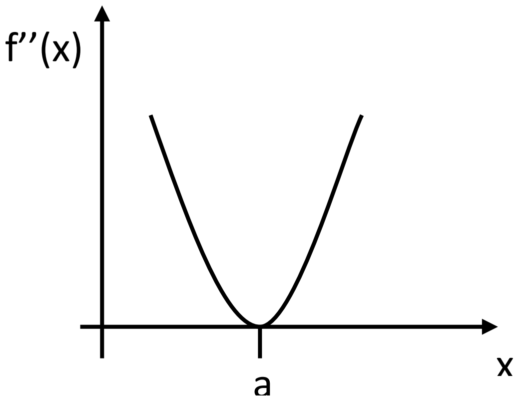
Possible f'(x) graphs are shown below.





Now think of drawing tangents to f'(x) graphs to obtain the graphs of f''(x).





To check that both cases are possible consider $f(x) = x^2 + 3$ and $f(x) = (x-2)^2 + 3$.

Drawing similar graphs around a maximum turning point or point of inflexion will show that f''(x) = 0 is possible.

However we do seem to be able to conclude the following:

If $f''(a)>0$, then $f(x)$ has a minimum turning point at $x=a$.

If $f''(a)<0$, then $f(x)$ has a maximum turning point at $x=a$.

If $f''(a)=0$, we have no information on the nature of the stationary point and must use a tale of sign as before.

The above applies only when $f'(a)=0$. So use the second derivative test when the second derivative is easy to find, if not then use a table of sign.

Examples

1 Sketch the graph of the function $f(x) = (x+2)(x-1)^2$.

$f'(x) = (x-1)^2 + 2(x+2)(x-1)$

For stationary points set $f'(x)=0$.

$$(x-1)^2 + 2(x+2)(x-1) = 0$$
$$(x-1)[(x-1)+2x+4] = 0$$
$$(x-1)(3x+3) = 0$$
$$3(x-1)(x+1) = 0$$
$$x=1 \text{ of } x=-1$$

Stationary points at (-1,4) and (1,0).

$f''(x) = 6x$

$f''(-1)$ is negative, therefore (-1,4) is a maximum turning point.

$f''(x)$ is positive, therefore (1,0) is a minimum turning point.

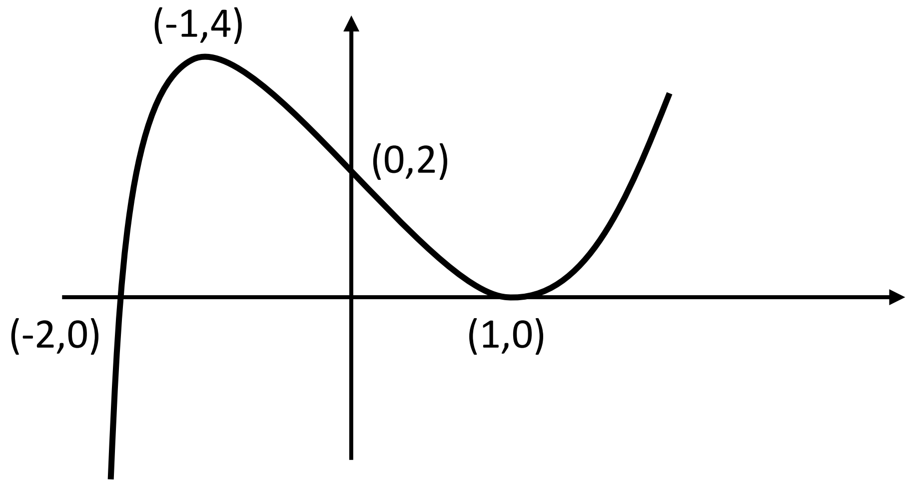
When $f(x) = 0$, $(x+2)(x-1)^2 = 0$

$$x=-2 \text{ or } x=1$$

The curve crosses the x-axis at (-2,0) and (1,0).

When $x = 0$, $f(0) = 2$.

The curve crosses the y-axis ay (0,2).



2 Find the co-ordinates and nature of the stationary points on the curve f(x) = $x^3 - 81\ln x$.

$f'(x) = 3x^2 - \frac{81}{x}$

For stationary points, set f'(x) = 0.

$3x^2 - \frac{81}{x} = 0$

$3x^3 - 81 = 0$

$x^3 = 27$

$x = 3,\ y = 27 - 81lnx$

$f''(x) = 6x + \frac{81}{x^2}$

f''(3) is positive, therefore (3,27-81lnx) is a minimum t.p.

3 Find the co-ordinates and nature of the stationary point on the curve.

f(x) = $e^x - 4x$

f'(x) = $e^x - 4$

For stationary points, set f'(x)=0.

$e^x - 4 = 0$

$e^x = 4$

x = ln4, y = 4 – 4ln4

f''(x) = e^x

f(ln4) is positive, therefore (ln4,4-4ln4) is a minimum t.p.

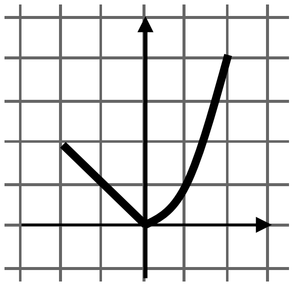
Exercise 2: Use the second derivative to find the stationary values and their nature for the following functions.

a) y = x – lnx b) y = xlnx c) y = xe^{-x} d) y = sinϴ + $\frac{1}{2}$sin2ϴ

Differentiable Functions

Not all functions are differentiable everywhere.
A maximum or minimum can occur at a point where f'(x) is not defined.
Consider the piecewise function detailed below and its graph

$$f(x) = \begin{cases} -x \quad when -2 \leq x \leq 0 \\ \\ x^2 \; when \; 0 \leq x \leq 2 \end{cases}$$



Consider the tangent at (0,0):

	To the Left	To the Right
	f(x) = -x	f(x) = x^2
	f'(x) = -1	f'(x) = 2x
	f'(0) = -1	f'(0) = 0

The left derivative is -1, the right derivative is 0. f'(x) does not exist at (0,0). From the graph, the minimum value of f(x) is 0, the maximum value occurs and an end point and is 4.

Exercise 3

1 $\quad f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

By sketching their graphs, show that f(x) and f'(x) are continuous but f''(x) is not.

2 $\quad f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$

Find the maximum value of f(x).

3 $\quad f(x) = \begin{cases} 2x^2 - 2x - 2, & 0 \leq x < 2 \\ 4 - x, & 2 \leq x < 4 \\ (x-4)(x-8), & 4 \leq x \leq 7 \end{cases}$

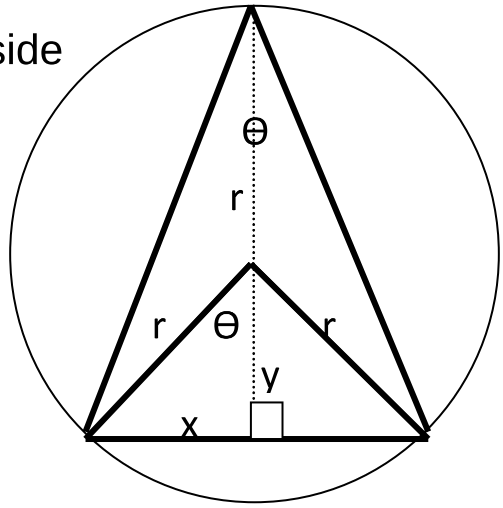
State the maximum and minimum values of f(x) and the coordinates of any point where f'(x) does not exist.

Optimisation

Example: An isosceles triangle is inscribed inside a circle with radius r. Show that the area of the triangle is

$$A=r^2\sin\Theta(1+\cos\Theta)$$

where Θ is the angle between the equal sides.
Find the maximum possible area of the triangle.



From $\sin\Theta = \frac{x}{r}$ $\Rightarrow$ $x=r\sin\Theta$ and from $\cos\Theta = \frac{y}{r}$ $\Rightarrow$ $y = r\cos\Theta$

The base of the triangle is $2r\sin\Theta$ and the height is $r+r\cos\Theta$

$$\text{Area} = \frac{1}{2}\times 2r\sin\Theta(r+r\cos\Theta)$$

$$A=r^2\sin\Theta(1+\cos\Theta)$$

$$A(\Theta)= r^2\sin\Theta + r^2\sin\Theta\cos\Theta$$

$$A(\Theta)= r^2\sin\Theta + \frac{1}{2}r^2\sin2\Theta$$

$A'(\Theta)= r^2\cos\Theta + r^2\cos2\Theta$ $\quad$ r^2 is a constant

For stationary points, set $A'(\Theta)=0$

$$r^2\cos\Theta + r^2\cos2\Theta=0$$

$$r^2(2\cos^2\Theta+\cos\Theta-1)=0$$

$$r^2(2\cos\Theta-1)(\cos\Theta+1)=0$$

$\cos\Theta = \frac{1}{2}$ or $\cos\Theta = -1$

$\Theta = \frac{\pi}{6}$ $\quad$ $\cancel{\Theta = \pi}$ n/a

Stationary value at $\Theta = \frac{\pi}{6}$

Remember your exact values.
All angles in Calculus questions are measured in radians.

$$A''(\Theta) = -r^2\sin\Theta - 2r^2\sin2\Theta$$

$A''(\frac{\pi}{6})$ is negative, there is a maximum stationary value at $\Theta = \frac{\pi}{6}$

The maximum area is : $A = r^2\sin\frac{\pi}{6}(1+\cos\frac{\pi}{6})$

$$A = r^2\frac{\sqrt{3}}{2}(1+\frac{1}{2})$$

$$A = \frac{3\sqrt{3}}{4}r^2$$

Exercise 4

1 Four squares each of side of s cm are cut from the corners of a metal square of side 16 cm. The metal is then bent to make an open topped tray of volume V cm^3.

a) Prove that V = $4s^3 - 64s^2 + 256s$

b) Find the value of s which makes the volume a maximum.

