St Andrew's Academy

Department of Mathematics

Advanced Higher

Course Notes

Book 4

Contents

FORMULAE LIST

Standard derivatives	
$f(x)$	$f'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\dfrac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax)+c$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)+c$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c$
$\dfrac{1}{x}$	$\ln\lvert x\rvert+c$
e^{ax}	$\dfrac{1}{a}e^{ax}+c$

Summations

(Arithmetic series) $$S_n = \frac{1}{2}n\left[2a+(n-1)d\right]$$

(Geometric series) $$S_n = \frac{a(1-r^n)}{1-r}$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \ldots$$

FORMULAE LIST (continued)

De Moivre's theorem

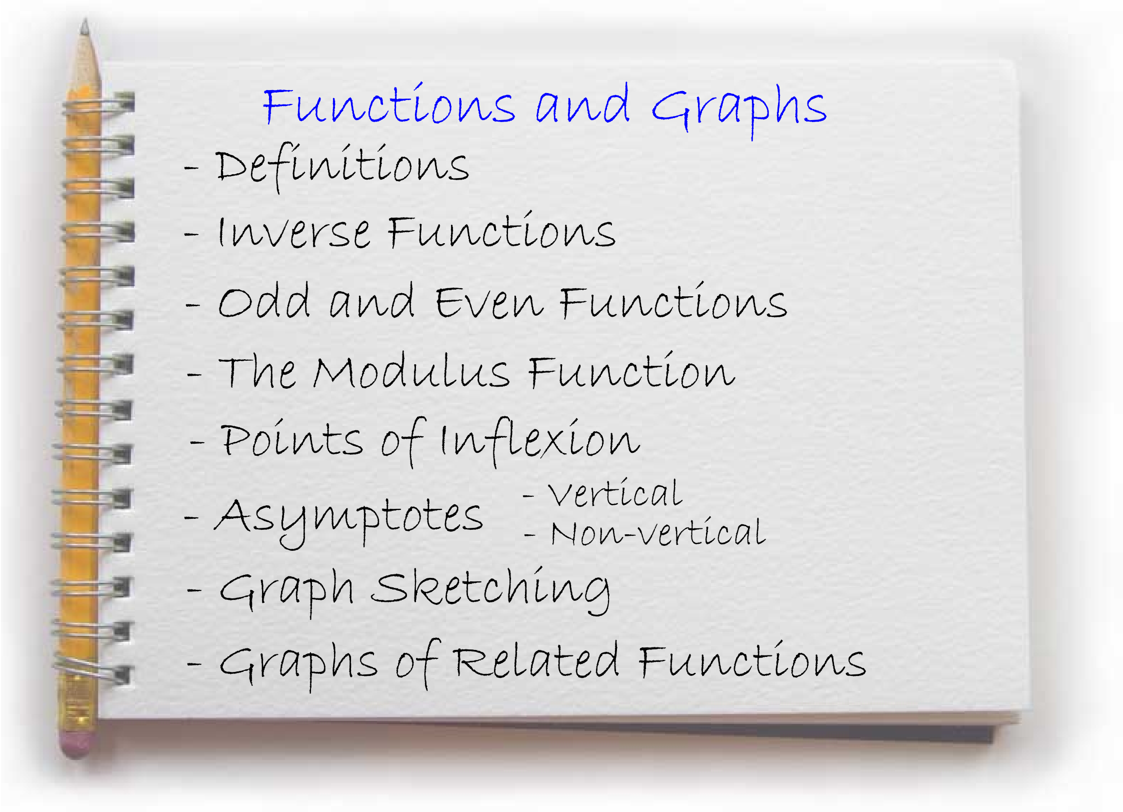
$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Vector product

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ about the origin, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

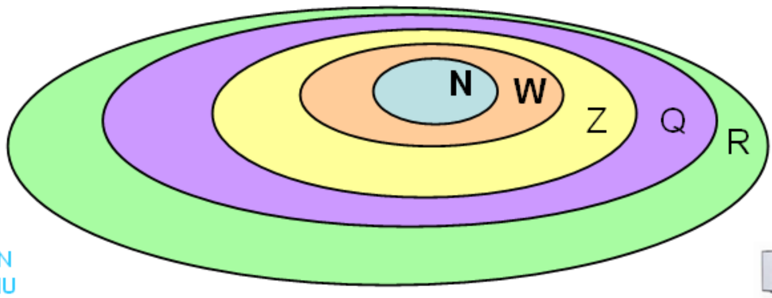


Definitions

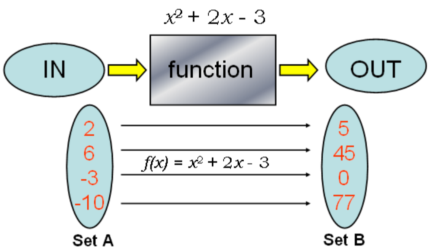
Functions and Graphs

Standard Sets

Natural Numbers, N	{ 1,2,3,4,…}
Whole Numbers, W	{ 0,1,2,3,4,…}
Integers, Z	{ …-2, -1, 0, 1, 2, …}
Rational Numbers, Q	{all number written as fractions}
Real Numbers, R	{all rational and irrational numbers}



MAIN MENU



Definitions

Function A function f is a rule which assigns each element of set A to exactly one element of set B. i.e. $f: A \rightarrow B$

Domain Set A is defined as the *domain* of the function

Co-domain Set B is defined as the *co-domain* - what could **possibly** come out of the function.

Range The subset of set B which are the actual values which come out of the function - the set of all *images* of the function.

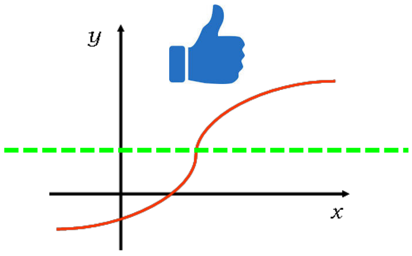
Choosing a domain and range

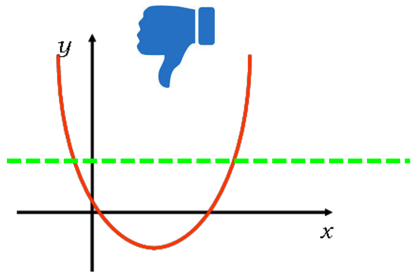
The domain of a function should be the largest set of x values possible for which the rule produces a valid result.

$f(x)$	Domain	Range
$f(x) = \sin x$	$\{x : x \in R\}$	{f(x) :
$f(x) = \sqrt{x-2}$		
$f(x) = x!$		
$f(x) = \tan x$		
$f(x) = x^2$		
$f(x) = 1 + \sin x$		
$f(x) = \sqrt{x^2}$		
$f(x) = \frac{1}{\sin x}$		

Inverse Functions

For a function to be reversible (i.e. have an inverse) it must have 'one-to-one correspondence'.





The rule which links the image $f(x)$ in set B back to set A is called the inverse of the function, denoted by f^{-1}

$$\boxed{y = f(x) \longleftrightarrow f^{-1}(y) = f^{-1}[f(x)] = x}$$

Inverse Functions - Algebraic Form

If an inverse exists, exchange x and y in the formula and then make y the subject of the formula.

Example 1: Find the inverse of the function $f(x) = 3x - 2$, and state a suitable domain and range.

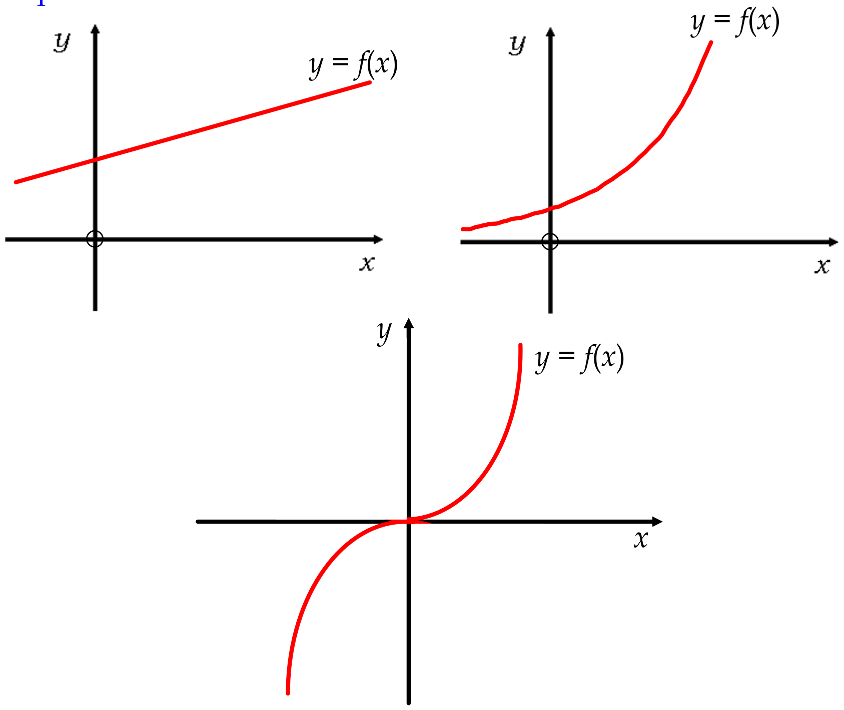
Example 2: Find the inverse of the function $f(x)=\frac{1}{2}e^{x+1}$,
and state a suitable domain and range.

Example 3: Find the inverse of the function $f(x)=-\frac{1}{8}x^3$,
and state a suitable domain and range.

Inverse Functions - Graphical Form

The graph of the inverse function is the reflection of the graph of the function in the line $y = x$

Example 4: Sketch the inverse of these functions:



Modulus Function

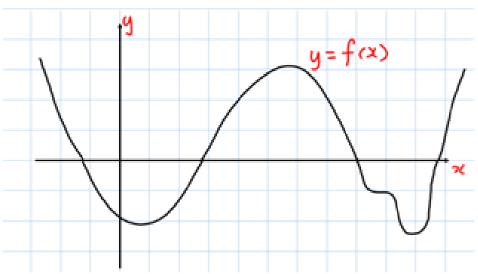
The absolute value of x is $|x|$, where $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

The graph of a modulus function is obtained by:

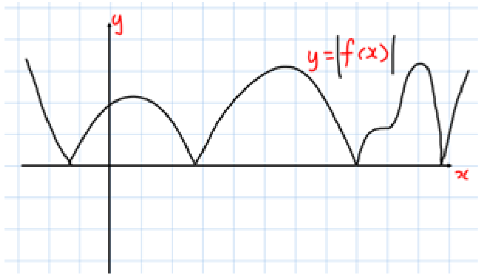
i) drawing the original function

ii) reflecting in the x-axis any part of the function which is below the x-axis.

