Advanced Higher

Mathematics

HSNe21S07

Exam Solutions – 2007

Question 1

$$\left(x-\frac{2}{x}\right)^4 = x^4 + 4x^3.\left(-\frac{2}{x}\right) + 6x^2\left(-\frac{2}{x}\right)^2 + 4x\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4$$

$$= x^4 - 8x^2 + 6x^2.\frac{4}{x^2} + 4x.\left(-\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$$

Coefficients from Pascal's triangle

If you can't remember the coefficients, you can always write them as $\binom{4}{0}$, $\binom{4}{1}$ etc. then use $\binom{4}{k} = \frac{4!}{k!(4-k)!}$

Question 2

(a) $f'(x) = \exp(\sin 2x).\frac{d}{dx}(\sin 2x)$

$= \exp(\sin 2x).\cos 2x.2$ using the chain rule again

$= 2\exp(\sin 2x)\cos 2x.$

(b) Taking ln on both sides:

$$\ln y = \ln 4^{(x^2+1)}$$
$$= (x^2+1)\ln 4$$
$$= x^2.\ln 4 + \ln 4$$

OR

$$\ln(4^{x^2}.4)$$
$$= \ln 4^{x^2} + \ln 4$$
$$= x^2 \ln 4 + \ln 4$$

Differentiating with respect to x...

$$\frac{d}{dy}(\ln y).\frac{dy}{dx} = 2x.\ln 4$$

Remember: ln4 is just a constant

$$\frac{1}{y}.\frac{dy}{dx} = 2x.\ln 4$$
$$\frac{dy}{dx} = y.2x.\ln 4$$
$$= 4^{(x^2+1)}.2x.\ln 4$$

Question 3

$$(3+3i)^3 - 18(3+3i) + 108$$
$$= 3^3 + 3.3^2.(3i) + 3.3(3i)^2 + (3i)^3 - 18.3 - 18.3i + 108$$
$$= 27 + 81i - 81 - 27i - 54 - 54i + 108$$
$$= 0. \quad \therefore z = 3+3i \text{ is a root.}$$

$\bar{z} = 3-3i$ is also a root, so $(z-3-3i)(z-3+3i)$ is a factor.

$$(z-3-3i)(z-3+3i) = z^2 - 6z + 18$$

<u>Method 1</u> Thus $z^3 - 18z + 108 = (z^2 - 6z + 18)(az + b)$

Equating coefficients: $a = 1$, $b = \frac{108}{18} = 6$.
Thus $z+6$ is a factor, so $z = -6$ is a root of the equation.

<u>Method 2</u> Using polynomial division...

$$\begin{array}{r|l} & z+6 \\ \hline z^2-6z+18 & z^3 \qquad - 18z + 108 \\ & \underline{z^3 - 6z^2 + 18z} \\ & \qquad 6z^2 - 36z + 108 \\ & \qquad \underline{6z^2 - 36z + 108} \\ & \qquad\qquad 0 \end{array}$$

So $z+6$ is a factor and $z = -6$ is a root of the equation

Question 4

$$\frac{2x^2-9x-6}{x(x^2-x-6)} = \frac{2x^2-9x-6}{x(x+2)(x-3)}.$$

Let $\dfrac{2x^2-9x-6}{x(x+2)(x-3)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-3}.$

$$\therefore\ A(x+2)(x-3) + Bx(x-3) + Cx(x+2) = 2x^2-9x-6.$$

Method 1 Let $x=0$ $\quad 6A = 6 \Leftrightarrow A = 1$

Let $x=-2$ $\quad 10B = 8+18-6 \Leftrightarrow B = 2$

Let $x=3$ $\quad 15C = 18-27-6 \Leftrightarrow C = -1$

$$\therefore\ \frac{2x^2-9x-6}{x(x^2-x-6)} = \frac{1}{x} + \frac{2}{x+2} - \frac{1}{x-3}$$

Method 2 $A(x^2-x-6) + B(x^2-3x) + C(x^2+2x) = 2x^2-9x+6$

Equating coefficients...