2 A sector of a circle with radius r cm has an area of 16cm^3.

a) Show that the perimeter P cm of the sector is given by

$$P(r) = 2\left(r + \frac{16}{r}\right)$$

b) Find the minimum value of P.

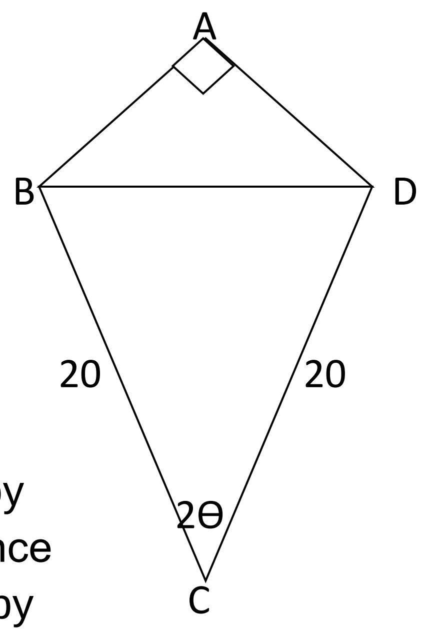
3 A cylindrical tank has a radius or r metres and a height of h metres. The sum of the radius and the height is 2 metres.

a) Prove that the volume, in m^3, is given by : V= $\pi r^2(2\text{-}r)$

b) Find the maximum volume.

4 ABCD is a kite which has AC as its axis.

Angle BAD is right angled and BC and DC are 20cm.



a) Show that the area of triangle BCD is given by the expression 200sin2Ө and find and expression for BD^2.

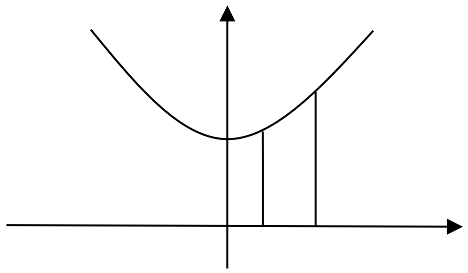
b) Use the expression for BD^2 to show that the area of triangle BAD is given by the expression 200-200cos2Ө and hence show that the area of the kite is given by the expression

A(Ө) = 200(1 – cos2Ө +sin2Ө)

c) Find the value of Ө which makes the area a maximum and find this maximum area.

The Volume of Revolution

The volume of revolution is formed when the area bounded by a curve y=f(x), x=a, x=b and the x-axis is rotated completely around the x-axis.



$$Volume = \int_a^b \pi y^2 dx$$

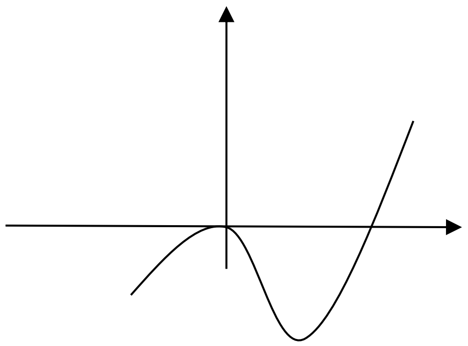
Example:
Find the volume generated, by rotating about the x-axis, the area enclosed by the curve y=x^3-2x^2 and the x-axis.

The curve crosses the x-axis where

x^3-2x^2=0

x^2(x-2)=0

x=0 and x=2



$$Volume = \int_0^2 \pi(x^3 - 2x^2)^2 dx$$

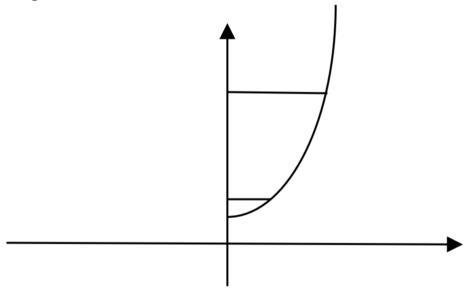
$$= \pi \int_0^2 (x^6 - 4x^5 + 4x^4) \quad dx$$

$$= \pi \left[\frac{x^7}{7} - \frac{2x^6}{3} + \frac{4x^5}{5}\right]_0^2$$

$$= \pi \left(\frac{2^7}{7} - \frac{2\times2^6}{3} + \frac{4\times2^5}{5}\right) - 0$$

$$= \frac{128}{105}\pi\ units^3$$

The volume of revolution can also be formed when the area bounded by a curve y=f(x) (x=f(y)), the y-axis, y=a and y=a is rotated completely about the y-axis.

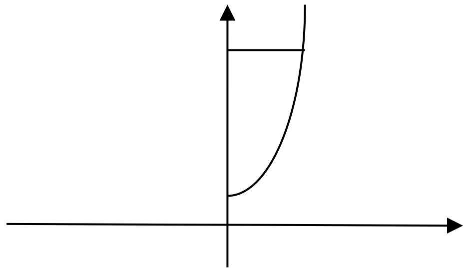


$$Volume = \int_a^b \pi x^2 dy$$

Example:

Find the volume generated, by rotating about the y-axis, the area enclosed by the curve $y=x^2+1$, $x>0$, the x-axis and the line $y=2$.

From $y=x^2+1$, we get $x^2=y-1$



$$Volume = \int_1^2 \pi x^2 dy$$

$$= \pi \int_1^2 (y-1)dy$$

$$= \pi \left[\frac{y^2}{2} - y\right]_1^2$$

$$= \pi \left\{\left(\frac{2^2}{2} - 2\right) - \left(\frac{1^2}{2} - 1\right)\right\}$$

$$= \frac{\pi}{2}\ units^3$$

Exercise 5

1 Find the volume of solids of revolution formed when the regions bounded by the following curves and the x-axis are rotated through one revolution about the x-axis.

a) $y = \frac{4}{x}, x = 1\ and\ x = 4$

b) $y = \sqrt{x}, x = 0\ and\ x = 4$

c) $x + 2y = 2, x = 0\ and\ x = 2$

d) $y = \frac{1}{x^2}, x = \frac{1}{3}\ and\ x = \frac{1}{2}$

e) $y = x(x-1)$

f) $y = \sqrt{9 - x^2}$

g) $y^2 = 8x, x = 0\ and\ x = 4$

h) $y = sinx, x = 0\ and\ x = 2\pi$

2 Find the volume of solids of revolution formed when the regions in the first quadrant bounded by the following curves and the y-axis are rotated through one revolution about the y-axis.

a) $x = \sqrt{y}\ and\ y = 4$

b) $x = y^2\ and\ y = 1$

c) $xy = 1, y = 3\ and\ y = 6$

d) $y = 4 - x^2, y = -4\ and\ y = 4$

e) $xy^2 = 2, y = 2\ and\ y = 4$

f) $x = y^2 + 1, y = -1\ and\ y = 1$

g) $y = lnx, y = 2\ and\ y = 5$

Summation and Mathematical Proof

Sigma ($\sum$) Notation

The sigma notation is used to write down a series. The series of square terms from 1^2 to 6^2 can be written as

$$\sum_{k=1}^{6} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

In general

$$\boxed{\sum_{k=1}^{n} f(k) = f(1) + f(2) + f(3) + f(4) + \cdots + f(n)}$$

Examples:

1 Write the following series in full

a)

$$\sum_{k=5}^{10} k(k+1)$$

$$\sum_{k=5}^{10} k(k+1) = 5(5+1) + 6(6+1) + 7(7+1) + 8(8+1) + 9(9+1) + 10(10+1)$$

$$= 30 + 42 + 56 + 72 + 90 + 110$$

b)

$$\sum_{k=1}^{4} (2k^2 - 1)$$

$$\sum_{k=1}^{4} (2k^2 - 1) = 2 \times 1^2 - 1 + 2 \times 2^2 - 1 + 2 \times 3^2 - 1 + 2 \times 4^2 - 1$$

$$= 1 + 7 + 17 + 31$$

2 Express $1+4+7+10+13+\ldots+298$ in $\sum$ notation.

$f(k)$ is the expression for the k^{th} term of the series.

This is an arithmetic series with $a=1$ and $d=3$.

The k^{th} $term = a+(k-1)d$

$$= 1+(k-1)3$$

$$= 3k-2$$

The final term is 298 and so $3k-2=298$

$$k=100$$

Therefore

$$1+4+7+10+13+\ldots+298 = \sum_{k=1}^{100}(3k-2)$$

Exercise 1

1 Write each of the following series in full

a) $$\sum_{k=1}^{5} k^2$$

b) $$\sum_{k=1}^{9}(2k-1)$$

c) $$\sum_{k=1}^{10}\frac{2520}{k}$$

2 Express each of the following series in the form

$$\sum_{k=1}^{n} f(k)$$

a) $1+2+3+4+\cdots+50$

b) $5+10+15+\cdots+30$

c) $3+5+7+\ldots+13$

d) $3+7+11+\ldots+199$

Special Summation

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Summation of a Series

The above rules can be used to evaluate other series. For example

$$\sum_{k=1}^{n}(ak+b) = a\sum_{k=1}^{n}k + \sum_{k=1}^{n}b = \frac{an(n+1)}{2} + bn$$

Similar expansions can be used for $f(k)$ containing k^2 and k^3.

Examples: Evaluate

a)

$$\sum_{k=1}^{10}(k+2)$$

$$\sum_{k=1}^{10}(k+2) = \sum_{k=1}^{10}k + \sum_{k=1}^{10}2$$

$$= \frac{n(n+1)}{2} + 2n$$

$$= \frac{10(10+1)}{2} + 2\times 10$$

$$= 75$$

b)

$$\sum_{k=1}^{20}(4k+5)$$

$$\sum_{k=1}^{20}(4k+5) = 4\sum_{k=1}^{20}k + \sum_{k=1}^{20}5$$

$$= \frac{4n(n+1)}{2} + 5n$$

$$= \frac{4\times 20(20+1)}{2} + 5\times 20$$

$$= 940$$

Exercise 2: Evaluate

a)

$$\sum_{k=1}^{10}k$$

b)

$$\sum_{k=1}^{20}2k$$

c)

$$\sum_{k=1}^{8}(2k+3)$$

d)

$$\sum_{k=1}^{20}(5k+2)$$

e)

$$\sum_{k=1}^{10}(3k-1)$$

Proof by Induction

The process is to firstly show that the conjecture is true for the first value. Then we assume the conjecture is true for a set value of n i.e. $n = k$. The next step is to use the previous step to prove the conjecture is true for $n = k + 1$.