BEFORE

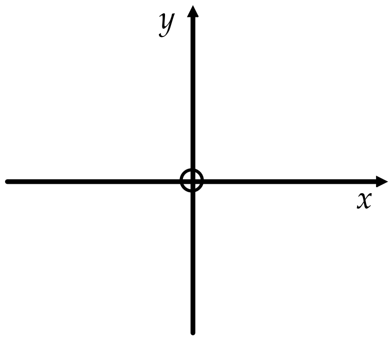


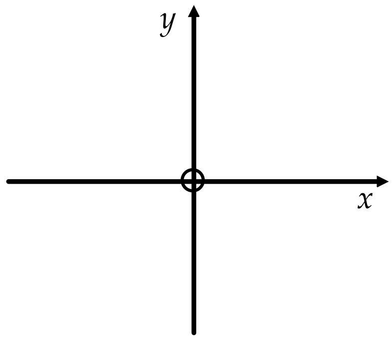
AFTER

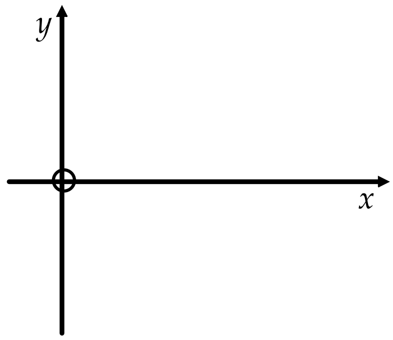


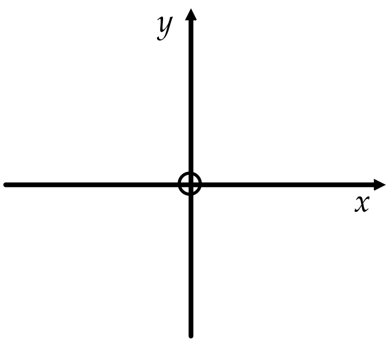
Example 5: Sketch the graphs of:

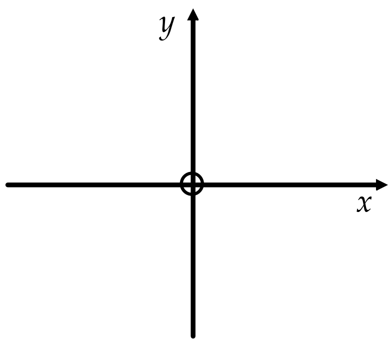
a) $y = |x|$ b) $y = |x^2 - 2|$ c) $y = |\sin 2x|$

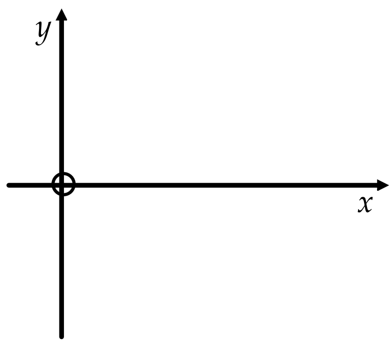












Practise: Sketch the graphs of:

1) $f(x) = |x - 1|$ 2) $f(x) = |2x - 1|$ 3) $y = |\cos x|$

4) $f(x) = |x^2 - 1|$ 5) $f(x) = |1 - x^2|$ 6) $y = |x^2|$

7) $f(x) = |x^3|$ 8) $f(x) = |\ln x|, x > 0$ 9) $y = |\tan x|$

Odd and Even Functions

A function is **even** if $f(-x) = f(x)$

- Even functions are symmetrical about the y-axis.
- Polynomial even functions contain only even powers of x.

Example 6: Prove that $f(x) = x^4 - 2x^2 + 5$ is an even function

Odd and Even Functions

A function is **odd** if $f(-x) = -f(x)$

- Odd functions have half turn symmetry about the origin.
- Polynomial odd functions contain only odd powers of x.

Example 7: Prove that $f(x) = 2x^3 + x$ is an odd function

Practise

1 **a** Prove that $2x^2 + 5$ is an even function.

b Prove that $3x^5 + 7x^3 - 4x$ is an odd function.

2 For each of the functions below, say whether it is odd or even, and sketch enough of its graph to illustrate its symmetry.

a $f(x) = x^6$ **b** $f(x) = x^5$ **c** $f(x) = \sin x$

d $f(x) = \cos x$ **e** $f(x) = x^2 - 1$ **f** $f(x) = x + x^3$

3 Classify each of the following functions as odd, even or neither.

a $f(x) = x$ **b** $f(x) = x^2 + x$ **c** $f(x) = \dfrac{x^2 + 1}{x^2}$

d $f(x) = x + \dfrac{1}{x}$ **e** $f(x) = 1 - \dfrac{1}{x}$ **f** $f(x) = \sin x \cos x$

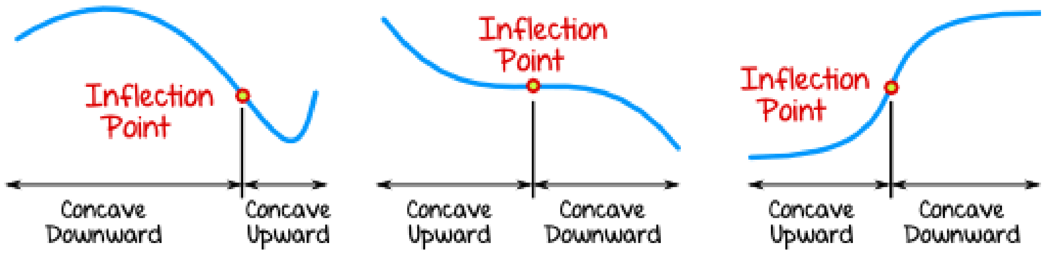
g $f(x) = \sin x + \cos x$ **h** $f(x) = x^3 + x^2$ **i** $f(x) = e^{x^2}$

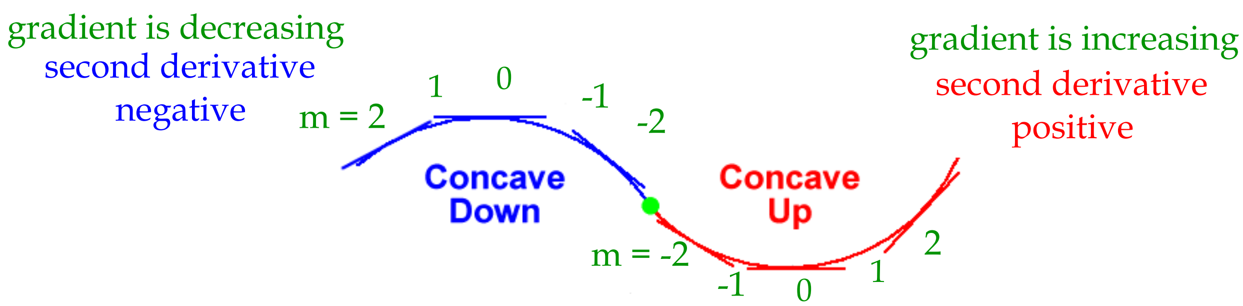
j $f(x) = e^x + e^{-x}$ **k** $f(x) = e^x - e^{-x}$ **l** $f(x) = \ln x$

Points of Inflexion

A point of inflexion can occur at a stationary point, when a function continues to increase or decrease on either side.

However, a point of inflexion can also occur simply when a curve changes in concavity.





A point of inflexion occurs when the **second derivative is zero.**

We need to examine the second derivative before and after the stationary point and check for a change of sign. This indicates a change of concavity, and therefore a point of inflexion.

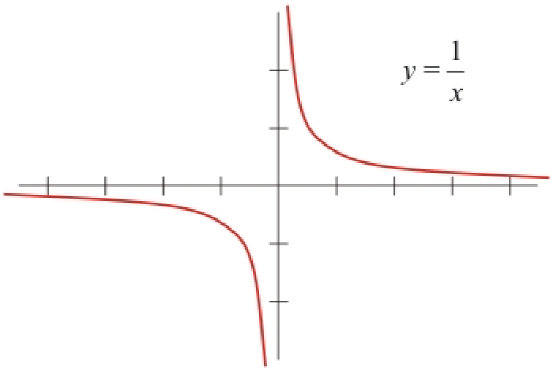
Example 8: Show if there is a point of inflexion for the functions:

a) $f(x) = x^4$ b) $f(x) = x^5$

Example 9: Find the coordinates of the point of inflexion of the function:
$f(x) = 3x^3 - 18x^2 - 7x$

Asymptotes

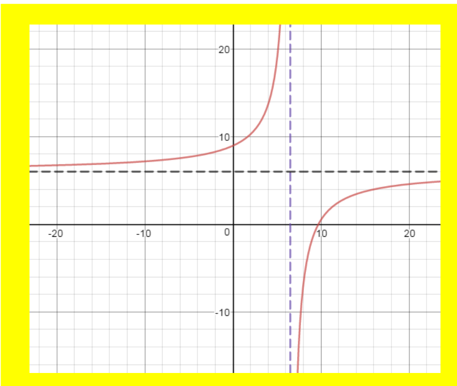
An asymptote is a (imaginary) straight line to which a curve approaches more and more closely as x or y approaches certain values.

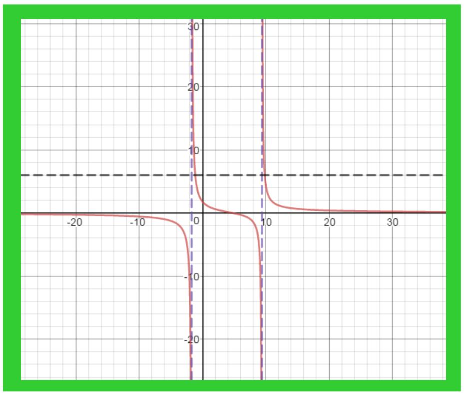


The function $y = \dfrac{1}{x}$ has:

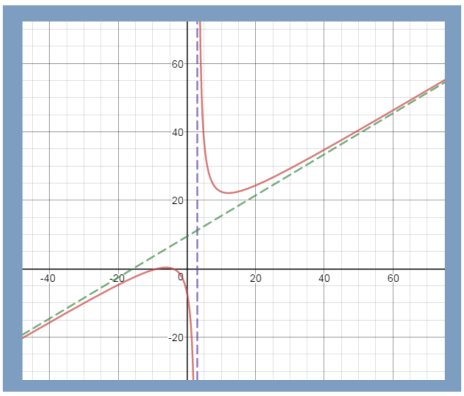
- a vertical asymptote, $x = 0$
- a horizontal asymptote, $y = 0$

Asymptotes which are neither horizontal nor vertical have the form $y = mx + c$ and are termed oblique asymptotes.





Interactive investigation of asymptotes



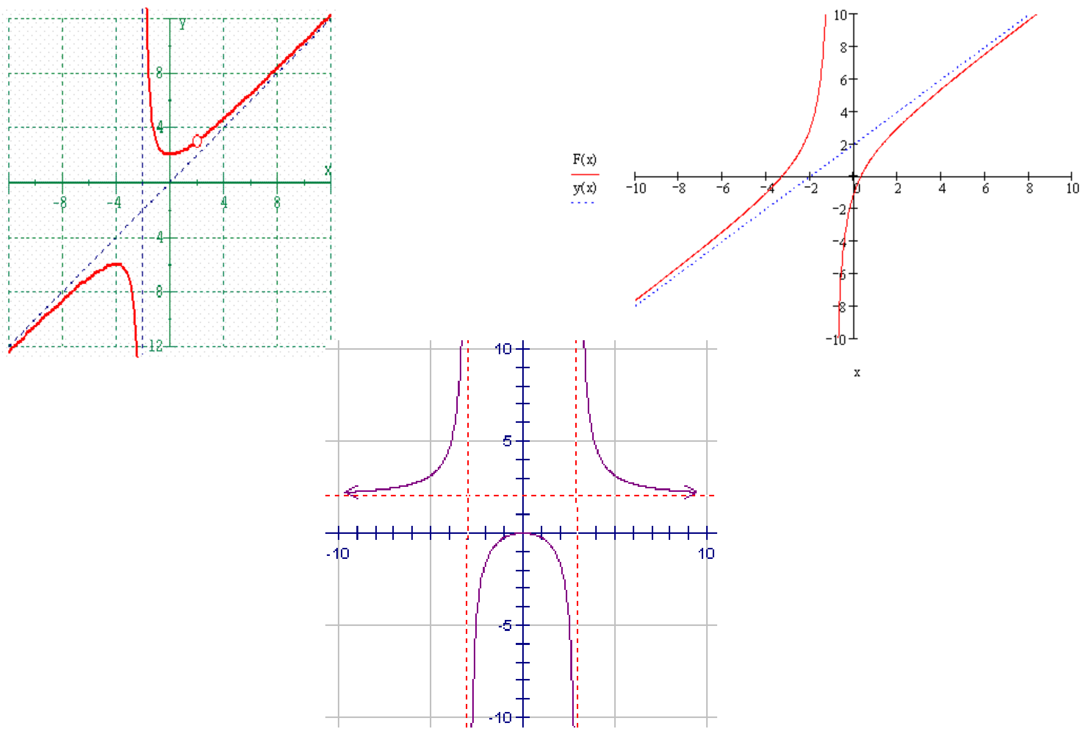
Vertical Asymptotes

For any rational function, $f(x)=\dfrac{g(x)}{h(x)}$, where g and h are polynomial functions,

vertical asymptotes, $x = a$, occur when $h(a) = 0$. i.e. the roots of the denominator

Like stationary points, asymptotes have a nature - how the curve behaves on either side of the asymptote.