$-6A = -6 \Leftrightarrow A = 1$

$-1 - 3B + 2C = -9 \Leftrightarrow -3B+2C = -8$ —— (1)

$1 + B + C = 2 \Leftrightarrow B + C = 1$ —— (2)

(1)+3×(2): $5C = -5 \Leftrightarrow C = -1$

So $B = 2.$

$$\therefore\ \frac{2x^2-9x-6}{x(x^2-x-6)} = \frac{1}{x} + \frac{2}{x+2} - \frac{1}{x-3}$$

$$\int_4^6 \frac{2x-9x-6}{x(x^2-x-6)}\,dx = \int_4^6 \left(\frac{1}{x} + \frac{2}{x+2} - \frac{1}{x-3}\right) dx$$

$$= \Big[\ln|x| + 2\ln|x+2| - \ln|x-3|\Big]_4^6$$

$$= \ln 6 + 2\ln 8 - \ln 3 - \left(\ln 4 + 2\ln 6 - \cancel{\ln 1}^{\,0}\right)$$

$$= -\ln 6 + \ln 8^2 - \ln 3 - \ln 4$$

$$= \ln \frac{64}{6\times3\times4}$$

$$= \ln\frac{8}{9} \qquad \therefore\ m=8 \text{ and } n=9.$$

Question 5

(a) $$AB = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix} = \begin{pmatrix} x & x & x \\ -6 & 6 & -1 \\ 0 & 0 & 8 \end{pmatrix}$$

(b) $$\det A = 1(2-(-1)) - 0 - 1(0) = 3.$$

$$\det AB = x(48) - x(-48) + x(0) = 96x.$$

$$\det AB = \det A \det B \Rightarrow \det B = \frac{96x}{3} = 32x.$$

Question 6

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$

So $$f(x) = \frac{1}{2}\cos 2x = \frac{1}{2}\left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \dots\right)$$

$$= \frac{1}{2} - x^2 + \frac{1}{3}x^4 - \dots$$

Thus $$f(3x) = \frac{1}{2} - (3x)^2 + \frac{1}{3}(3x)^4 - \dots = \frac{1}{2} - 9x^2 + 27x^4 - \dots$$

Question 7

$$599 = 11.53 + 16$$
$$53 = 3.16 + 5$$
$$16 = 3.5 + 1$$

So $1 = 16 - 3.5$

$$= 599 - 11.53 - 3(53 - 3.16)$$
$$= 599 - 14.53 + 9(599 - 11.53)$$
$$= 10.599 - 113.53$$

$\therefore\ p = 10$ and $q = -113$.

Question 8

A.E. $\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)(\lambda+3) = 0$

C.F. $y = Ae^{-3x} + Bxe^{-3x}$

P.I. Try $y = Ce^{2x}$ $\therefore\ \dfrac{dy}{dx} = 2Ce^{2x}$; $\dfrac{d^2y}{dx^2} = 4Ce^{2x}$

The equation says: $4Ce^{2x} + 12Ce^{2x} + 9Ce^{2x} = e^{2x}$

i.e. $25C = 1$

$C = \dfrac{1}{25}$

The general solution is $y = \text{C.F.} + \text{P.I.}$

$$= Ae^{-3x} + Bxe^{-3x} + \frac{1}{25}e^{2x}.$$

Question 9

Method 1

$$\sum_{r=1}^{n}(4-6r) = \sum_{r=1}^{n}4 - 6\sum_{r=1}^{n}r$$
$$= 4n - 6\cdot\frac{n(n+1)}{2}$$
$$= 4n - (3n^2 + 3n)$$
$$= n - 3n^2$$

Remember:
$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Method 2

$$\sum_{r=1}^{n}(4-6r) = \frac{n}{2}\left(2a+(n-1)d\right) \quad \text{where } a=-2,\ d=-6$$
$$= \frac{n}{2}\left(-4-6(n-1)\right)$$
$$= -2n - 3n(n-1)$$
$$= -2n - 3n^2 + 3n$$
$$= n - 3n^2$$

Putting $n=2q$: $\displaystyle\sum_{r=1}^{2q}(4-6r) = 2q - 3(2q)^2 = 2q - 12q^2.$

$$\sum_{r=q+1}^{2q}(4-6r) = \sum_{r=1}^{2q}(4-6r) - \sum_{r=1}^{q}(4-6r)$$
$$= (2q-12q^2) - (q-3q^2)$$
$$= q - 9q^2.$$

Question 10

Let $u = 1 + x^2$ so $du = 2x\,dx$. When $x=0$ $u=1$
$x=1$ $u=2$

$$\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx = \int_1^2 \frac{x^3}{u^4}\cdot\frac{du}{2x} = \frac{1}{2}\int_1^2 \frac{u-1}{u^4}\,du = \frac{1}{2}\int_1^2 \left(u^{-3} - u^{-4}\right)du$$

$$= \frac{1}{2}\left[-\frac{u^{-2}}{2} + \frac{u^{-3}}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{1}{3u^3} - \frac{1}{2u^2}\right]_1^2$$

$$= \frac{1}{2}\left(\frac{1}{24} - \frac{1}{8} - \left(\frac{1}{3} - \frac{1}{2}\right)\right)$$

$$= \frac{1}{24}.$$

Volume of solid of revolution is $V = \pi\int_0^1 y^2\,dx$

$$= \pi\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$$

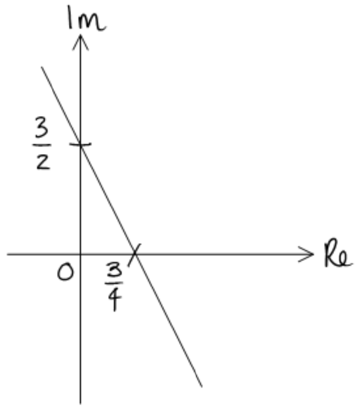
$$= \frac{\pi}{24} \text{ cubic units}$$