If the conjecture is true for $n = k$ then it is true for $n = k + 1$. Since we have shown that it is true for the first value of n say $n = 1$, then it will be true for $n = 2$. If it is true for $n = 2$, then it will be true for $n = 3$. If it is true for $n = 3$, then it will be true for $n = 3$ and so on. Therefore by induction the statement is true for all n.

Example: Use proof by induction to prove the following statements

a) Prove that $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n + 1)$

Prove true for $n = 1$: $LHS = 1$ $RHS = \frac{1}{2} \times 1 \times (1 + 1) = 1$

So the statement is true for $n = 1$.

Assume true for $n = k$: $1 + 2 + 3 + 4 + \cdots + k = \frac{1}{2}k(k + 1)$

Prove true for $n = k + 1$: $1 + 2 + 3 + 4 + \cdots + k + (k + 1)$

$$= \frac{1}{2}k(k + 1) + (k + 1)$$
$$= \frac{1}{2}k(k + 1) + \frac{2}{2}(k + 1)$$
$$= \frac{1}{2}(k + 1)(k + 2)$$
$$= \frac{1}{2}(k + 1)\big((k + 1) + 1\big)$$

For $n = k + 1$:

$$\frac{1}{2}n(n + 1)$$
$$= \frac{1}{2}(k + 1)((k + 1) + 1)$$

If true for $n = k$ then true for $n = k + 1$. Since we have shown that it is true for $n = 1$, then true for $n = 2$. Since it is true for $n = 2$, then true for $n = 3$ and so on. Therefore by induction the statement is true for all $n \in \mathbb{N}$.

b) Prove that $3+8+13+18+\cdots+(5n-2)=\frac{1}{2}n(5n+1)$

Prove true when $n=1$: $LHS=1$ $RHS=\frac{1}{2}\times 1\times(1+1)=1$

So the statement is true for $n=1$.

Assume true for $n=k$: $3+8+13+18+\cdots+(5k-2)=\frac{1}{2}k(5k+1)$

Prove true for $n=k+1$: $3+8+13+18+\cdots+(5k-2)+(5(k+1)-2)$

$$=\frac{1}{2}k(5k+1)+\frac{2}{2}(5k+3)$$
$$=\frac{1}{2}[k(5k+1)+2(5k+3)]$$
$$=\frac{1}{2}[5k^2+k+10k+6]$$
$$=\frac{1}{2}[5k^2+11k+6]$$
$$=\frac{1}{2}(k+1)(5k+6)$$
$$=\frac{1}{2}(k+1)(5(k+1)+1))$$

If true for $n=k$ then true for $n=k+1$. Since we have shown that it is true for $n=1$, then true for $n=2$. Since it is true for $n=2$, then true for $n=3$ and so on. Therefore by induction the statement is true for all $n\in\mathbb{N}$.

c) Prove that 8^n is a factor of $(4n)!$ for all $n\in\mathbb{N}$.

Prove true when $n=1$: $8^1=8$ $(4)!=24$

8 is a factor of 24.

So the statement is true for $n=1$.

Assume true for $n=k$: 8^k is a factor of $(4k)!$

Prove true for $n=k+1$: $(4(k+1))!$

$$=(4k+4)!$$
$$=(4k+4)(4k+3)(4k+2)(4k+1)(4k)!$$
$$=(4k+4)(4k+3)(4k+2)(4k+1)(4k)!$$
$$=4(k+1)(4k+3)2(2k+1)(4k+1)(4k)!$$
$$=8(4k)!\,(k+1)(4k+3)(2k+1)(4k+1)$$

$8(4k)!$ is a factor of $(4(k+1))!$

Now 8^k is a factor of $(4k)!$

So 8×8^k is a factor of $8(4k)!$

Therefore $8\times 8^k=8^{k+1}$ is a factor of $(4k+4)!$

If true for $n = k$ then true for $n = k + 1$. Since we have shown that it is true for $n = 1$, then true for $n = 2$. Since it is true for $n = 2$, then true for $n = 3$ and so on. Therefore by induction the statement is true for all $n \in \mathbb{N}$.

d) Prove that $2^n > n^2$ for all $n > 4$, $n \in \mathbb{N}$.

Prove true when $n = 5$: $2^5 = 32$ $\quad 5^2 = 25$

$32 > 25$

So the statement is true for $n = 5$.

Assume true for $n = k$: $2^k > k^2$

Prove true for $n = k + 1$: $2^k > k^2$

$$2 \times 2^k > 2k^2$$

$$2^{k+1} > k^2 + k^2$$

$$2^{k+1} > k^2 + k \times k$$

$$2^{k+1} > k^2 + 4k \quad \text{since } k > 4$$

$$2^{k+1} > k^2 + 2k + 2k$$

$$2^{k+1} > k^2 + 2k + 1 \quad \text{since } k > 4$$

$$2^{k+1} > (k + 1)^2$$

If true for $n = k$ then true for $n = k + 1$. Since we have shown that it is true for $n = 5$, then true for $n = 6$. Since it is true for $n = 7$, then true for $n = 8$ and so on. Therefore by induction the statement is true for all $n \in \mathbb{N}$.

e) Prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$

Prove true for $n = 1$

$LHS = \sum_{r=1}^{1} r^2 = 1^2 = 1$ $\quad RHS = \frac{1}{6}(1)(1 + 1)(2 \times 1 + 1) = 1$

True for $n = 1$

Assume true for $n = k,\ k \geq 1$

$$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k + 1)(2k + 1)$$

Prove true for $n = k + 1$

$$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k + 1)^2$$

$$= \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2$$

$$= \frac{1}{6}k(k+1)(2k+1) + \frac{6}{6}(k+1)^2$$
$$= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]$$
$$= \frac{1}{6}(k+1)[2k^2 + 7k + 6]$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$
$$= \frac{1}{6}(k+1)\big((k+1)+1\big)(2(k+1)+1)$$

Hence if true for k, it is true for $n = k+1$.
True for $n = 1 \Rightarrow$ True for $n = 2$ since $k \geq 1$.
True for $n = 2 \Rightarrow$ True for $n = 3$ and so on for all n.
Hence true for all n, by induction.

f) Prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Prove true for $n = 1$

$LHS = \sum_{r=1}^{1} r^3 = 1^3 = 1$ $\quad$ $RHS = \frac{1}{4} \times 1^2(1+1)^2 = 1$

True for $n = 1$

Assume true for $n = k,\ k \geq 1$

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$$

Prove true for $n = k+1$

$$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^3 + (k+1)^3$$
$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$
$$= \frac{1}{4}k^2(k+1)^2 + \frac{4}{4}(k+1)^3$$
$$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)]$$
$$= \frac{1}{4}(k+1)^2[k^2 + 4k + 4]$$
$$= \frac{1}{4}(k+1)^2(k+2)^2$$
$$= \frac{1}{4}(k+1)^2\big((k+1)+1\big)^2$$

Hence if true for k, it is true for $n = k+1$.
True for $n = 1 \Rightarrow$ True for $n = 2$ since $k \geq 1$.
True for $n = 2 \Rightarrow$ True for $n = 3$ and so on for all n.
Hence true for all n, by induction.

g) For all integers $n \geq 10,\ n^2 \geq 10n$.

Prove true for $n = 10$

$n^2 = 100$ $\qquad 10n = 100$

$n^2 = 10n$ so true for $n = 10$

Assume true for $n = k$

$k^2 \geq 10k$

Prove true for $n = k + 1$

$$
\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&\geq 10k + 2k + 1 \\
&\geq 10k + 2k + 10 - 9 \\
&\geq 10(k+1) + 2k - 9 \\
&\geq 10(k+1) + 2k - 9 \ \text{ since } 2k - 9 \geq 11
\end{aligned}
$$

$(k+1)^2 \geq 10(k+1)$

Hence, it true for $n = k$, true for $n = k + 1$.

True for $n = 10 \Rightarrow$ True for $n = 11$ since $k \geq 10$.

True for $n = 11 \Rightarrow$ True for $n = 12$ and so on for all n.

Hence true for all n, by induction.

h) Show that $4^n + 6n - 1$ is divisible by 9 for all $n \geq 1$.

Prove true for

$4^1 + 6 \times 1 - 1 = 9$

9 is divisible by 9. True for $n = 1$

Assume true for $n = k$

$4^k + 6k - 1 = 9t$

Prove true for $n = k + 1$

$$
\begin{aligned}
&4^{k+1} + 6(k+1) - 1 \\
&= 4 \times 4^k + 6k + 5 \\
&= (4 \times 4^k + 24k - 4) - 18k + 9 \\
&= 4(4^k + 6k - 1) - 18k + 9 \\
&= 4(9p) - 9(2k+1) \\
&= 9(4p - 2k - 1)
\end{aligned}
$$

$4^{k+1} + 6(k+1) - 1$ is divisible by 9

Hence, it true for $n = k$, true for $n = k + 1$.

True for $n = 1 \Rightarrow$ True for $n = 2$ since $k \geq 1$.

True for $n = 2 \Rightarrow$ True for $n = 3$ and so on for all n.

Hence true for all n, by induction.

i) Prove that S_n of the series $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots = \frac{n}{n+1}$

Prove true for $n = 1$

$LHS = \frac{1}{1\times2} = \frac{1}{2}$ $\qquad$ $RHS = \frac{1}{1+1} = \frac{1}{2}$

True for $n = 1$

Assume true for $n = k$

$$\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{k\times(k+1)} = \frac{k}{k+1}$$

Prove true for $n = k + 1$

$$\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{k\times(k+1)} + \frac{1}{(k+1)\times(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)\times(k+2)}$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{k+1}{(k+1)+1}$$

Hence, it true for $n = k$, true for $n = k + 1$.