An investigation of h(a$^-$) and h(a$^+$) will determine if $f(x) \to \pm\infty$



Example 10: Investigate the vertical asymptotes of $f(x) = \dfrac{x-2}{x^2+2x-3}$

Example 11: Investigate the vertical asymptotes of $f(x) = \text{cosec } x \quad [0,2\pi)$

Horizontal and Oblique Asymptotes

In general, for a rational function: $f(x) = \dfrac{g(x)}{h(x)}$

i) If order of g(x) < order of h(x) then **horizontal asymptote**, $y = 0$

ii) If order of g(x) = order of h(x) then **horizontal asymptote**, $y = c$

iii) If order of g(x) = order of h(x) +1 then **oblique asymptote**, $y = mx + c$

Example 12: Investigate the non-vertical asymptotes of $f(x)=\dfrac{x-2}{(x+3)(x-1)}$

Algebraic division of two functions of equal order always produces a constant and a remainder fraction in x, hence the horizontal asymptote is always y = c.

Example 13: Investigate the non-vertical asymptotes of $f(x)=\dfrac{2x+3}{x-1}$

Example 14: Investigate all the asymptotes of $f(x)=\dfrac{2x^2+3x-2}{x+3}$

Graph Sketching

To sketch a graph as fully as possible consider:

a) asymptotes and their nature

b) stationary points and their nature

c) intercepts of x and y axes, where they exist

The Second Derivative

When considering the nature of stationary point, there is an alternative technique to using a nature table - the second derivative.

For any function:

First derivative $\longrightarrow$ rate of change of the function (gradient)

Second derivative $\longrightarrow$ rate of change of gradient (concavity)

If the value of the second derivative is positive $\longrightarrow$ MIN TP

If the value of the second derivative is negative $\longrightarrow$ MAX TP

(If the value of the second derivative is zero, revert to a nature table.)

Example 15: Investigate the nature of the stationary point of $f(x) = x^2 + x - 12$

Example 16: Find the stationary points and their nature, given:

$$f(x) = x^3 + 3x^2$$

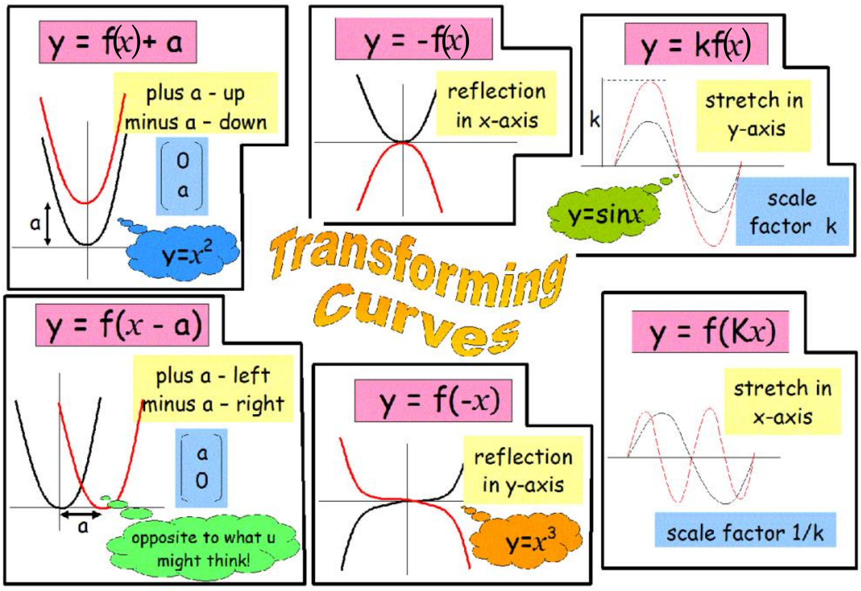
Example 17: Sketch the graph of $f(x) = \dfrac{5}{x - 3}$

Example 18: Sketch the graph of $f(x) = \dfrac{2x^2 + x - 1}{x - 1}$

Graphs of Related Functions

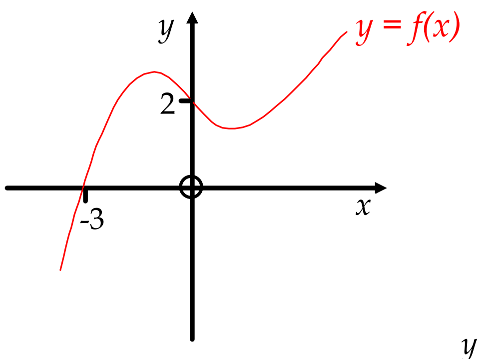
We can combine existing knowledge from Higher with modulus and inverse functions to develop the theory of related functions.

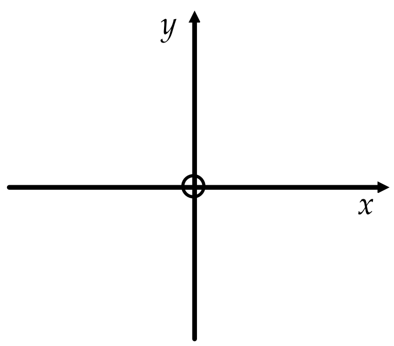
Review of Higher Work

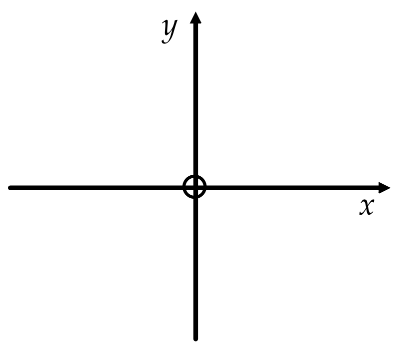


Example 18: Given the function $y = f(x)$ below, sketch the curves of

(a) $y = 2f(x)$ and (b) $y = f(\frac{x}{3})$



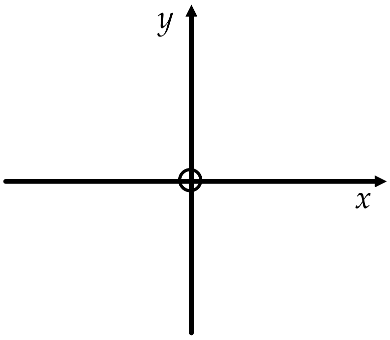


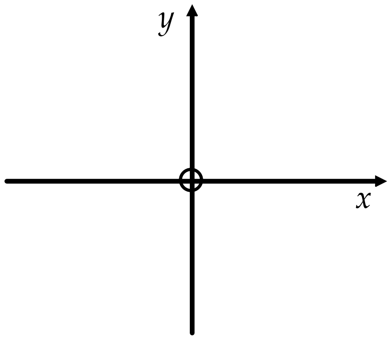


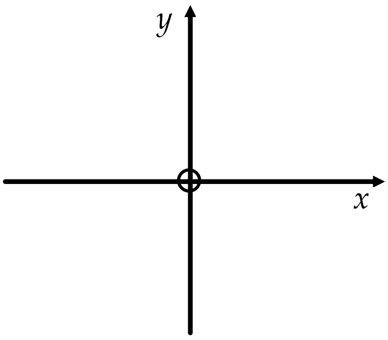
Example 20: Given $f(x) = e^x$, sketch the graphs of :

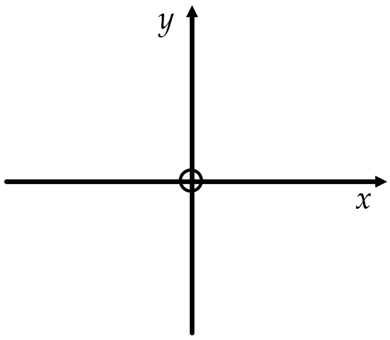
(a) $y = f(x)$ (b) $y = f(-x)$ (c) $y = f(x + 2)$

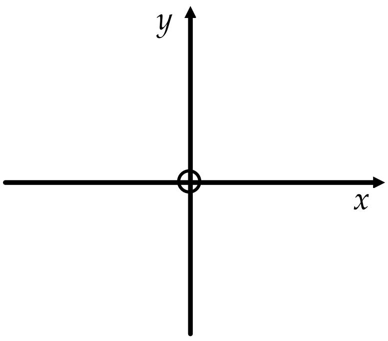
(d) $y = f^{-1}(x+2)$ (e) $y = |f^{-1}(x+2)|$ (f) $y = |f^{-1}(x+2)| + 3$

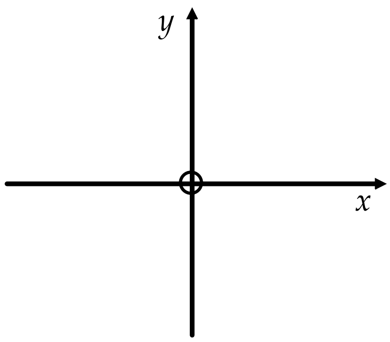












Past Paper Questions

Functions and Graphs

2001

A function is defined by $f(x)=\frac{x^2+6x+12}{x+2}$, $x\neq -2$.

(a) Express $f(x)$ in the form $ax+b+\frac{b}{x+2}$ stating the values of a and b.

(b) Write down an equation for each of the two asymptotes.

(c) Show that $f(x)$ has two stationary points.

Determine the coordinates and the nature of the stationary points.

(d) Sketch the graph of f.

(e) State the range of values of k such that the equation $f(x)=k$ has no solution.

(2, 2, 4, 1, 1 marks)

2002

Express $\frac{x^2}{(x+1)^2}$ in the form $A+\frac{B}{x+1}+\frac{C}{(x+1)^2}$, $(x\neq -1)$, stating the values of the constants A, B and C.

A curve is defined by $y=\frac{x^2}{(x+1)^2}, (x\neq -1)$.

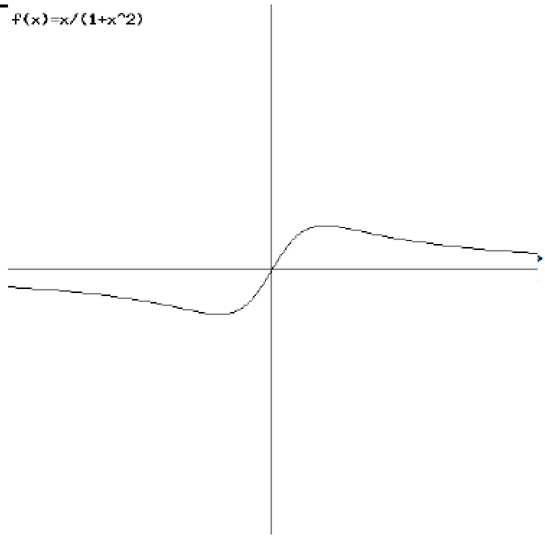
(i) Write down equations for its asymptotes.

(ii) Find the stationary point and justify its nature.

(iii) Sketch the curve showing clearly the features found in (i) and (ii).

(3, 2, 4, 2 marks)

2003



The diagram shows the shape of $y=\frac{x}{1+x^2}$.

Obtain the stationary points of the graph.

Sketch the graph of $y=\left|\frac{x}{1+x^2}\right|$ and identify its three critical points.

(4, 3 marks)

2004

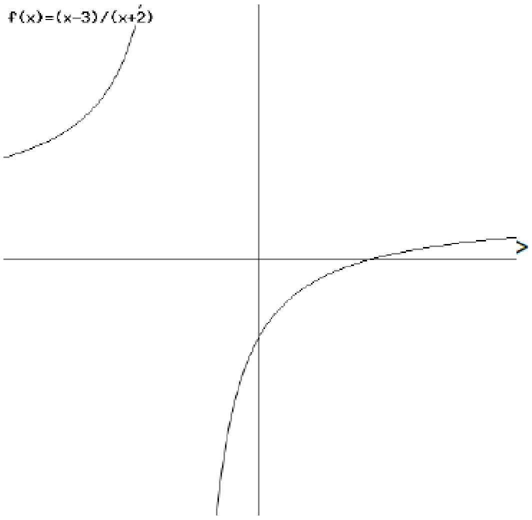
Determine whether the function $f(x)=x^4\sin 2x$ is odd, even or neither.

Justify your answer.

(3 marks)

2004

The function f is defined by $f(x)=\frac{x-3}{x+2}$, $x\neq -2$, and the diagram shows part of its graph.



(a) Obtain algebraically the asymptotes of the graph of f.

(b) Prove that f has no stationary values.