Question 11

$$|z-2| = |z+i|$$
$$|x-2+iy| = |x+(1+y)i|$$
$$\sqrt{(x-2)^2+y^2} = \sqrt{x^2+(1+y)^2}$$
$$\Rightarrow x^2-4x+4+y^2 = x^2+1+2y+y^2$$
$$-4x+4 = 1+2y$$
$$4x+2y-3=0 \Leftrightarrow y=-2x+\tfrac{3}{2}.$$

Remember: $|a+ib|=\sqrt{a^2+b^2}$

The locus is the line with the above equation in the Complex Plane.



Question 12

$\underline{n=1}$ $\quad (1+a)^1 = 1+a = 1+na$. So true for $n=1$.

Assume true for $n=k$, i.e. $(1+a)^k \geq 1+ka$.

Let $n=k+1$.

$$\begin{aligned}(1+a)^{k+1} &= (1+a)(1+a)^k \\ &\geq (1+a)(1+ka) \\ &= 1+ka+a+ka^2 \\ &\geq 1+ka+a \\ &= 1+(k+1)a.\end{aligned}$$

Hence the statement is true for $n=k+1$.

Therefore by the Principle of Mathematical Induction, the statement is true for all positive integers n.

Question 13

(a) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2\cos 2t}{-2\sin 2t} = -\cot 2t.$

$\boxed{\cot u = \dfrac{1}{\tan u}}$

When $t = \frac{\pi}{8}$, $\dfrac{dy}{dt} = -\cot\frac{\pi}{4} = -\dfrac{1}{\tan\frac{\pi}{4}} = -1$

$\therefore$ the gradient of the tangent is -1.

When $t = \frac{\pi}{8}$, $x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Pt $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Equation of tangent: $y - \frac{1}{\sqrt{2}} = -\left(x - \frac{1}{\sqrt{2}}\right)$

$x + y - \sqrt{2} = 0.$

(b)
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\cot 2t)$$
$$= \frac{d}{dt}(-\cot 2t)\cdot\frac{dt}{dx} \quad \text{using the Chain Rule}$$
$$= \operatorname{cosec}^2 2t \cdot 2 \cdot \frac{1}{-2\sin 2t}$$
$$= -\operatorname{cosec}^3 2t$$

$\boxed{\dfrac{dt}{dx} = \dfrac{1}{dx/dt}}$

So
$$\sin 2t \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\sin 2t \operatorname{cosec}^3 2t + \cot^2 2t$$
$$= -\frac{1}{\sin^2 2t} + \frac{\cos^2 2t}{\sin^2 2t}$$
$$= -\frac{1 - \cos^2 2t}{\sin^2 2t}$$
$$= -1$$

$\boxed{1 - \cos^2 u = \sin^2 u}$

$\therefore k = -1.$

Question 14

(a)

$$\frac{dG}{dt} = \frac{25k-G}{25} \Rightarrow \int \frac{1}{25k-G}\,dG = \int \frac{1}{25}\,dt$$

$$-\ln|25k-G| = \frac{t}{25} + c$$

$$|25k-G| = Ae^{-t/25} \quad \text{where } A > 0$$

If $G < 25k$ then $25k - G = Ae^{-t/25}$

If $G > 25k$ then $25k - G = -Ae^{-t/25}$

Hence $G = 25k + Ae^{-t/25}$ where A is an arbitrary constant

When $t=0$, $G=0$ so $A = -25k$.

$\therefore G = 25k\left(1 - e^{-t/25}\right)$

(b) When $t = 5$, $G = 0{\cdot}6$ so:

$$25k\left(1-e^{-1/5}\right) = 0{\cdot}6$$

$$k = \frac{0{\cdot}6}{25\left(1-e^{-1/5}\right)}$$

$$= 0{\cdot}132 \quad \text{(to 3 d.p.)}$$

(c) After 10 years, the growth is $25 \times 0{\cdot}132\left(1-e^{-10/25}\right) = 1{\cdot}088$ (to 3d.p.)

$\therefore$ the claim is justified.

(d) As $t \to \infty$, $G \to 25k$.

So the height approaches $0{\cdot}3 + 25k = 3{\cdot}60$ m (to 2 d.p.)