True for $n = 1 \Rightarrow$ True for $n = 2$ since $k \geq 1$.

True for $n = 2 \Rightarrow$ True for $n = 3$ and so on for all n.

Hence true for all n, by induction.

Exercise 3: Prove the following statements by induction, $n \in \mathbb{N}$.

a) $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$

b) $5 + 7 + 9 + \cdots + (2n - 1) = (n - 2)(n + 2),\ n \geq 3$.

c) $2^n > n$

d) $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^{n+1} - 1$

e) $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

f) $3^{2n} - 1$ is a multiple of 8

g) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$

h) $3^n > n^3$ for all $n \geq 4$

i) De Moivre's Theorem

j) Binomial Theorem

k) $\sum_{r=1}^{n} r(r + 1) = \frac{1}{3}n(n + 1)(n + 2)$

l) $\sum_{r=1}^{n} r(r + 1)(r + 2) = \frac{1}{4}n(n + 1)(n + 2)(n + 3)$

m) S_n of the series $\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \cdots = \frac{n}{2n+1}$

n) S_n of the series $\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \cdots = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$

o) $n^3 + 3n^2 - 10n$ is divisible by 3

p) $7^n + 4^n + 1^n$ is divisible by 6

q) For all integers $n > 2,\ 2^n > 2n$

r) For all integers $n \geq 4,\ 3^n > n^3$

2 Use the results of $\sum_{r=1}^{n} r,\ \sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ to prove by direct method

a) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

b) $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

Other Methods of Proof

Notation and Terminology

A statement of the form "If.....then....." is called an implication.
For example 1) If $x > 5$ then $2x > 10$
2) If $x = 5$ then $x^2 = 25$
These can be written in shorthand
1) $x > 5 \Rightarrow 2x > 10$
2) $x = 5 \Rightarrow x^2 = 25$
We note that both of these statements are true, but not all statements (or implications) will be true.
For examples $x = 4 \Rightarrow 2x = 9$ is false.

Negation of a Statement

Statement: A rhombus has four equal sides. (can be true or false)
Negation: A rhombus does not have four equal sides.

If a statement is true then the negation is false.
If a statement if false then the negation is true.

Given that p represents a statement, the negation is written $-p$ (reads "not p"), and is such that is p is true, $-p$ is false; is p is false, $-p$ is true.

Examples: Negate the following statements

a) "All cats have tails."
p: All cats have tails.
Since the given statement is taken to be true for every cat, the negation must assert that at least one cat has no tail.
$-p$: Some cats have tails.

b) "Some pilots are women."

p: Some pilots are women.
Since the given statement asserts that there is at least one pilot who is a women, the negation must assert that all pilots are not women.
$-p$: No pilots are women.

The negation of “all” is “some”.
The negation of “some” is “no”.
The negation of “no” is “some”.

Exercise 1

1 Which of the following is a negation of “All boys are adventurous”?
a) No boys are adventurous.
b) All boys are unadventurous.
c) Some boys are not adventurous.
d) No boys are adventurous.

2 Which of the following is a negation of
“No visitors may walk on the grass”?
a) All visitors may walk on the grass.
b) Some visitors may not walk on the grass.
c) All visitors may not walk on the grass.
d) Some visitors may walk on the grass.

3 Write down the negation of each of the following statements:
a) For all real x, x^2 is positive.
b) Some pupils find mathematics difficult.
c) No dogs like cats.
d) There exists a positive integer x such that $x + 3 > 0$.
e) Every parallelogram has half turn symmetry.
f) No schoolboy lies.
g) A number which has zero in the units place is divisible by five.
h) All numbers of the form $2^n - 1$, (n an integer), are prime.

The Converse of a Statement

Statement: If a triangle is right-angled, then the square of the hypotenuse is equal to the sum of the squares on the other two sides.

Converse: If the square in the longest side of a triangle is equal to the sum of the squares on the other two sides, then the triangle is right angled.

If $p \Rightarrow q$ then the converse is $q \Rightarrow p$.

In the cases of a statement and its converse:

a) The statement may be true and the converse false.
b) The statement may be false and the converse true.
c) Both may be false or both may be true.

Exercise 2: State the converse of each of the following, and show by a counter example that the converse is false.

a) If a number ends in 0, it is divisible by 5.
b) All primes greater than 2 are odd numbers.
c) If a quadrilateral is a square, its diagonals intersect at right angles.
d) $x = 3 \ \Rightarrow \ x^3 = 9$.
e) If two numbers are odd then their sum is even.
f) If two integers are even, then their product is even.

Equivalent Statements

If a statement and its converse are true then the implication can be replaced by the two-way implication sign $\Leftrightarrow$.

If $p \Rightarrow q$ and $q \Rightarrow p$ are both true then we can write $p \Leftrightarrow q$.
This is sometimes read as "p (is true) if and only if q (is true)" or "p iff q".

If $p \Rightarrow q$, we say that p is a sufficient condition for q; if $q \Rightarrow p$, we say that p is a necessary condition for q.

Examples: In each of the following, say whether the first statement is

i) a necessary condition
ii) a sufficient condition
iii) both necessary and sufficient
iv) neither

for the second condition.

a) p: John plays the piano. q: John is a concert pianist.
Since $q \Rightarrow p$, the first statement is a necessary statement for the second as John must be able to play the piano to be concert pianist, but since $p \nRightarrow q$, it is not a sufficient condition.

b) p: ABCD is a rhombus. q: The diagonals of ABCD bisect each other.
Since $p \Rightarrow q$, the first statement is a sufficient condition for the second as the diagonals of a rhombus bisect each other. Since $q \nRightarrow p$, it is not a necessary condition.

Exercise 3

In each of the following, say whether the first statement is

i) a necessary condition
ii) a sufficient condition
iii) both necessary and sufficient
iv) neither

for the second condition.

a) p: There are more than 8 people in this room.
q: There are 9 people in this room.

b) p: ABCD is a parallelogram.
q: The diagonals of ABCD are perpendicular.

2 Which of the following statements are necessary or/and sufficient for the statement q: “Natural number n is divisible by 6” to be correct?

a) p: n is divisible by 3

b) p: n is divisible by 9

c) p: n is divisible by 12

d) p: n^2 is divisible by 12

e) p: $n = 384$

f) p: n is even and divisible by 3

g) : $n = m(m+1)(m+2)$, where m is some natural number

Contrapositive of a Statement

If $p \Rightarrow q$ then the contrapositive is $-q \Rightarrow -p$.
The negation of q implies the negation of p.
If $p \Rightarrow q$ is true then $-q \Rightarrow -p$ must also be true.
Similarly, if $p \Rightarrow q$ is false then $-q \Rightarrow -p$ must also be false.
This is important logical method of proving by indirect proof.

Disproving Conjectures – Using Counter Examples

A conjecture is a statement someone thinks may be true but it may be based on incomplete evidence. We can disprove some conjectures and statements by producing a single counter example.

Examples: Disprove the following conjectures by finding a counter example.

a) $n^2 + n + 41$ is prime number for all $n \in \mathbb{N}$.

Take $n = 41$: $n^2 + n + 41 = 41^2 + 41 + 41$

$= 41(41 + 1 + 1)$

$= 41 \times 43$ which is not prime.

So the statement is false.

b) For any real numbers a and b: $a^2 > b^2 \Rightarrow a > b$.

Take $a = -3$ and $b = 1$.

Then $(-3)^2 > 1^2$ because $9 > 1$ but -3 is not greater than 1.

So the statement is false.

c) For any real numbers a, b, c: $a > b \Rightarrow ac > bc$.

Take $a = 2,\ b = 1,\ c = -3$

Then $a > b$ since $2 > 1$

$ac = 2 \times (-3) = -6, \quad bc = 1 \times (-3) = -3$

so ac is not greater that bc since $-6 < -3$

So the statement is false.

d) For all real numbers a and b: $|a| + |b| \leq |a + b|$.

Take $a = 1,\ b = -2$

$|a| + |b| = |1| + |-2| = 1 + 2 = 3$

$|a + b| = |1 + (-2)| = |-1| = 1$

But 3 is not less than or equal to 1.

So the statement is false.

Exercise 4

1 Find another counter example for the examples above.

2 Find a counter example to disprove the following conjectures

a) For all real values of x: $x^2 - x \geq 0$

b) For any real numbers $a,\ b,\ c$ and d:

$a > b$ and $c > d \ \Rightarrow ac > bd$

c) For any real numbers a and b: $\frac{a}{b} > 1 \Rightarrow a = b$

Converse and Two Way Implication

Consider the converse of $> 5 \Rightarrow 2x > 10$. $2x > 10 \Rightarrow\ x > 5$. This is also true. So we can write $x > 5 \Leftrightarrow 2x > 10$. This statement can be read from left to right and from right to left.

We can read the statement as

$x > 5$ if and only if $2x > 10$ [if and only if can be written as iff]

or $x > 5$ is equivalent to $2x > 10$

Not all converses are true.
The converse of $x = 5 \Rightarrow x^2 = 25$ is
$x^2 = 25 \Rightarrow x = 5$ but x could equal -5. Therefore the statements are not equivalent.

Exercise 5

1 State in words the converse of each statement and say if it is true of false. If the converse is false, give a counter example.

a) If a number ends in zero then it is divisible by 5.

b) If n is a prime number greater than 2 then n is an odd number.

c) $x = 3 \Rightarrow x^2 = 9$

d) If a and b are odd numbers then $a + b$ is even.

e) If 3 is a root of $x^2 + x - k = 0$ then k is a multiple of 3.

2 State whether the implication can be replaced by the two way implication.

a) $a = b \Rightarrow a + c = b + c$

b) $x = y \Rightarrow -x = -y$

c) $n = -3 \Rightarrow n^2 = 9$

d) If n is odd then n^2 is odd $n \in \mathbb{Z}$

e) If a triangle ABC is right angled at A $\Rightarrow a^2 = b^2 + c^2$.

f) If $y = 1 - x^2$ then $\frac{dy}{dx} = -2x$

The Form of Numbers

What are even natural numbers? We can list them: $2, 4, 6, 8, 10 \ldots$
We can say that these are all of the form $2k$ where k is a natural number.
We can say that a number, $n,$ is even if and only if $n = 2k$ for some $k \in \mathbb{N}$.