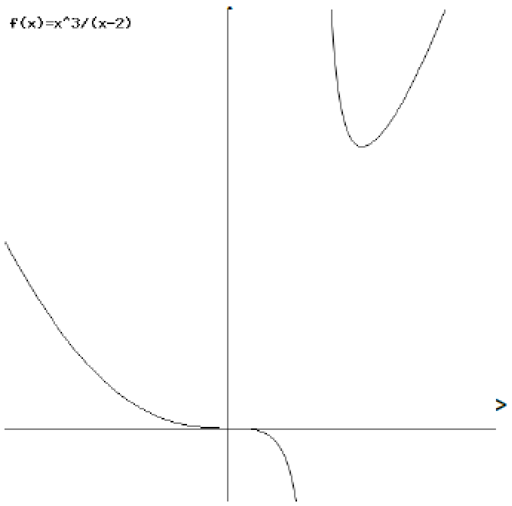
(c) Does the graph of f have any points of inflexion? Justify your answer.

(d) Sketch the graph of the inverse function f^{-1}.

State the asymptotes and domain of f^{-1}. **(3, 2, 2, 3 marks)**

2005

The diagram shows part of the graph of $y=\frac{x^3}{x-2}$, $x\neq 2$.



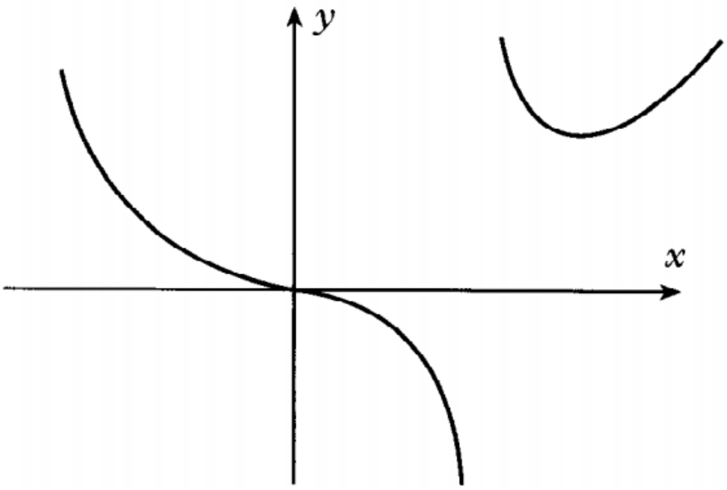
(a) Write down the equation of the vertical asymptote.

(b) Find the coordinates of the stationary points of the graph of $y=\frac{x^3}{x-2}$.

(c) Write down the coordinates of the stationary points of the graph of $y=\left|\frac{x^3}{x-2}\right|+1$.

(1, 4, 2 marks)

2006



The diagram shows part of the graph of a function f which satisfies the following conditions:

(i) f is an even function;

(ii) two of the asymptotes of the graph $y = f(x)$ are $y = x$ and $x = 1$.

Copy the diagram and complete the graph. Write down equations for the other two asymptotes.

(3 marks)

2009

The function $f(x)$ is defined by $f(x) = \dfrac{x^2 + 2x}{x^2 - 1}$ $(x \neq \pm 1)$.

Obtain equations for the asymptotes of the graph of $f(x)$.

Show that $f(x)$ is a strictly decreasing function.

Find the coordinates of the points where the graph of $f(x)$ crosses

(i) the x-axis and

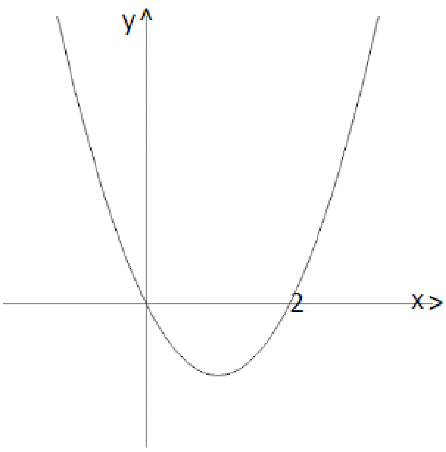
(ii) the horizontal asymptote.

Sketch the graph of $f(x)$, showing clearly all relevant features. **(3, 3, 2, 2 marks)**

2010

The diagram below shows part of the graph of a function $f(x)$.

State whether $f(x)$ is odd, even or neither. Fully justify your answer.



(3 marks)

2012

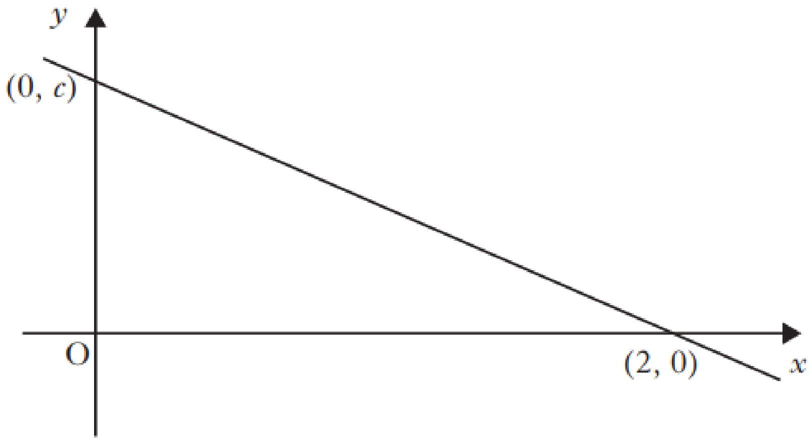
A function is defined by $f(x)=|x+2|$ for all x.

(a) Sketch the graph of the function for $-3 \le x \le 3$.

(b) On a separate diagram, sketch the graph of $f'(x)$. **(2, 2 marks)**

2013

Part of the straight line graph of a function $f(x)$ is shown.



(a) Sketch the graph of $f^{-1}(x)$, showing points of intersection with the axes.

(b) State the value of k for which $f(x)+k$ is an odd function.

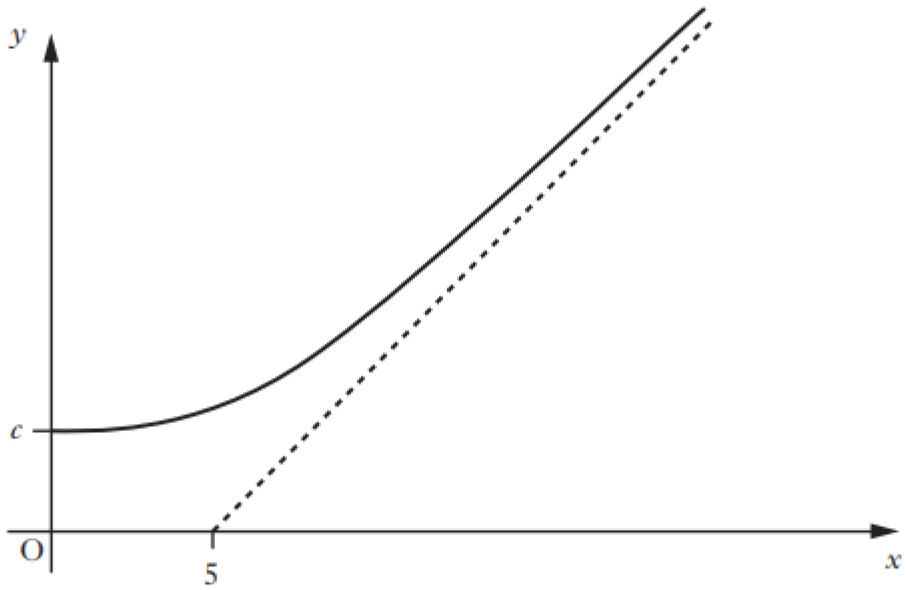
(c) Find the value of h for which $|f(x+h)|$ is an even function. **(2, 1, 2 marks)**

2014

The function $f(x)$ is defined for all $x \geq 0$.

The graph of $y = f(x)$ intersects the y-axis at $(0, c)$ where $0 < c < 5$.

The graph of the function and its asymptote, $y = x - 5$, are shown below.



(a) Copy the diagram above.

On the same diagram, sketch the graph of $y = f^{-1}(x)$.

Clearly show any points of intersection and any asymptotes.

(b) What is the equation of the asymptote of the graph $y = f(x+2)$?

(c) Why does your diagram show that the equation $x = f(f(x))$ has at least one solution?

(4, 1, 1 marks)

2015

For some function, f, define $g(x) = f(x) + f(-x)$ and

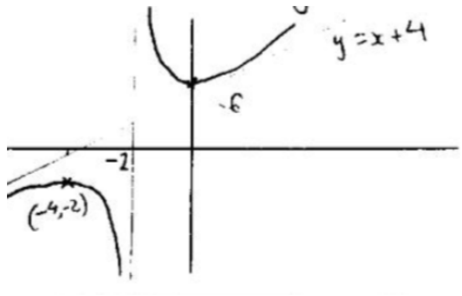
$$h(x) = f(x) - f(-x).$$

Show that $g(x)$ is an even function and that $h(x)$ is an odd function.

Hence show that $f(x)$ can be expressed as the sum of an even and an odd function.

(4 marks)

Past Paper Answers
Functions and Graphs

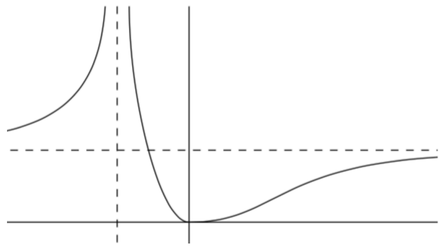


2001 (Q.A8)

(a) $f(x) = x + 4 + \frac{4}{x+2}$ so a = 1 and b = 4.
(b) V.A: x = -2, O.A: y = x+ 4
(c) SPs: Min TP at (0,6) Max TP at (-4,-2)
(d) see diagram
(e) -2 < k < 6

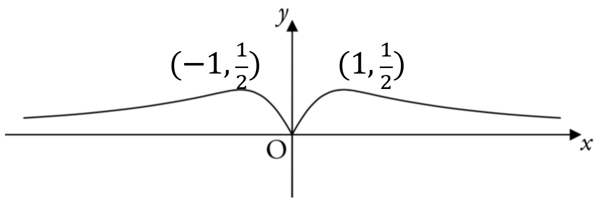
2002 (Q.A8)

(a) $y = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$ i.e. A = 1, B = -2, C = 1
(b)(i) V.A.: x = -1, H.A.: y = 1
(ii) Min TP at (0,0)
(iii) see diagram



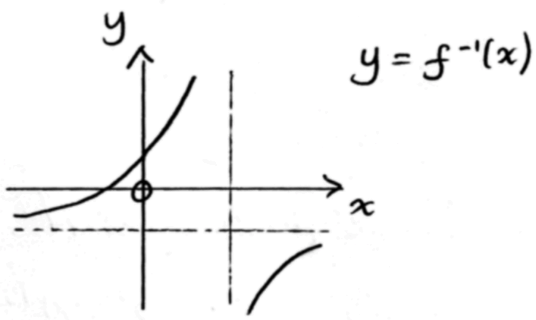
2003 (Q.A7)

(a) SPs: $\left(1,\frac{1}{2}\right)$, $\left(-1,-\frac{1}{2}\right)$
(b) see diagram



2004 (Q.10)

Function is odd as *f(-x) = -f(x)*



2004 (Q.13)

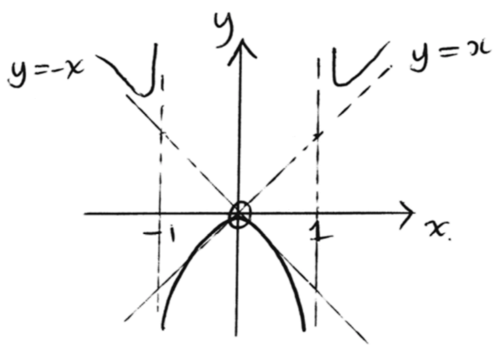
(a) V.A.: x = -2, H.A.: y = 1
(b) No stationary points as *f ′(x)* is never zero.
(c) No points of inflexion.
(d) See diagram: V.A.: x=1, H.A.: y = -2
Domain: $\{x: x \in \mathbb{R}, x \neq 1\}$

2005 (Q.10)