Question 15

(a) We want to show there are no solutions of the system:

$$\left.\begin{array}{lrcl} ① & 2+s &=& 1-2t \\ ② & -s &=& t \\ ③ & 2-s &=& 2+3t \end{array}\right\}$$

Method 1

$$\begin{aligned} s + 2t &= -1 \\ s + t &= 0 \\ s + 3t &= 0 \end{aligned}$$

Write these as: $\left(\begin{array}{cc|c} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right)$

There are no solutions of the system because row 3 is inconsistent.

Method 2 Using ②, ① $\Rightarrow$ $2+s = 1+2s$ i.e. $s = 1$

③ $\Rightarrow$ $2-s = 2-3s$ i.e. $s = 0$.

These cannot both be the case, so there are no solutions of the system.

(b) L_1: $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$; L_2: $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

So $\underline{d}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\underline{d}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$.

$$\begin{aligned} \underline{d}_3 &= \underline{d}_1 \times \underline{d}_2 \quad \text{since } L_3 \text{ is perpendicular to } L_1, L_2 \\ &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{vmatrix} \\ &= \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \quad \text{or equivalently } \underline{d}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Check...
$\underline{d}_1 . \underline{d}_3 = 0$
$\underline{d}_2 . \underline{d}_3 = 0$

$\therefore$ L_3 has parametric equations $x = 1+2u$, $y = 1+u$, $z = 3+u$.

(c) Solve
$$\left.\begin{aligned} 1+2u &= -1-2t \\ 1+u &= t \\ 3+u &= 2+3t \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} u+t &= -1 \quad \text{— } ① \\ u-t &= -1 \quad \text{— } ② \\ u-3t &= -1 \quad \text{— } ③ \end{aligned}\right.$$

Method 1
$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -3 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & -4 & 0 \end{array}\right)$$

So $t=0$ and $u=-1$.

Method 2 ① + ②: $2u=-2 \Leftrightarrow u=-1$. So $t=0$.

These values also satisfy ③.

Put $t=0$, $u=-1$ into L_2 (or L_3), $\therefore Q(-1,0,2)$

Put $(x,y,z)=(1,1,3)$ into L_1: $\left.\begin{aligned} 1 &= 2+s \\ 1 &= -s \\ 3 &= 2-s \end{aligned}\right\} \Leftrightarrow s=-1$.

Hence P lies on L_1.

(d) $PQ = \sqrt{(1+1)^2+1^2+1^2} = \sqrt{6}$ units.

Question 16

(a) The asymptotes are $y = \pm\frac{\pi}{2}$.

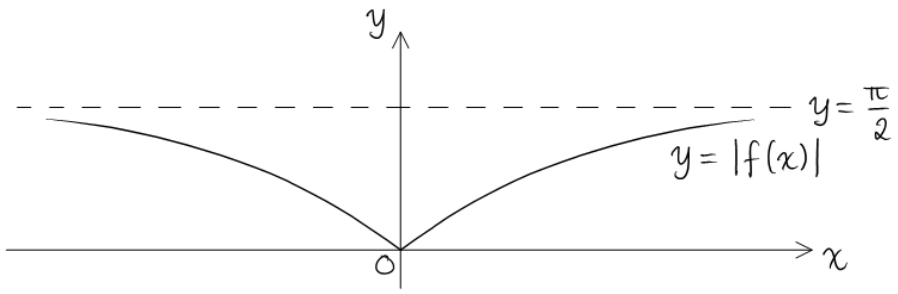
$\left(-\frac{\pi}{2} < \tan^{-1}(u) < \frac{\pi}{2}\right)$

(b)
$$\int_0^{1/2} 1 \cdot \tan^{-1} 2x \, dx = \left[x \cdot \tan^{-1} 2x\right]_0^{1/2} - \int_0^{1/2} x \cdot \frac{2}{1+4x^2} \, dx$$
$$= \frac{1}{2}\tan^{-1}(1) - \left[\frac{2}{8}\ln\left|1+4x^2\right|\right]_0^{1/2}$$
$$= \frac{\pi}{8} - \frac{1}{4}\left(\ln 2 - \cancel{\ln 1}^{\,0}\right)$$
$$= \frac{\pi}{8} - \frac{1}{4}\ln 2$$

Because...
$$\frac{d}{dx}\left(\ln\left|1+4x^2\right|\right) = \frac{1}{1+4x^2} \cdot 8x$$

$\therefore$ the area is $\frac{\pi}{8} - \frac{1}{4}\ln 2$ square units.

(c) All points below the x-axis are reflected above:



Since $|f(x)|$ is even, the area is $2\left(\frac{\pi}{8} - \frac{1}{4}\ln 2\right) = \frac{\pi}{4} - \frac{1}{2}\ln 2$ sq. units.