The odd numbers are $1, 3, 5, 7, 9$... These are all 1 less than an even number, so we can say that a number, n, is odd if and only if $n = 2k - 1$ for some $k \in \mathbb{N}$. Alternatively we can write them as $n = 2k - 1$ for some $k \in \mathbb{W}$. In this case we are thinking of the odd numbers as one more than the even numbers. Therefore all whole numbers can be written as $2k$ or $2k + 1$ for some $k \in \mathbb{W}$.

> To show n is even we must show it to be written as two times a natural number.
> To show n is odd we must show it to be written as two times a natural number plus (or minus) one.

Example: $n = (2k-1)^2 + 3(2k-1)$ for some $k \in \mathbb{N}$, is n even or odd?

$$n = (2k-1)^2 + 3(2k-1)$$
$$\Rightarrow n = 4k^2 - 4k + 1 + 6k - 3$$
$$\Rightarrow n = 4k^2 + 2k - 2$$
$$\Rightarrow n = 2(2k^2 + 2k - 1)$$
$\Rightarrow n = 2t$ for some $k \in \mathbb{N}$, since $2k^2 + 2k - 1 \in \mathbb{N}$
$\Rightarrow n$ is even.

We can also express all whole numbers in terms of 3 or 4 or 5 etc.

All whole numbers take one of the forms: $3k, 3k+1, 3k+2$ for some $k \in \mathbb{W}$. Note that $3k + 3 = 3(k+1)$ and this is back to the original form.
If a number, n, is divisible by 3 we can write it in the form $n = 3k$ for some $k \in \mathbb{W}$. If a number is not divisible by 3 then it must be in the form $3k + 1$ or $3k + 2$.

All whole numbers take one of the forms: $4k, 4k+1, 4k+2, 4k+3$ for some $k \in \mathbb{W}$. If a number, n, is divisible by 4 we can write it in the form $n = 4k$ for some $k \in \mathbb{W}$. If a number is not divisible by 4 then it must be in the form $4k+1, 4k+2$ or $4k+3$.

Methods of Proof

Direct Proof

Use algebra to represent the values presented, and manipulate them to achieve the desired conclusion.

1) The sum of any two even integers is even.
2) The negative of any even integer is even.
3) The sum of any two odd numbers is even.
4) If n is odd then n^2 is even.
5) The product of two odd integers is odd.
6) If n is prime then n^2 is prime.
7) The difference of an even integer and an odd integer is odd.
8) The sum of two rational numbers is rational.
9) The product of two rational numbers is rational.
10) If n is even then $7n + 4$ is even, $n \in N$
11) If m is even and n is odd then $m+n$ is odd, $m, n \in N$
12) $n^3 - n$ is always divisible by 6.
13) $6^n + 4$ is always divisible by 10, $n \in N$
14) If n is an odd integer, then $n^2 - 1$ is divisible by 8.

Proof by Contradiction

To prove a theorem by direct means is sometimes difficult and sometimes impossible. In such cases we can proof by contradiction where we begin by assuming the opposite of the implication is true.

Examples: Use proof by contradiction to show these statements are true.

a) Let n be a natural number. If $7n$ is even then n is even.

Proof Assume $7n$ is even and n is odd.

If n is odd then $n = 2k - 1$ for some $k \in \mathbb{W}$.

$$
\begin{aligned}
7n &= 7(2k-1) \\
&= 14k - 7 \\
&= 14k - 6 - 1 \\
&= 2(7k-3) - 1 \\
&= 2t - 1 \quad \text{for some } t \in \mathbb{W}.
\end{aligned}
$$

$\Rightarrow 7n$ is odd

This is a contradiction, since we assumed that $7n$ was even.

Hence result.

b) Let m be a natural number. If m^2 is even then m is even.

Proof Assume m^2 is even and m is odd.

If m is odd then $m = 2k - 1$ for some $k \in \mathbb{W}$.

$$
\begin{aligned}
m^2 &= (2k-1)^2 \\
&= 4k^2 - 4k + 1 \\
&= 2(2k^2 - 2k) + 1 \\
&= 2t + 1 \quad \text{for some } t \in \mathbb{W}.
\end{aligned}
$$

$\Rightarrow m^2$ is odd

This is a contradiction, since we assumed that m^2 was even.

Hence result.

c) Let n be a natural number. If $9n$ is odd $\Rightarrow n$ is odd.

Proof Assume $9n$ is odd and n is even.

If n is even then $n = 2k$ for some $k \in \mathbb{W}$.

$$\begin{aligned} 9n &= 9(2k) \\ &= 2(9k) \\ &= 2t \quad \text{for some } t \in \mathbb{W}. \end{aligned}$$

$\Rightarrow 9n$ is even

This is a contradiction, since we assumed that $9n$ was odd.

Hence result.

d) Let m and n be integers. If mn is odd then m and n are both odd.

Proof Assume that mn is odd and that m and n are not both odd i.e. one of m or n is even, say n.

If n is even then $n = 2k$ for some $k \in \mathbb{W}$.

$$\begin{aligned} mn &= m \times 2k \\ &= 2mk \\ &= 2t \quad \text{for some } t \in \mathbb{W} \end{aligned}$$

$\Rightarrow mn$ is even

This is a contradiction, since we assumed that mn was odd.

Hence result.

e) Prove that $\sqrt{2}$ is irrational.

Proof Assume that $\sqrt{2}$ is rational.

If $\sqrt{2}$ is rational then it can be written as a fraction in its lowest terms i.e. $\sqrt{2} = \frac{m}{n}$, where m and n have no common factors.

$$\sqrt{2} = \frac{m}{n}$$

$$\Rightarrow 2 = \frac{m^2}{n^2}$$

$$\Rightarrow 2n^2 = m^2$$

$\Rightarrow m^2$ is even

$\Rightarrow m$ is even

$\Rightarrow m = 2k$ for some $k \in \mathbb{Z}$

$\Rightarrow 2n^2 = (2k)^2$ since $2n^2 = m^2$

$\Rightarrow 2n^2 = (2k)^2$

$\Rightarrow 2n^2 = 4k^2$

$\Rightarrow n^2 = 2k^2$

$\Rightarrow n^2$ is even

$\Rightarrow n$ is even

Therefore m and n have a factor of 2 in common. This is a contradiction, hence result.

f) Prove that the set of primes is infinite.

Proof Assume the set of primes is finite and that there are n primes

$p_1, \ p_2, \ p_3, \ p_4, \ldots p_n$

Consider $N = p_1 \times p_2 \times p_3 \times p_4 \times \ldots \times p_n$

The next number after N is $N + 1$

$N + 1 = p_1 \times p_2 \times p_3 \times p_4 \times \ldots \times p_n + 1$

$N + 1 = p_1 k + 1$ so $N + 1$ is not divisible by p_1.

$N + 1 = p_2 k + 1$ so $N + 1$ is not divisible by p_2.

Similarly $N + 1$ is not divisible by $p_3, \ p_4, \ldots p_n$.

So $N + 1$ must be prime and it is greater than p_n.

This is a contradiction, hence result.

Exercise 6: Use proof by contradiction to prove these results

a) For $n \in \mathbb{N}$, if $5n$ is even then n is even.

b) For $n \in \mathbb{N}$, if $3n$ is odd then n is odd.

c) For $n \in \mathbb{N}$, if n^2 is odd then n is odd.

d) For $n \in \mathbb{N}$, if n^3 is even then n is even.

e) For $n \in \mathbb{N}$, if 3 divides n^2 then 3 divides n.

f) $\sqrt{3}$ is irrational.

g) For $x, y \in \mathbb{R}$, if $x + y$ is irrational then at least one of x and y is irrational.

h) That if x and y are integers such that $x + y$ is odd, then one of them must be odd and one must be even

i) That if x and y are real numbers such that $x + y$ is irrational, then at least one of $x,\ y$ is irrational

j) That if m and n are integers such that mn^2 is even, then at least one of m or n is even

k) That if $\sin\theta \neq 0$, then $\theta \neq k\pi$ for any integer k

Proof by Contrapositive

1 Prove that if x and y are integers and $xy = 100$, then either $x \leq 10$ or $y \leq 10$.

p: x and y are integers and $xy = 100$

q: either $x \leq 10$ or $y \leq 10$

$-p$: x and y are integers and $xy \neq 100$

$-q$: $x > 10$ and $y > 10$

Proof by contradiction requires us to show $-q \Rightarrow -p$

$x > 10$ and $y > 10$

$\Rightarrow \quad xy > 100$

$\Rightarrow \quad xy \neq 100$

The contrapositive is true and hence the statement is true.

2 Prove that if 7 is a factor of n^2 then 7 is a factor n.

p: 7 is a factor of n^2

q: 7 is a factor n

$-p$: 7 is not a factor of n^2

$-q$: 7 is not a factor n

Proof by contradiction requires us to show $-q \Rightarrow -p$

7 is not a factor n

$\Rightarrow \quad n = 7m + t$ for some integers m and t

$\Rightarrow \quad n^2 = (7m + t)^2$

$= 49m^2 + 14mt + t^2$

$= 7(7m^2 + 2mt) + t^2$

$\Rightarrow \quad n^2 = 7k + remainder$

$\Rightarrow \quad$ 7 is not a factor of n^2

The contrapositive is true and hence the statement is true.

Exercise 7: Prove, using the contrapositive.

a) That if x and y are integers and xy is odd, then both x and y are odd

b) That every prime number greater than 3 is of the form $6n \pm 1$, where n is a positive integer

c) That if n is a natural number such that n^2 is even, then n is even.