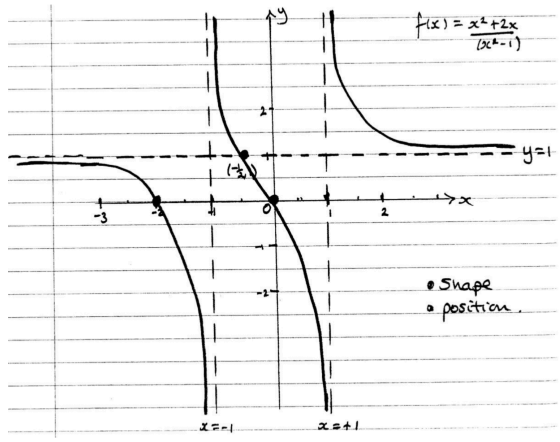
(a) V.A.: x = 2
(b) SPs: (0,0) and (3,27)
(c) SPs: (0,1) and (3,28)

2006 (Q.12)

See diagram. Asymptotes y = -x and x = -1



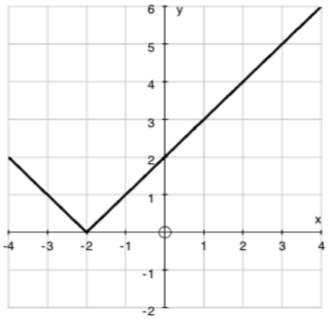
2009 (Q.13)
(a) V.A.s: x = -1, x = 1, H.A.: y = 1
(b) As $f'(x)$ always negative, function always decreasing.
(c) (i) x= -2, x=0, (ii) $x = -\frac{1}{2}$
(d) see diagram



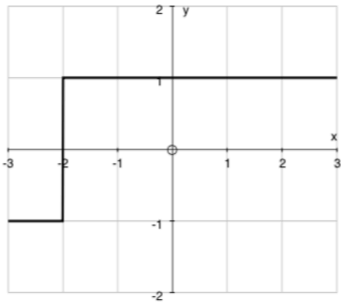
2010
Function is neither even nor odd

2012 (Q.7)

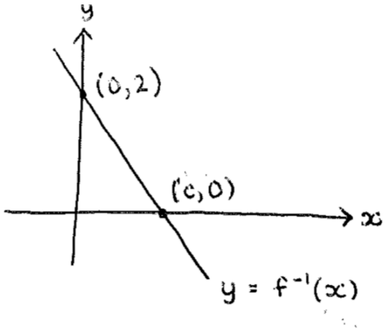
(a)



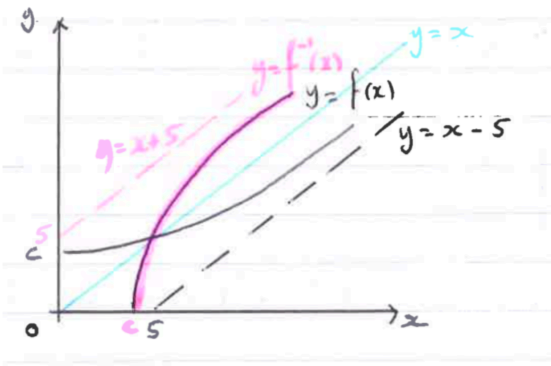
(b)



2013 (Q.13)
(a) see diagram
(b) k = -c
(c) h = 2



2014 (Q.11)
(a) see diagram
(b) y = x – 3
(c) As $f^{-1}(x) = f(x)$ there will be a solution on line y=x



2015
Proofs

Advanced Higher Mathematics
Formal Homework Assignment

Functions and Graphs

1. Prove whether the following functions are odd, even or neither.

(a) $f(x) = x^6 - 6x^2 - 1$ (b) $f(x) = 3\sin x$

2. Find the domain and range of the following functions

(a) $f(x) = \sqrt{x-3}$ (b) $f(x) = \dfrac{2}{\cos x}$

3. Given $f(x) = 2x^2 - 5x - 3$, sketch the graph of $h(x) = |f(x)|$

4. Identify the coordinates and nature of any points of inflexion on the curve defined by the function $y = 4x^3 - 3x^2$.

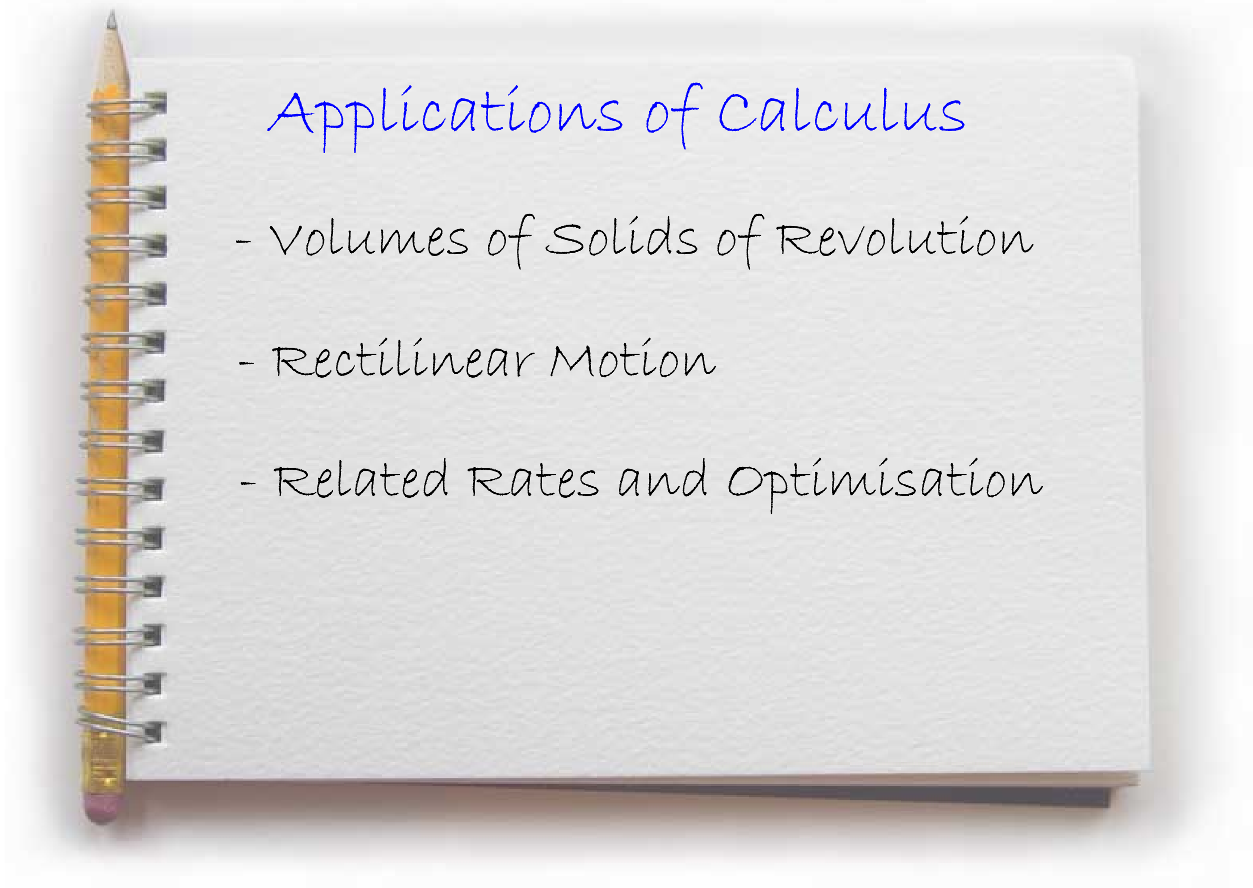
5. A function is defined by the equation $y = \dfrac{5}{x^2 + 3x - 4}$

(a) Write down the equations of the vertical asymptotes and the coordinates of the y-intercept.
(b) Find the equation of the non-vertical asymptote.
(c) Sketch the curve showing all the main features.

6. A function is given as $f(x) = \dfrac{x^2 - x - 5}{x+2}$, $x \neq -2$

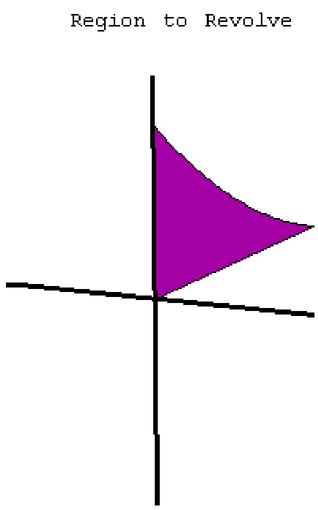
(a) Write down the equation of the vertical asymptote and the coordinates of the *y*- intercept.
(b) Find the equation of the non-vertical asymptote.
(c) Using calculus, find the coordinates and justify the nature of any stationary points.
(d) Sketch the curve showing all the main features.

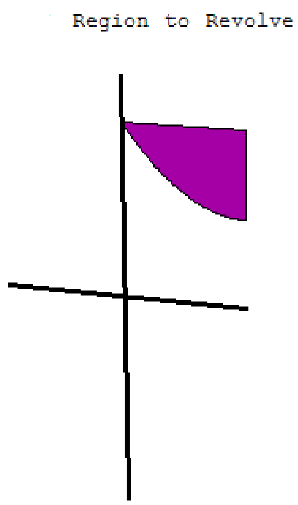
7. Reflect on your understanding of functions and graphs, particularly in relation to: (a) connections to your existing knowledge and
(b) any elements of this new area which interest you.



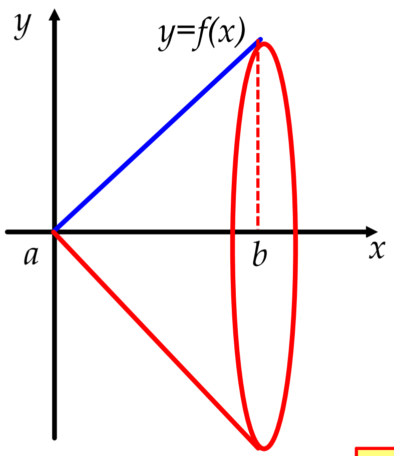
Volumes of Solids of Revolution

The area under a curve $y=f(x)$ (or between 2 curves) can be rotated about the x-axis to form a solid of revolution.





The line $y=x$ is the simplest example:



A straight line through the origin, where rotated about the x-axis, forms a cone.

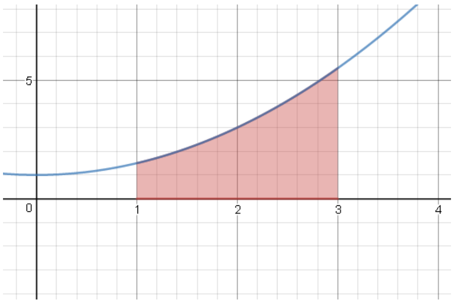
The volume of revolution here is regarded as an infinite number of circular disks which are added together.

$$\boxed{V = \int_a^b \pi y^2 \, dx}$$

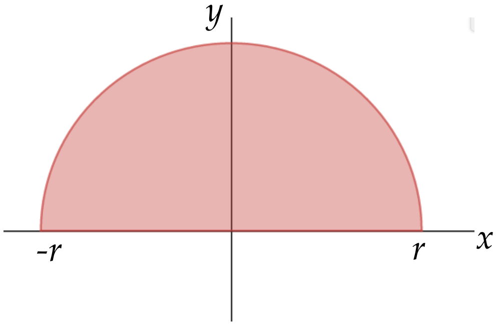
For the solid above, $V = \int_a^b \pi x^2 \, dx$

Example 1: A section of the line $y = x$ between $x = 0$ and $x = 4$ is rotated around the x-axis. Find the volume of the resultant solid.

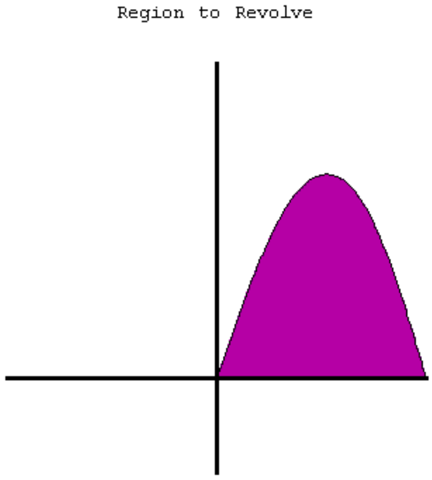
Example 2: The area between the curve $y = \frac{1}{2}x^2 + 1$ and the x-axis between x=1 and x=3 is rotated around the x-axis.
Find the volume of the resultant solid.

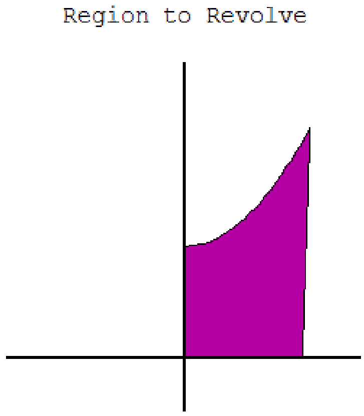


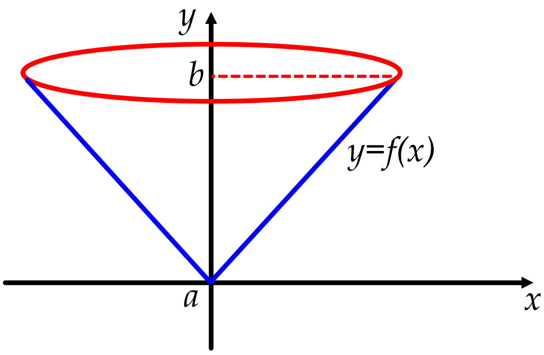
Example 3: Calculate the volume of the solid formed when a semicircle radius r units is rotated about the x-axis.



A similar process can be used to determined the volume of revolution for curves which are rotated around the y-axis.



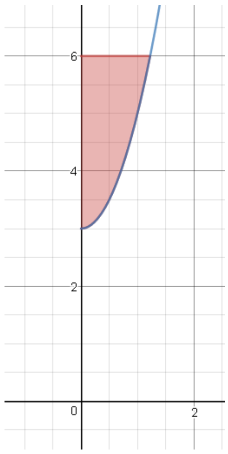




The volume of the solid becomes:

$$\boxed{V = \int_a^b \pi x^2 dy}$$

Example 4: The area between the curve $y = 2x^2+3$ and the line $y=6$ is rotated around the y-axis Find the resultant volume created.



Rectilinear Motion

As the velocity of a body is the rate of change of displacement, s(t), then:

$$v(t) = s'(t)$$

Also, as acceleration is the rate of change of velocity, then:

$$a(t) = v'(t)$$

These relationships are reversed by integration.

$$v(t) = \int a(t)dt \qquad s(t) = \int v(t)dt$$

As with other differential equations, solutions may be general or particular depending on available data.

Example 5: A particle starts from rest and its acceleration, a, after t seconds is given by: $a = 6t + 2$ ms^{-1}.
Determine the velocity of the particle and its displacement from the start after 3 seconds.

Example 6: Two particles, A and B, move towards each other in a straight line and collide when $a_A = a_B$.
At the point of collision, v_A=120 ms^{-1} and v_B=140 ms^{-1}.
Given that $a_A(t) = 8t$ ms^{-2} and $a_B(t) = 5(t+3)$ ms^{-2} ,
calculate the initial velocity of both particles.

Related Rates

It is possible to derive expressions for the rate of change of connected quantities, using an application of the chain rule.

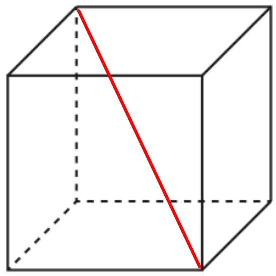
If V is the volume of an expanding sphere of radius r at time t, then we can derive the rate of change of the radius as it expands.

$$\frac{dr}{dt} = \frac{dr}{} \times \frac{}{dt}$$

As we know the formula for V, we can find $\frac{dV}{dr}$ and, if we know the rate of change of volume, we can then calculate the rate of change of radius.

Example 7: A sphere of radius r cm is expanding at the rate of 100cm^3/s. Calculate the rate of increase of the radius when r = 6cm.

Example 8: A block of ice is in the shape of a cube.
At what rate is the space diagonal decreasing if the edges of the cube are melting at a rate of 2cm/s?

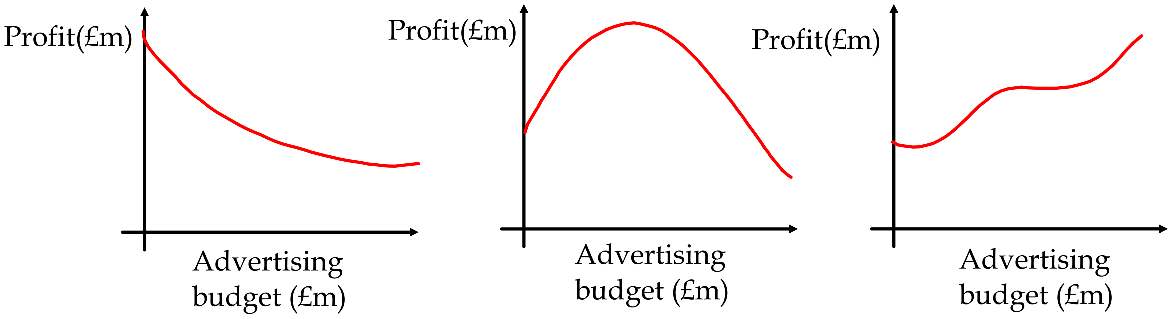


Example 9: A cylindrical tank of diameter 20cm is filled at a rate of 2 litres per second. How fast does the water level rise?

Optimisation

Optimisation is the process of maximising or minimising the value of a function (quantity), usually within a given domain.

In addition to studying stationary points, it may be necessary to consider the end-points of a function, as these may provide an optimal solution.

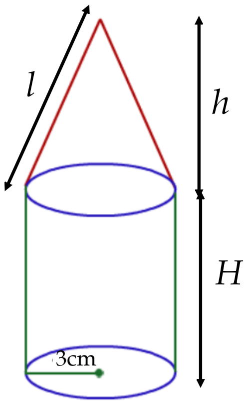


Example 10: A communications cable has a copper core with a sheath of insulating material. If x is the ratio of the radius of the core to the thickness of the insulating sheath, the speed of the signal along the cable is given by:

$$S = 8x^2 \ln\left(\frac{1}{2x}\right)$$

Find the value of x that gives maximum signal speed.

Example 11: A pencil case is in the shape of a cylinder with a conical end. The total volume of the container is 900cm³.



a) Find an expression for H in terms of h.

b) Show that the surface area, Scm², of the container is given by:

$$S = 600 - 2\pi h + 9\pi + 3\pi\sqrt{h^2 + 9}$$

c) Find the value of h for which the total surface area is a minimum.

Note: $V_{cone} = \frac{1}{3}\pi r^2 h$ Curved $SA_{cone} = \pi r l$

Example 12:

The turning effect, T, of a power boat is given by the formula

$$T = 8\cos x \sin^2 x, \quad 0 < x < \frac{\pi}{2},$$

where x is the angle (in radians) between the rudder and the central line of the boat. Find the size of x which maximises the turning effect.

Past Paper Questions

Applications of Calculus

2004

② A solid is formed by rotating the curve $y = e^{-2x}$ between $x = 0$ and $x = 1$ through $360°$ about the x-axis. Calculate the volume of the solid that is formed. **(5 marks)**

2007

② Use the substitution $u = 1 + x^2$ to obtain $\int_0^1 \frac{x^3}{\left(1+x^2\right)^4}\,dx$.

A solid is formed by rotating the curve $y = \frac{x^{3/2}}{\left(1+x^2\right)^2}$ between $x = 0$ and $x = 1$ through $360°$ about the x-axis. Write down the volume of this solid. **(5, 1 marks)**

2010

② A new board game has been invented and the symmetrical design on the board is made from 4 identical "petal" shapes. One of these petals is the region enclosed between the curves $y = x^2$ and $y^2 = 8x$ as shown shaded in diagram 1 below.

Calculate the area of the complete design, as shown in diagram 2.

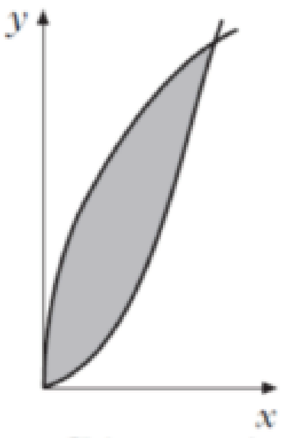


Diagram 1

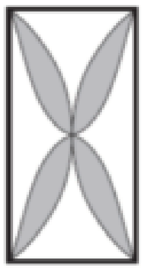


Diagram 2

The counter used in the game is formed by rotating the shaded area shown in diagram 1 above, through $360°$ about the y-axis.
Find the volume of plastic required to make one counter. **(5, 5 marks)**

2013

① The velocity, v, of a particle P at time t is given by

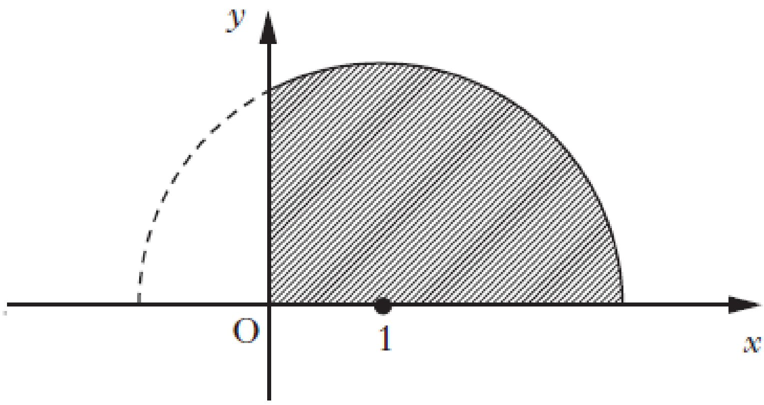
$$v = e^{3t} + 2e^{t}$$

(a) Find the acceleration of P at time t.

(b) Find the distance covered by P between $t = 0$ and $t = \ln 3$. **(2, 3 marks)**

2014

① A semi-circle with centre $(1, 0)$ and radius 2, lies on the x-axis as shown.
Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis.



(5 marks)

2012

① The radius of a cylindrical column of liquid is decreasing at the rate of $0 \cdot 02ms^{-1}$, while the height is increasing at the rate of $0 \cdot 01ms^{-1}$.
Find the rate of change of the volume when the radius is $0 \cdot 6$ metres and the height is 2 metres.
(Recall that the volume of a cylinder is given by $V = \pi r^2 h$.) **(5 marks)**