Number Theory

The Fundamental Theorem of Arithmetic states that any integer, $n > 1$, can be expressed uniquely as a product of prime numbers
i.e. 6=2x3, 21=3x7, 36=2x2x3x3 etc

Example: Express 430 as a product of its prime number
430=2x215
430=2x5x43

Exercise 5: Express the following numbers as a product of primes
a) 490 b) 1125 c) 2728

The Division Algorithm

If a is a non-negative integer and b a positive integer, then there exists unique non-negative integers q and r such that

$$a = bq + r \quad \text{and} \quad 0 \leq r < b$$

Proof On the real number line, the integers $a,\ a-b,\ a-2b,\ a-3b,$.. form a decreasing sequence of integers.
Since only finitely many of these are ≥ 0, there is a unique integer $q \geq 0$ for which

$$a-(q+1)b < 0 \leq a - bq$$

and so $0 \leq a - bq < b$.
If we write $r = a - bq$, then $a = bq + r$ with $0 \leq r < b$.
Thus we have found non-negative integers q and r for which

$$a = bq + r \quad \text{and} \quad 0 \leq r < b \quad \text{both hold.}$$

To show that q and r are unique, suppose that

$$a = bq_1 + r_1 \quad \text{and} \quad 0 \leq r_1 < b.$$

Then $r_1 = a - bq_1$ and $0 \leq a - bq_1 < b$.

It follows that $a-(q_1+1)b<0\leq a-bq_1$ so that q_1 is the integer determined above and $r_1=a-bq=r$.
Thus the theorem is proved.

Examples

1 $\quad a=193$ and $b=17$

$193=11.17+6 \qquad q=11,\ r=6$

2 $\quad a=581$ and $b=23$

$581=25.23+6 \qquad q=25,\ r=6$

Exercise 2: Use the division identity for the following

1 $\quad a=75$ and $b=12$
2 $\quad a=327$ and $b=13$
3 $\quad a=392$ and $b=19$

If $r=0$ then we say that b is a divisor of a.
The notation used is $b|a$ which means b is a divisor of a.

Euclidean Algorithm

The Euclidean Algorithm is used to find the greatest common divisor (G.C.D.) or two or more integers where this cannot be done simply.

For integers a and b, $\quad a=bq_1+r_1$ and $0\leq r_1<b$

$b=r_1q_2+r_2$ and $0\leq r_2<r_1$

$r_1=r_2q_3+r_3$ and $0\leq r_3<r_2$

and so on until $\quad r_{n-2}=r_{n-1}q_n+r_n$ and $0\leq r_{n-1}<r_{n-1}$

$r_{n-1}=r_nq_{n+1}+0$ i.e. r eventually becomes 0

To find the G.C.D. for small numbers, you use factorisation as follows.

Example: Find the G.C.D. of 15 and 24

$15=3x5$ and $24=2^3x3$

So the G.C.D. of 15 and 24 is 3

Notation (15,24)=3 means the G.C.D. if 15 and 24 is 3

To find the G.C.D. for large numbers, use the Euclidean Algorithm.

Example: Find the Euclidean Algorithm to find the G.C.D. of (1147,851)
Use repeated application of the division identity until $r = 0$.
The last non-zero remainder is the G.C.D.

1147 = 1x851 + 296
851 = 2x296 + 259
296 = 1x259 + 37
259 = 7 x 37 + 0
Hence (1147,851) = 37

Exercise 3:

1 Find the G.C.D. of

a) (15,27)	b) (16,42)	c) (72,108)
d) (111,481)	e) (451, 168)	f) (679, 388)
g) (756, 714)	h) (1470, 1330)	i) (1498, 535)

2 Use the Euclidean Algorithm to find the G.C.D. of

a) (1219,901)	b) (4277,2821)	c) (5213,2867)
d) (2172, 1267)	e) (1692, 684)	f) (34034, 51051)

Expressing the G.C.D. of Two Positive Integers as a Linear Combination of the Two Integers

Having found the G.C.D. of two positive integers a and b, it is possible, by working backwards, to express the divisor (d) in terms of the two integers in the form of a linear combination

$d = xa + yb$ where x and y are integers.

Example

Use the Euclidean Algorithm to find the G.C.D. of 1147 and 851 hence find the integers x and y to write this G.C.D. in the form $x.1147 + y.851$

1147 = 1x851 + 296 (1)
851 = 2x296 + 259 (2)
296 = 1x259 + 37 (3)
259 = 7 x 37 + 0
Hence (1147,851) = 37

From (3) 37 = 296 – 1x259
From (2) 37 = 296 – 1x(851 – 2x296)
= 296 – 1x851 + 2x296
= 3x296 – 1x851
From (1) 37 = 3x(1147 – 1x851) – 1x851
= 3x1147 – 3x851 –1x851
37 = 3x1147 – 4x851

$x = 3$ and $y = -4$

Exercise 4:

1 a) Use the Euclidean Algorithm to find the G.C.D. of 345 and 285.
b) Hence find the integers x and y to write this G.C.D. in the form $345x+285y$

2 Calculate (583, 318) and express it in the form 583s + 318t, where $s, t \in Z$

3 a) Evaluate d = (1292, 1558)
b) hence express d in the form 1292s + 1558y where $s, t \in Z$

4 a) Use the Euclidean Algorithm to find the G.C.D. of 7293 and 798.
b) Hence find the integers x and y to write this G.C.D. in the form $7293x+798y$

5) Find a and b such that 248a + 261b = 1, $a, b \in Z$

6) 5612x + 540y = 4. Assuming x and y are integers, find their values.

Expressing Base 10 Integers in Other Bases

Our number system works on a base of 10. We have 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, the next number requires going up to the next column (the tens column). Other common bases are base 2 (binary) and base 16 (hexadecimal). You have to be able to write numbers in other bases.

Notation 352_{ten} means 352 in the base 10

Examples

1 Write 235_{ten} in the base 6.

$235 = 6 \times 39 + 1$

$39 = 6 \times 6 + 3$

$6 = 6 \times 1 + 0$

$1 = 6 \times 0 + 1$

Reading the remainders in reverse gives 1031_{six}

Or $235 = 6 \times 39 + 1$

$= 6 \times (6 \times 6 + 3) + 1$

$= 6 \times 6 \times 6 + 3 \times 6 + 1$

$= 6 \times 6 \times 6 + 0 \times 6 \times 6 + 3 \times 6 + 1$

$= 6^3 + 0 \times 6^2 + 3 \times 6 + 1$

$= 1031_{six}$

2 Write 423_{ten} in the base 8.

$423 = 8 \times 52 + 7$

$52 = 8 \times 6 + 4$

$6 = 8 \times 0 + 6$

Reading the remainders in reverse gives 647_{eight}

Or $423 = 8 \times 52 + 7$

$= 8 \times (8 \times 6 + 4) + 7$

$= 8 \times 8 \times 6 + 4 \times 8 + 7$

$= 6 \times 8^2 + 4 \times 8 + 7$

$= 647_{eight}$

Exercise 5

1 a) Express 81 to base 2 b) Express 579 to base 5.

c) Express 1064 to base 7. d) Express 15287 to base 9.

e) Express 333 to base 4. f) Express 1727 to base 12.

2 Express in base 10:

a) 1234_7 b) 777_8 c) 110110_2 d) $A81B_{12}$

3 Express a) 626_7 in base 5 b) 401_6 in base 7

c) $CC5_{12}$ in base 6

Answers

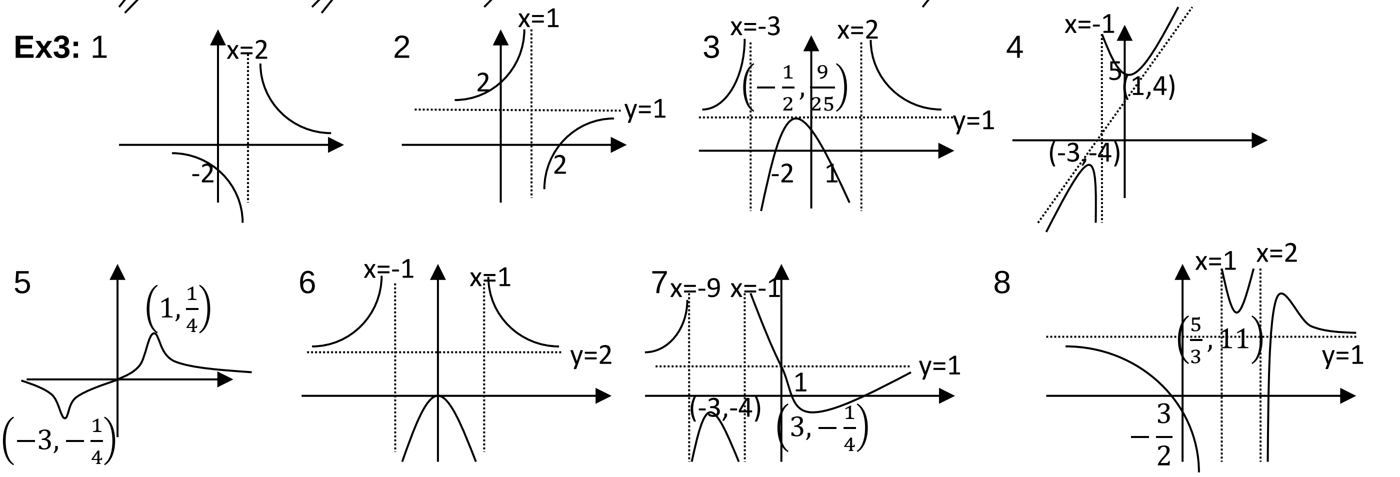
Properties of Function

Ex1: 1 x=2 2 x=-3 x=1 3 x=-1 x=3 4 x=2 5 x=-2 x=2 6 x=-2 x=1 7 x=0 8 x=1 9 none 10 none