Past Paper Answers
Applications of Calculus

2004 (Q.11)
$V = \frac{\pi}{4} - \frac{\pi}{4e^4}$ = 0.7710 to 4 dp

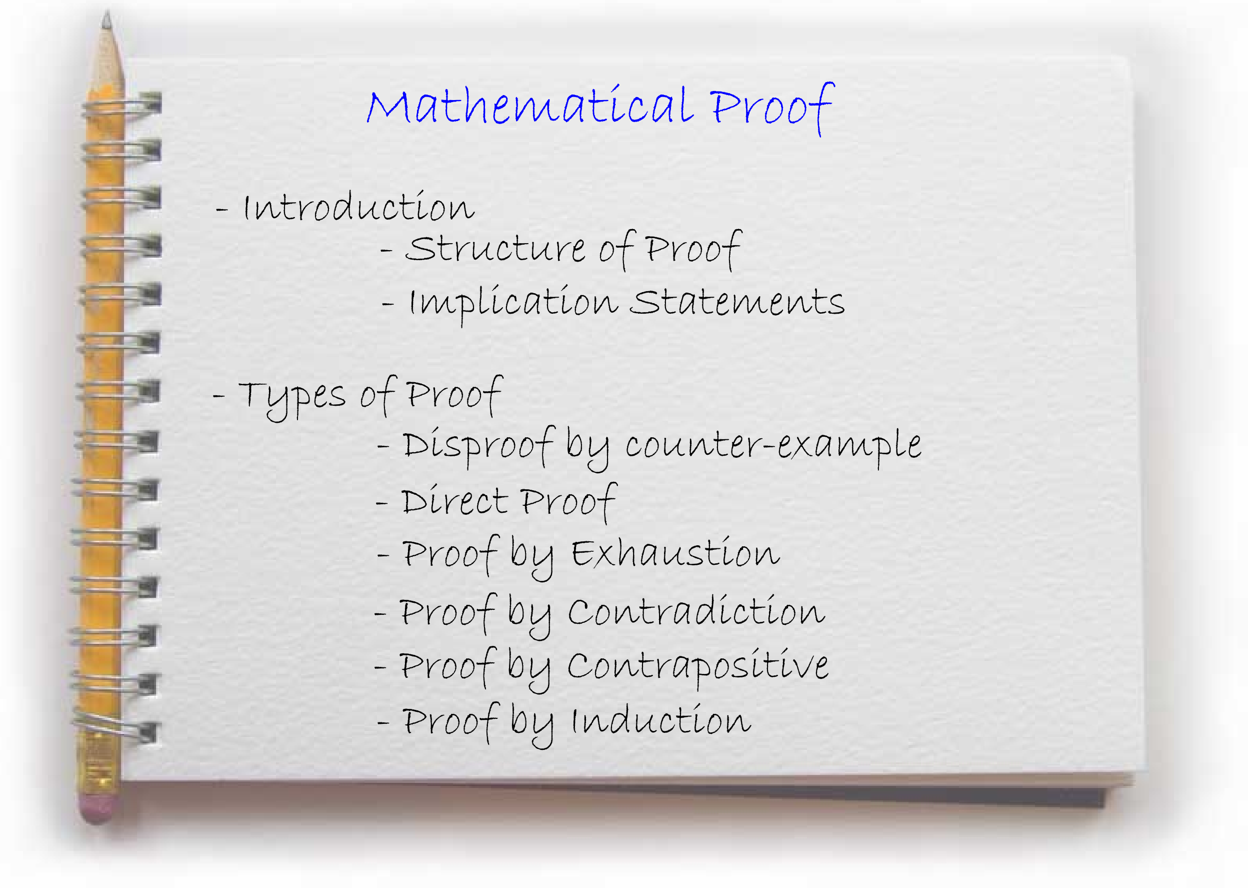
2007 (Q.10)
(a) $\frac{1}{24}$ (b) V = $\frac{\pi}{24}$ cubic units

2010 (Q.15)
(a) $A = \frac{32}{3}\ or\ 10\frac{2}{3}$ square units (b) $V = \frac{24\pi}{5}$ cubic units

2012 (Q.12)
$\frac{dV}{dt} = -0.0444\pi\ m^3/s$

2013 (Q.4)
(a) $a(t) = 3e^{3t} + 2e^t$
(b) $\frac{38}{3}\ or\ 12\frac{2}{3}$ units

2014 (Q.10)
$V = 9\pi$ units



Introduction

Mathematics depends on the absolute certainty that the rules used in any branch of mathematics are precise and true. If a false assumption is made, any new rules or theories produced become invalid. Every brick in the wall must be solid.

A mathematical principle can only be said to be true if it can be proved to be true at all times - that is, without exception.

The Structure of Proof

DEFINITION

The precise meaning to a word or concept

A *perfect square* is any natural number which is the square of another natural number..........1.

CONJECTURE

A precise statement or proposition

16 is a perfect square...........2.

PROOF

A complete logical argument
proving the conjecture is true

We know that 4 is a natural number. 4 x 4 = 16. 16 is a natural number.
Therefore 16 is a perfect square ....3.

THEOREM

A proven conjecture

16 is a perfect square...........4.

Implication Statements

Implication statements usually read, "If......then....."

=> is the mathematical implication symbol.

e.g. If $x = 4$ then $x^2 = 16$

becomes $x = 4 \Rightarrow x^2 = 16$ ____(1)

The converse produces the symbol <=, "..implied by.."

e.g. $x + 2 = 7 \Leftarrow x = 5$ ____(2)

Some statements are true both ways - these are said to be equivalent.

e.g. $x + 2 = 7 \Leftrightarrow x = 5$

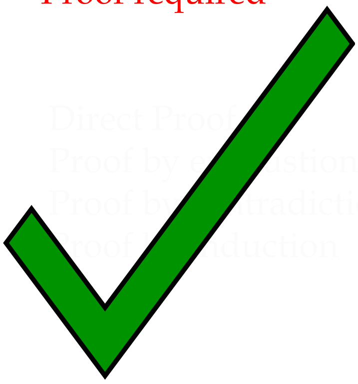
$x + 2 = 7$ if and only if (iff) $x = 5$

Note: (1) is not equivalent

Methods of Proof

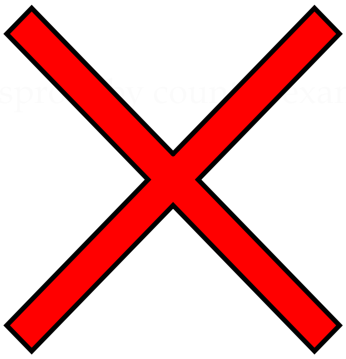
A conjecture is either
TRUE or UNTRUE

Proof required



Direct Proof
Proof by exhaustion
Proof by contradiction
Proof by induction

Only one example to
demonstrate is required.



Disproof by counter-example

Disproof by counter-example

If a conjecture can be proven untrue even in one instance, then the entire conjecture is false.

Example 1:
Disprove the following statement by finding a counter-example.
"For all real numbers a and b, if $b^2 > a^2$, then $b > a$"

Let $a = 3$ and $b = -4$.

Then $a^2 = 3^2 = 9$ and $b^2 = (-4)^2 = 16$, and so $b^2 > a^2$.

But $b < a$ since $-4 < 3$.

Therefore the conjecture is false. $\square$

Example 2:

Disprove the following statement by finding a counter-example.

"For all real numbers x, y and z, if $x > y$, then $xz > yz$."

Let $x = 10$, $y = 8$ and $z = -3$.

Then $x > y$, since $10 > 8$.

But $xz = 10 \times (-3) = -30$ and $yz = 8 \times (-3) = -24$.

$-30 < -24$, and hence $xz < yz$.

Therefore the conjecture is false. $\square$

Direct Proof

This usually involves algebraic manipulation to arrive at a general rule.

It has a common shape:

~ any number of linked implications

~ given implies goal

~ and if given is true

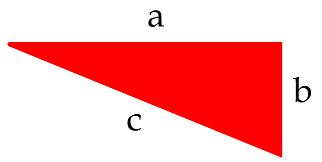
~ this proves goal to be true

Example 3:

Prove that the product of two consecutive numbers is always even.

Example 4:

Provide a direct proof of Pythagoras' Theorem



Use direct proof to prove the following conjectures

1) Prove that if n is even then $7n + 4$ is even, $n \in \mathbb{N}$

2) Prove that if m is even and n is odd then $m + n$ is odd, $m,n \in \mathbb{N}$

3) Prove that if a, b and c are integers such that a divides b and b divides c then a divides c.

4) Prove than $n^3 - n$ is always divisible by 6.

5) Prove that $6^n + 4$ is always divisible by 10, $n \in \mathbb{N}$
   (consider the units digit only)

6) By considering the factors of $3^{2n} - 1$,
   prove that $3^{2n} + 7$ is always divisible by 8, $n \in \mathbb{N}$

7) If n is an odd integer, prove that $n^2 - 1$ is divisible by 8.

Proof by Exhaustion

Where a proof involves the consideration of a (usually) small number of possibilities, this method shows the conjecture is true for each of these possibilities.

Example 5:

Prove that every integer which is a perfect cube is either a multiple of 9, or 1 more or less than a multiple of 9.

Example 6:

Prove than if n is a positive integer, then n^7-n is divisible by 7.

Proof by Contradiction

It may be impossible, or very difficult, to prove a conjecture directly. It may be easier to assume the logical negation (opposite!) of the result we wish to prove, and reach a contradiction, proving that the original conjecture must be true.

That is, when trying to prove:
"If P, then Q"

the method of contradiction states:
"Assume P and Not Q"

For any statement Q, not Q (~Q) is its negation.

Negation Statements

The negation of "all" is "some".

The negation of "some" is "no".

The negation of "no" is "some".

The negation of "If a then b" is "Assume a and ~b".

Example 7: Write the negations of:

(a) All men cry
(b) Some women can't read maps.
(c) No prime numbers are even.
(d) If the sum of two real numbers is irrational then at least one of the numbers must be irrational.

Proof by Contradiction

Example 8: Prove that if a and b are integers and ab is an odd integer, then a and b must both be odd.

Example 9:

Prove that if $n^3 + 5$ is odd, then n is even, $n \in Z$.

Example 10: Prove that $\sqrt{2}$ is irrational.

Proof by Contrapositive

Considering the statement,

$$\mathbf{p} \Rightarrow \mathbf{q}$$

we can create a logically equivalent statement using negations:

$$\mathbf{\sim q} \Rightarrow \mathbf{\sim p}$$

$$(\text{NOT } q \Rightarrow \text{NOT } p)$$

If we can prove this conjecture, then, logically,
the original conjecture must also be true.

Example 11: Prove that if 5 is a factor of n^2 then 5 is a factor of n.

Example 12: Prove that if n^3 is not even then n is not even.

Sigma Notation

Sigma notation can be used to represent a mathematical series, if the n^{th} term formula is known.

Example 13a: Write in sigma notation: $3 + 7 + 11 + 15 + \ldots\ldots$

Similarly, we can expand an expression in sigma notation:

Example 13b: Expand fully: $\sum_{k=1}^{6} k^3 - k$

Practise: Write in sigma notation:

(a) 11 + 18 + 25 + 32 + ... (b) 7 + 12 + 17 + 22 + ... + 52

Expand:

(c) $\sum_{n=1}^{\infty} 6n - 1$ (d) $\sum_{n=1}^{8} 3n + 7$ (e) $\sum_{k=4}^{7} 2k^2 - 3k + 1$

Formulae for Common Series

There are three common series for which we can use standard formulae:

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$\boxed{\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}}$$

Calculating the Value of a Finite Series

Example 14: Evaluate:

(a) $\displaystyle\sum_{k=1}^{5} 2k$

(b) $\displaystyle\sum_{k=1}^{8} 5k-2$

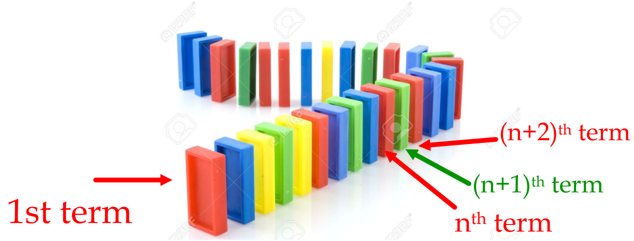
Example 15: Evaluate: $\sum_{k=1}^{10} k^3 + 3k^2 - 2k + 5$

Example 16: Evaluate: $\sum_{k=10}^{20} k^2 + 2k - 3$

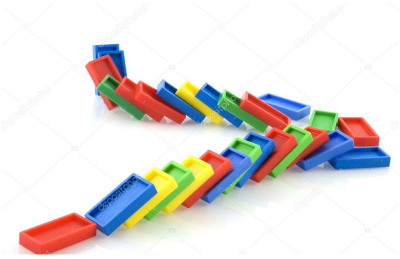
Proof by Induction

The method of proof by induction involves a demonstration that if a statement is true for the n^{th} term, then it is also true for the next term (n+1).

Once this is established, logically, the statement must be true for the next term again (n+2) and so on. It creates a line of dominoes....



What we need is to topple this first domino - so we prove the rule is true for the smallest possible value. Then watch the dominoes tumble!



Proof by Induction

This method is used to prove a given formula is true for all possible values: "...for all n".

Initial Step

Prove the statement is true for the smallest possible value of n

Inductive Step

Assume true for $n = k$

Prove true for $n = k + 1$

Logical Argument

If true for $n = k$ then also true for $n = k + 1$

As true for first value (eg $n = 1$) then, by induction, true for all n.