11 x=1 12 x=-1 x=1

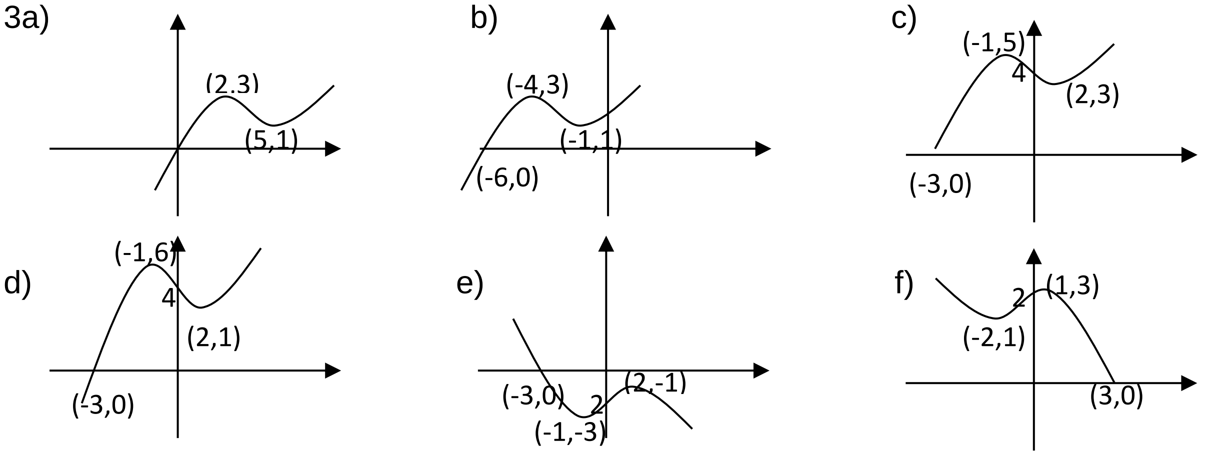
Ex2: 1 y=0 2 y=0 3 y=0 4 y=1 5 y=-1 6 y=1
7 y=x+3 8 y=x+1 9 y=x 10 y=0 11 y=x+1 12 y=2

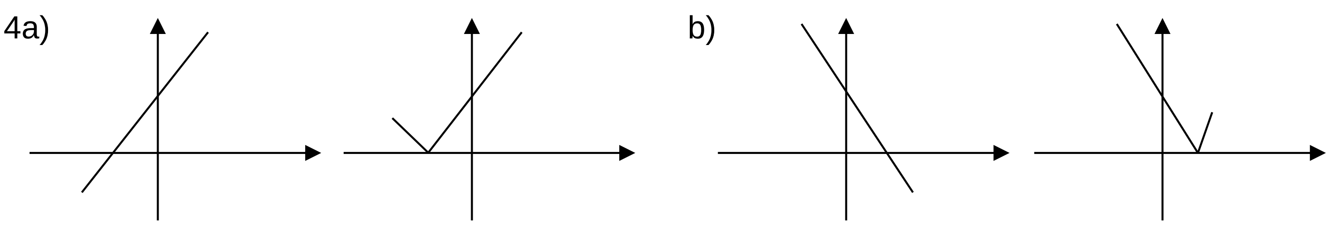
Ex3:



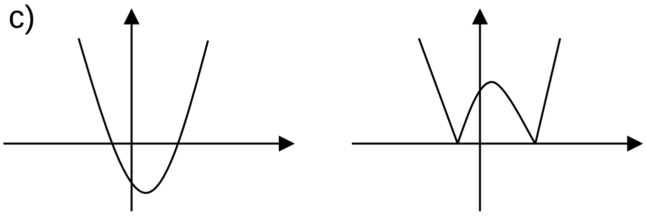
Ex4: 1) Proof   2) Point of Inflexion at x=½ (½,-½)   3) Point of Inflexion at (-2,4); concave down before x= -2, concave up after x= -2   4) Point of Inflexion at (-2.52,0); concave up before x= -2.52, concave down after x=-2.52

Ex5: 1a) $\frac{1}{2}x$ b) 2-x c) $\frac{2}{x}$ d) $\log_2 x$ e) $\frac{1}{2}(1-x)$ f) e^x+2 2a) $\frac{\pi}{3}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{6}$ e) $\frac{5\pi}{6}$ f) $\frac{\pi}{6}$

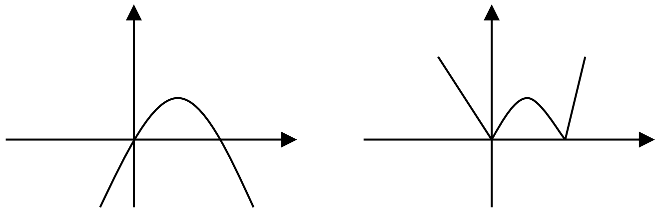




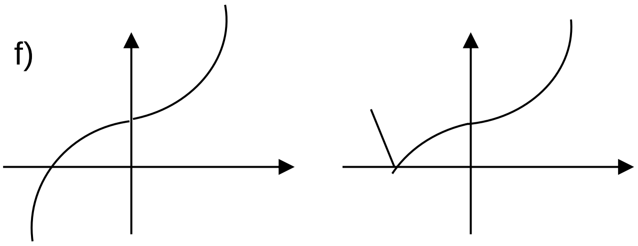
c)



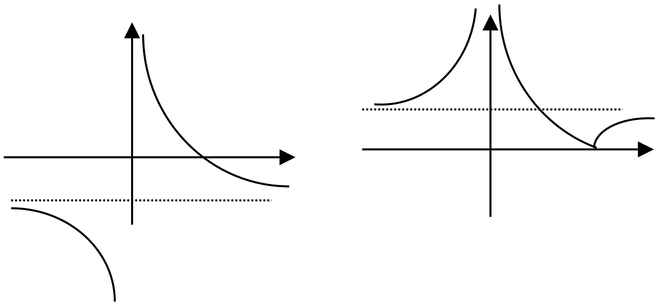
d)



f)



g)



5 a) neither b) even c) odd d) odd e) odd f) odd g)odd
h) even i) odd j) even k) neither l) neither

Motion and Optimisation

Ex1 1)a)3m b)3m c) 2m/s d) $2m/s^2$ 2)a) 88m/s, 166m/s b)$66m/s^2$, $90m/s^2$ c) 127m/s d) $78m/s^2$ 3)a)$\frac{1}{2}m/s^2$ b) $2\frac{1}{2}m/s^2$ 4) a) 0m/s, $18m/s^2$ b) -27m/s, $-36m/s^2$

Ex2 a) min at (1,1) b) min at $\left(\frac{1}{e},-\frac{1}{e}\right)$ c) max at $\left(1,\frac{1}{e}\right)$ d) max $\left(\frac{\pi}{3},\frac{3\sqrt{3}}{4}\right)$ PoI $(\pi,0)$ min $\left(\frac{5\pi}{3},\frac{-3\sqrt{3}}{4}\right)$

Ex3 1) see graphs 2) max is 1 3) max is 2, min is -4, f'(x) does not exist (0,-2), (2,2), (4,0), (7,-3)

Ex4 1)a)proof b) $\frac{8}{3}$cm 2)a)proof b)16cm 3)a)proof b)$\frac{32\pi}{27}cm^3$ 4)a)+b) proof c)$\frac{8\pi}{3}$, A=200(1+$\sqrt{2}$)

Ex5: 1a) 12π b) 8π c) $\frac{2\pi}{3}$ d) $\frac{19\pi}{3}$ e) $\frac{\pi}{30}$ f) 18π g) 64π h) $\frac{1}{2}\pi^2$

2a) 8π b) $\frac{\pi}{5}$ c) $\frac{\pi}{6}$ d) 32π e) $\frac{7\pi}{48}$ f) $\frac{56\pi}{15}$ g) $\frac{1}{2}(e^{10}-e^{6})\pi$

Summation and Proof by Induction

Ex1 1a) $1^2+2^2+3^2+4^2+5^2$ b) 1+3+5+7+9+11+13+15+17 c) $\frac{2520}{1}+\frac{2520}{2}+\frac{2520}{3}+\cdots\frac{2520}{10}$

2a) $$\sum_{1}^{50} k$$ b) $$\sum_{1}^{6} 5k$$ c) $$\sum_{1}^{6} 2k+1$$

Ex2 a) 55 b) 420 c) 96 d)1150 e) 155

Ex3 parts of proofs are shown below

a) $1+3+5+...+[2(k+1)+1]=k^2+2k+1$
$= (k+1)^2$

b) $5+7+...+(2k-1)+[2(k+1)-1]$
$=(k-2)(k+2)+2k+1$
$=k^2-4+2k+1$
$=(k-1)(k+3)$
$=(k+1-2)(k+1+2)$

c) $2^k>k$
$2\times2^k>2k$
$2\times2^k>k+k$
$2^{k+1}>k+1$ since k>1

d) $1+2+4+...+2^k+2^{k+1}$
$=2^{k+1}-1+2^{k+1}$
$=2\times2^{k+1}-1$
$=2^{k+2}-1$

e) $\frac{1}{1\times2}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$
$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$
$=\frac{k(k+2)+1}{(k+1)(k+2)}$

f) $3^{2k}-1=8t$ for $t\epsilon\mathbf{N}$
$3^{2(k+1)}-1=3^{2k+2}-1$
$=9\text{x}3^{2k}-9+8$
$=9(3^{2k}-1)+8$
$=9\text{x}8t+8$
$=8(9t+1)$

g) $1^3+2^3+3^3+...+k^3+(k+1)^3$
$=\frac{1}{4}k^2(k+1)^2+(k+1)^3$
$=\frac{1}{4}(k+1)^2[k^2+4(k+1)]$
$=\frac{1}{4}(k+1)^2(k+2)^2$

h) $3^k>k^3$
$3\text{x}3^k>3k^3=k^3+2k^3=k^3+k^2\text{x}2k$ since $k\geq4$
$>k^3+8k^2 = k^3+3k^2+5k^2= k^3+3k^2+k\text{x}5k$
$> k^3+3k^2+20k= k^3+3k^2+3k+17k$
$> k^3+3k^2+3k+1$
$3^{k+1}>(k+1)^3$

i) See Nelson 2 p99

j) See Nelson 2 p99

Other Forms of Proof

Ex1 1) c 2) d
3a) For some real x, x^2 is not positive
b) No pupils find mathematics difficult
c) Some dogs like cats
d) There is no positive integer x such that $x+3>0$
e) Some parallelogram do not have half turn symmetry
f) Some school boys lies
g) Some numbers with a zero in the units place are not divisible by five.
h) Some numbers of the form 2^n-1, (n an integer), are not prime.