Proofs of Summation Statements

The basic premise of all of these proofs is the same:

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1)$$

That is,

given $\sum_{r=1}^{4} r = 10$, $\quad \sum_{r=1}^{5} r = \sum_{r=1}^{4} r + 5$

$=$

$=$

Example 17

Given $\sum_{r=1}^{k} 2r - 1 = k^2$, find an expression for $\sum_{r=1}^{k+1} 2r - 1$

Proof by Induction

Example 18

Prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ for $n \geq 1$

Example 19

Prove that $\sum_{r=1}^{n} r\,r! = (n+1)! - 1$ for $n \geq 1$

Further Proofs

Prove, by induction, that the following are true:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{j=1}^{n} 2^{j-1} = 2^n - 1$$

$$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x}, \qquad \text{for } x \neq 1 \text{ and integers } n \geq 0\,.$$

Proofs of Other Statements

For other proofs, the structure remains the same, but the method of proving true for n=k+1 is different.
Again, it is important to have a goal to aim for.

Example 20

Prove: If $n \geq 2$, then $n^3 - n$ is divisible by 3.

Example 21

Prove by induction that $n < 2^n$ for all natural numbers n

Further Proofs

Prove, by induction, that the following are true:

$5^n - 1$ is divisible by 4

$3^n > n^2$

$9^n + 3$ is divisible by 4

$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Past Paper Questions

Mathematical Proof

2001

Prove by induction , that for all integers $n \geq 1$,

$$2+5+8+...+(3n-1)=\frac{1}{2}n(3n+1).$$

(5 marks)

2002

Prove by induction that $4^n - 1$ is divisible by 3 for all positive integers n.

(5 marks)

A matrix $A=\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. Prove by induction that

$$A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix},$$

where n is any positive integer. **(6 marks)**

2003

1 Given that $p(n) = n^2 + n$, where n is a positive integer, consider the statements:

A $p(n)$ is always even

B $p(n)$ is always a multiple of 3.

For each statement, prove if it is true or, otherwise, disprove it.

(4 marks)

2 (a) Prove by induction that for all natural numbers $n \geq 1$

$$\sum_{r=1}^{n} 3(r^2 - r) = (n-1)n(n+1).$$

(b) Hence evaluate $\sum_{r=11}^{40} 3(r^2 - r)$. **(4, 2 marks)**

2004

Prove by induction that $\frac{d^n}{dx^n}(xe^x) = (x+n)e^x$ for all integers $n \geq 1$. **(5 marks)**

2005

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

State the value of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. **(5, 1 marks)**

2006

① For all natural numbers n, prove whether the following results are true or false.

(a) $n^3 - n$ is always divisible by 6.

(b) $n^3 + n + 5$ is always prime. **(5 marks)**

② The square matrices A and B are such that $AB = BA$. Prove by induction that $A^n B = BA^n$ for all integers $n \geq 1$.

5 marks)

2007

Prove by induction that for $a > 0$,

$$(1+a)^n \geq 1 + na$$

For all positive integers n. **(5 marks)**

2008

For each of the following statements, decide whether it is true or false and prove your conclusion.

A For all natural numbers m, if m^2 is divisible by 4 then m is divisible by 4.

B The cube of any odd integer p plus the square of any even integer q is always odd.

(5 marks)

2009

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}.$$

(5 marks)

2010

① (a) Prove that the product of two odd integers is odd.

(b) Let p be an odd integer. Use the result of (a) to prove by induction that p^n is odd for all positive integers n.

(2, 4 marks)

② Prove by contradiction that if x is an irrational number, then $2 + x$ is irrational.

(4 marks)

2011

Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 **for all integers** $n \geq 2$. **(5 marks)**

2012

Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$. **(6 marks)**

2013

① Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n}(4r^3 + 3r^2 + r) = n(n+1)^3$$

(6 marks)

② Let n be a natural number.
For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A If n is a multiple of 9, then so is n^2.
B If n^2 is a multiple of 9, then so is n.

(4 marks)

2014

Given A is the matrix $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$,

Prove by induction that

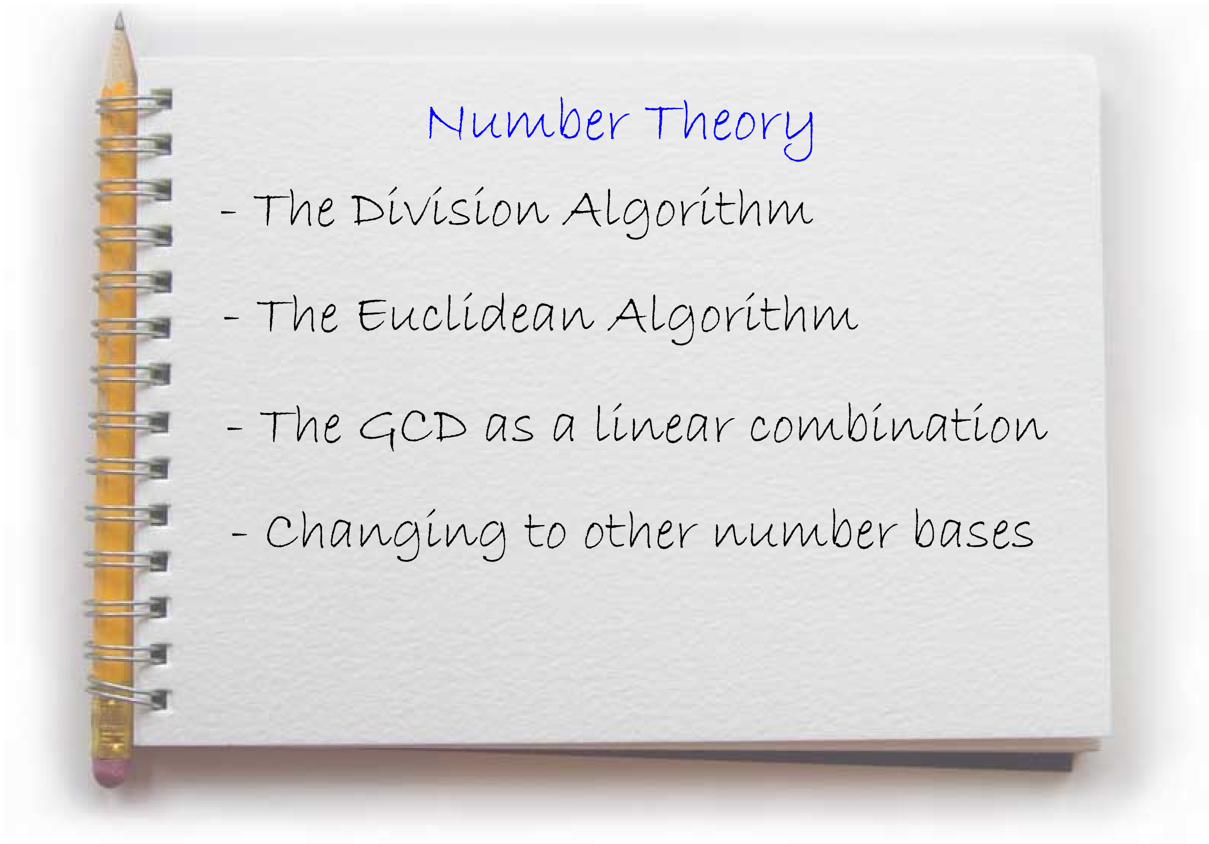
$$A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}, \quad n \geq 1.$$

(4 marks)

2015

Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8.

(3 marks)



The Division Algorithm

When we divide a positive integer, a, by another, b, where $a > b$, then we can always deduce that:

$$a = qb + r$$

q = quotient
r = remainder
$0 \quad r < b$

Examples:

1) If $a = 97$, $b = 8$

2) If $a = 3281$, $b = 107$

The form also holds for negative integers, **as long as r is positive.**

Given a = -42 and b = 5,

Given a = 50 and b = -3,

The division algorithm states:

For any integers, a and b, $b \neq 0$, there are two unique integers such that $a = qb + r$, $0 \leq r < |b|$

Given a = 187 and b = 14,

Given a = -45 and b = 6,

Given a = 4 and b = 10,

The Euclidean Algorithm

The division algorithm forms the basis for the Euclidean Algorithm, which allows the calculation of the GCD (greatest common divisor) of 2 or more positive integers.

Example 1: Find the GCD of 48 and 112

The last non-zero remainder is the GCD.

Example 2: Find the GCD of 1980 and 3696

Example 3: Simplify: $\dfrac{3024}{5184}$

Practise: Simplify: a) $\dfrac{396}{576}$ b) $\dfrac{1270}{4826}$

The GCD as a linear combination (Bezout's Identity)

If $\text{GCD}(a, b) = d$ where a and b are positive integers

then $xa + yb = d$ where x and y are integers

Example 4: Find x and y such that $140x + 252y = \text{GCD}(140,252)$

Example 5: Find x and y such that $585x + 104y = 13$

Practise:

1) Find a and b such that $254a + 32b = 2$

2) Find s and t such that $74s + 383t = 1$

3) Find x and y such that $7544x + 115y = 23$

4) Find p and q such that $24p + 687q = 3$

Number Bases

The decimal (base 10) number system was developed in India and the wider Middle East between the 1st and 9th century and was adopted in the western world in the Middle Ages (10th-12th century).

The other common number base in use today is the binary (base 2) system, use in computing to represent the 'on-off' states of computer processors.

In all bases, the place values are powers of the base number.
The digit in that place represents the multiple of that value.

BASE 10

10^4	10^3	10^2	10^1	10^0
10000	1000	100	10	1

Number Bases

BASE 5

5^4	5^3	5^2	5^1	5^0
625	125	25	5	1

$2403_5 =$

BASE 2

2^4	2^3	2^2	2^1	2^0

$10110_2 =$

BASE 6

6^4	6^3	6^2	6^1	6^0

$524_6 =$

Converting to other bases

We can use the Euclidian algorithm to convert numbers from base 10 to other bases.

Example 7: Write 192 in base 6.

Example 8: Write 212 in base 5.

Practise: Write: (a) 235 in base 6

(b) 423 in base 8

(c) 81 in base 2

(d) 579 in base 5

(e) 713 in base 16

Diaphantine Equations

Diaphantine equations are indeterminate equations with integer solutions.
(These equations have either infinite or no solutions)

Consider the equation: $ax + by = c$

For integer solutions to exist, c must be a multiple of GCD(a,b)

i.e. $\mathbf{c \mid GCD(a,b)}$

If one solution is identified (x,y), then additional solutions include:

$(x+b, y-a)$, $(x+2b, y-2a)$,$(x+3b, y-3a)$ etc

and $(x-b, y+a)$, $(x-2b, y+2a)$,$(x-3b, y+3a)$ etc

Example 9: Find solutions, if they exist, for the equation:

$$3x + 6y = 5$$

Example 10: Find solutions, if they exist, for the equation:

$$4x + 3y = 6$$

Example 11: Find solutions, if they exist, for the equation:

$$30x - 42y = 66$$

Practise: Find 5 solutions, if they exist, for the equations:

a) $4x + 12y = 7$

b) $5x + 7y = 10$

c) $8x + 40y = 9$

d) $6x + 2y = 8$

e) $15x + 27y = 18$

Past Paper Questions

Number Theory

2001

Use the Euclidean algorithm to find integers x and y such that $149x+139y=1$.

(4 marks)

2004

Use the Euclidean algorithm to show that $(231,17)=1$ where (a,b) denotes the highest common factor of a and b.

Hence find integers x and y such that $231x+17y=1$. **(4 marks)**

2007

Use the Euclidean algorithm to find integers p & q such that $599p+53q=1$.

(4 marks)

2012

Use the division algorithm to express 1234_{10} in base 7. **(3 marks)**

2013

Use the Euclidean algorithm to obtain the greatest common divisor of 1204 and 833, expressing it in the form $1204a+833b$, where a and b are integers.

(4 marks)

2015

Use the Euclidean algorithm to find integers p and q such that

$$3066p+713q=1.$$ **(4 marks)**

Past Paper Answers
Number Theory

2001 (Q.B1)
(a) Proof
(b) x = 14, y = -15

2004 (Q.10)
(a) Proof
(b) x=-5, y = 68

2007 (Q.7)
p=10, q = -113

2012 (Q.10)
$1234_{10} = 3412_7$

2013 (Q.5)
(9)(1204) – (13)(833) = 7, i.e. a = 9, b = -13

2015 (Q.7)
p = 10, q = -43