Ex2 1a) If a number is divisible by 5, it ends in zero (e.g. 15)
b) If a number is odd, it is a prime number greater than 2 (e.g. 9)
c) If the diagonals of a quadrilateral intersect at right angles, the quadrilateral is a square (e.g. rhombus)
d) If $x^2=9, x=3$ (e.g. -3)
e) If the sum of two numbers is even then the numbers are odd (e.g. 4 and 6)
f) If the product of two numbers is even then the numbers are even (e.g. 6 and 5)

Ex3 1) necessary 2) neither 3a) necessary b) none c) sufficient d) necessary and sufficient e) sufficient f) necessary and sufficient g) neither h) sufficient

Ex4 1a) n=82 b) a=-4, b=1 c) a=3, b=1, c=-2 d) a=1, b=-3 2a) $x=\frac{1}{2}$ b) a=3, b=2, c=-2, d=-3 c) a=-6, b=-3 d) a=5, b=2, x=0 Other examples are possible!

Ex5 1a) If a number is divisible by by 5, then it ends in 0. False e.g. 15
b) If n is an odd number greater than 2, then it is prime. False e.g. 15
c) $x^2 = 9 \Rightarrow x = 3$. False e.g. x=-3
d) If a+b is even then a and b are odd. False e.g. 2 and 4
e) If k is a multiple of 3 then 3 is a root of $x^2+x-k=0$. False e.g. $x^2+x-3=0$
2a) yes b) yes c) no d) yes e) yes f) no

Ex6

(a) If $5n$ is even then n is even, $n \in N$

Assume that $5n$ is even and n is odd.

Then $n = 2k+1$, $k \in N$

$n = 2k+1 \Rightarrow 5n = 5(2k+1)$
$= 10k+5$
$= 2(5k+2) +1$
⇨ $5n$ is odd

This is a contradiction, as $5n$ was assumed to be even. Therefore the original conjecture is true.

(b) If $3n$ is odd then n is odd, $n \in N$

Assume that $3n$ is odd and n is even.

Then $n = 2k$, $k \in N$

$n = 2k \Rightarrow 3n = 3(2k)$
$= 6k$
$= 2(3k)$
⇨ $3n$ is even

This is a contradiction, as $3n$ was assumed to be odd Therefore the original conjecture is true.

(c) If n^2 is odd then n is odd, $n \in N$

Assume that n^2 is odd and n is even.

Then $n = 2k$, $k \in N$

$n = 2k \Rightarrow n^2 = (2k)^2$
$= 4k^2$
$= 2(2k^2)$
⇨ n^2 is even

This is a contradiction, as n^2 was assumed to be odd Therefore the original conjecture is true.

(d) If n^3 is even then n is even, $n \in N$

Assume that n^3 is even and n is odd.

Then $n = 2k+1$, $k \in N$

$n = 2k+1 \Rightarrow n^3 = (2k+1)^3$
$= 8k^3+ 12k^2 + 6k+1$
$= 2(4k^3+6k^2 + 3k) +1$
⇨ n^3 is odd

This is a contradiction, as n^3 was assumed to be even. Therefore the original conjecture is true.

(e) If 3 divides n^2 then 3 divides n, $n \in N$

Assume that 3 divides n^2 and 3 does not divide n.

Then n = 3k+1 or n=3k+2, $k \in N$

n = 3k+1 => $n^2 = (3k+1)^2$
$= 9k^2 + 6k+1$
$= 3(3k^2 + 2k) +1$
=> n^2 is not divisible by 3

n = 3k+2 => $n^2 = (3k+2)^2$
$= 9k^2 + 12k+4$
$= 3(3k^2 + 4k+1) +1$
=> n^2 is not divisible by 3

This is a contradiction, as n^2 was assumed to be divisible by 3. Therefore the original conjecture is true.

(f) $\sqrt{3}$ is irrational

Assume $\sqrt{3}$ is rational, i.e. $\sqrt{3} = \frac{p}{q}$ where p and q are integers with no common factors.

$\sqrt{3} = \frac{p}{q}$

$3 = \frac{p^2}{q^2}$

$3q^2=p^2$ => p^2 is a multiple of 3

=> p is a multiple of 3 i.e. p =3m, $m \in Z$

$3 = \frac{p^2}{q^2}$

$3 = \frac{(3m)^2}{q^2}$

$3q^2=9p^2$

$q^2=3p^2$ => q^2 is a multiple of 3

=> q is a multiple of 3 i.e. q =3n, $m \in Z$

Thus, 3|q and 3|q, but p and q are asumed to have no common factors. This is a contradiction and so the original conjecture is true.

(g) If x + y is irrational then at least one of x and y is irrational.

Assume that x+y is irrational and that x and y are both rational.

Then $x = \frac{a}{b}$ and $y = \frac{c}{d}$; a, b, c, $d \in Z$

So x+y = $\frac{a}{b} + \frac{c}{d}$

= $\frac{ad+bc}{bd}$

As $ad+bc$ and bd have no common factors, then $x+y$ is rational.

This is a contradiction, x+y was assumed to be irrational. Therefore the original conjecture is true.

(h) If x and y are integers such that x + y is odd, then one of them must be odd and one must be even.

Assume that x + y is odd and x and y are both odd or even.

If x and y are both even, x = 2m and y = 2n, $m,n \in Z$

So x+ y = 2m + 2n

= 2(m+n)

=> x + y is even

If x and y are both odd, x = 2m+1 and y = 2n+1, $m,n \in Z$

So x+ y = 2m+1 + 2n+1

= 2(m+n+1)

=> x + y is even

In both cases, x+y is shown to be even.

This is a contradiction, as x + y is assumed to be odd.

Therefore the original conjecture is true.

(i) same as (g)

(j) If m and n are integers such that mn^2 is even then at least one of m or n is even.

Assume that mn^2 is even and both m and n are odd.

Then $m=2p+1$ and $n=2q+1$, $p,q \in Z$

So $mn^2 = (2p+1)(2q+1)^2$

$=(2p+1)(4q^2+4q+1)$

$= 8pq^2+8pq+2p+4q^2+4q+1$

$= 2(4pq^2+4pq+p+2q^2+2q)+1$

=> mn^2 is odd.

This is a contradiction as mn^2 was assumed to be even. Therefore the original conjecture is true.

(k) If $sin\theta \neq 0$ then $\theta \neq k\pi$ for any integer k.

Assume $sin\theta \neq 0$ and $\theta = k\pi$

Then $sin\theta = sink\pi$

$= 0$

This is a contradiction as it was assumed $sin\theta \neq 0$.

Therefore the original conjecture is true.

Ex7

(a) If x and y are integers and xy is odd, then both x and y are odd.

p: xy is odd ~p: xy is even
q: both x and y are odd ~q: at least one of x and y is even

Contrapositive conjecture: (~q => ~p)
If at least one of x and y is even, then xy is even

As either x or y is even, let $x = 2k,\ k \in Z$

$$x = 2k \Rightarrow xy = 2kx$$
$$= 2(kx)$$
$$\Rightarrow xy \text{ is even} \quad \text{i.e if } x \text{ is even, } xy \text{ is even}$$

So the contrapositive conjecture is true, therefore the original conjecture is also true.

(b) Every prime number greater than 3 is of the form $6n\pm1$, $n \in W$

p: given a prime number greater than 3
~p: given a composite number greater than 3
q: the number is of the form $6n\pm1$
~q: the number is not of the form $6n\pm1$

Contrapositive conjecture: (~q => ~p)
If a number is not of the form $6n\pm1$, it is a composite number (>3)

Let m be a number is not of the form $6n\pm1$,
So $m= 6n$ or $m=6n+2$ or $m=6n+3$ or $m=6n+4$

Case 1: $m = 6n$
⇨ m is composite (6 is a factor)

Case 2: $m = 6n+2$
$= 2(3n+1)$
⇨ m is composite (2 is a factor)

Case 3: $m = 6n+3$
$= 3(2n+1)$
⇨ m is composite (3 is a factor)

Case 4: $m = 6n+4$
$= 2(3n+2)$
⇨ m is composite (2 is a factor)

In all cases, m is composite, so the contrapositive conjecture is true, therefore the original conjecture is also true.

(c) If n is a natural number such that n^2 is even, then n is even.

p: n^2 is even ~p: n^2 is odd
q: n is even ~q: n is odd

Contrapositive conjecture: (~q => ~p)
If n is odd then n^2 is odd

Let $n = 2k+1, \ k \in Z$
$n = 2k+1 \Rightarrow n^2 = (2k+1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 4k)+1$
$\Rightarrow n^2$ is odd i.e if n is odd then n^2 is odd

So the contrapositive conjecture is true, therefore the original conjecture is also true.

Number Theory

Ex1 a) $2x5x7^2$ b) 3^2x5^3 c) $2^3x11x31$

Ex2 1) 75=6.12+3 2) 327=25.15+2 3) 392=20.19+12

Ex3 1a) 3 b) 2 c) 36 d) 37 e) 1 f) 97 g) 42 h) 70 i) 107
2a) 53 b) 91 c) 1 d) 181 e) 36 f) 17017

Ex4 1a) 15 b) x = 5, y = -6 2) 53; s =2, t = -1 3) a) 38 b)s = 5, t = -6
4a) 3 b) x=-115 y=1051 5) a=20, b = -19 6) x = -28, y=291

Ex5 1 a) 1010001_2 b) 4304_5 c) 3050_7 d) 22865_9 e) 11031_4 f) BBB_{12}
2a) 466 b) 511 c) 54 d) 18455 3a) 32224_5 b) 2657 c) 11125_6