St Andrew's Academy

Department of Mathematics

Advanced Higher

Course Notes

Book 3

Contents

FORMULAE LIST

Standard derivatives	
$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)+c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
e^{ax}	$\frac{1}{a}e^{ax}+c$

Summations

(Arithmetic series) $$S_n = \frac{1}{2}n\left[2a+(n-1)d\right]$$

(Geometric series) $$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \ldots$$

De Moivre's theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Vector product

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ about the origin, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Vectors

- Vectors are named using a directed line segment, eg $\overline{AB}$, or a bold letter, eg $\boldsymbol{u}$, written by hand as $\underline{u}$
- A component vector is in the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ – also known as a column vector
- The magnitude of a vector, denoted $|\overline{AB}|$ or $|\boldsymbol{u}|$, is $\sqrt{a^2+b^2+c^2}$
- Multiplication by a scalar is $k\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$.

  If $\boldsymbol{u} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$ then $2\boldsymbol{u} = \boldsymbol{v}$

  this means that $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, but $\boldsymbol{v}$ is twice as long as $\boldsymbol{u}$
- The zero vector is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

  eg $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

Position Vectors

- $\overline{AB} = \boldsymbol{b} - \boldsymbol{a}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the position vectors of A and B.

Collinearity

- If $\overline{AB} = k\overline{BC}$, where k is a scalar, then $\overline{AB}$ is parallel to $\overline{BC}$. If B is a common point then A, B and C are collinear.
- To find the coordinates of a point B which divides $\overline{AC}$ in the ratio 2:3, use

  $\frac{\overline{AB}}{\overline{BC}} = \frac{2}{3}$

- $\boldsymbol{u}+\boldsymbol{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$
- $\boldsymbol{u}-\boldsymbol{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$
- $\overline{BA}$ is the negative of $\overline{AB}$.

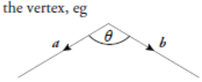
  eg $\overline{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \overline{BA} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$
- This is the same line segment, but it points in the opposite direction

The Scalar Product

$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

- $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ (given in the exam)

Remember: the vectors must point away from the vertex, eg



- $\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$ (given in the exam)
- $\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$
- $\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}||\boldsymbol{b}|}$
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then $\boldsymbol{a}\cdot\boldsymbol{b} = 0$ since $\cos 90° = 0$
- $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$ and $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}$

Example 1: Given A(5,2,6) and B(1,-4,2), find the size of angle AOB.

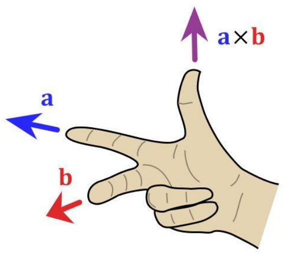
Example 2: Given A(5, -1, 4) and B(9, 6, 0), find a unit vector, $\underline{u}$, which is parallel to $\overrightarrow{AB}$.

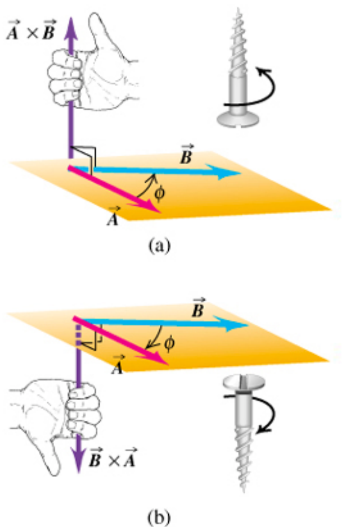
The Vector (Cross) Product

The vector (cross) product is a vector produced by the multiplication of 2 vectors.

The resultant vector is perpendicular to both vectors.

We mostly use a right-handed system to define the direction of the resultant vector.

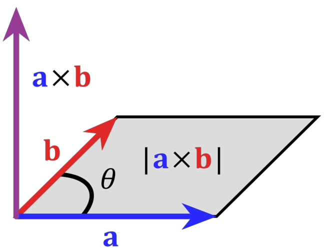




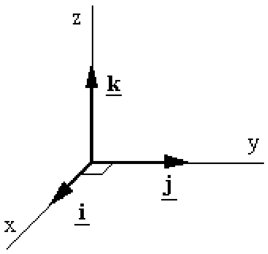
Properties of the Vector Product

For 2 three-dimensional vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$:

- $\underset{\sim}{a} \times \underset{\sim}{b}$ is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (the 'normal' vector)
- If $\underset{\sim}{a} \times \underset{\sim}{b} = 0$, $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$
- $\underset{\sim}{a} \times \underset{\sim}{b} = -(\underset{\sim}{b} \times \underset{\sim}{a})$
- The distributive law holds: $\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = (\underset{\sim}{a} \times \underset{\sim}{b}) + (\underset{\sim}{a} \times \underset{\sim}{c})$
- $|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin\theta$
- $|\underset{\sim}{a} \times \underset{\sim}{b}|$ is the area of the parallelogram drawn from vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$



The Vector Product of Unit Vectors



$\underset{\sim}{i} \times \underset{\sim}{j} =$

$\underset{\sim}{i} \times \underset{\sim}{k} =$

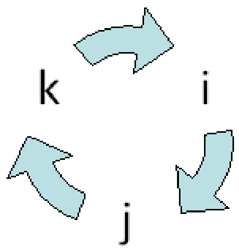
$\underset{\sim}{j} \times \underset{\sim}{i} =$

$\underset{\sim}{j} \times \underset{\sim}{k} =$

$\underset{\sim}{k} \times \underset{\sim}{i} =$

$\underset{\sim}{k} \times \underset{\sim}{j} =$

We can summarise these results in a cyclic diagram:



clockwise - positive

anti clockwise - negative

The Vector Product in Component Form

For two vectors, $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 3: Given $\underline{a} = \underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{b} = 3\underline{i} - \underline{j} + \underline{k}$, find $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$

Example 4: Find the area of a parallelogram with vertices:
A(3, 2, -1), B(4, 0, 1) and C(-2, 3, 3)

Example 5: Given $\underline{u} = 2\underline{i} + 2\underline{j} - \underline{k}$, $\underline{v} = 3\underline{i} - \underline{j} - 2\underline{k}$ and $\underline{w} = 2\underline{i} - 3\underline{j} + 4\underline{k}$,
evaluate $(\underline{u} \times \underline{v}).(\underline{u} \times \underline{w})$

Example 6: Find the area of a triangle with vertices:
A(1, 3, -2), B(4, 3, 0) and C(2, 1, 1)

Example 7: Find a unit vector perpendicular to both

$\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + 2\underline{k}$

Equations of Straight Lines

A line in 3-dimensional space can be expressed in one of 3 ways:

Vector form

Parametric form

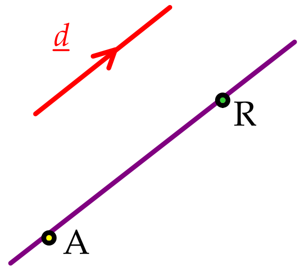
Symmetric form

A line is either specified as being:

a) through a point in a given direction
b) through two points

Vector form

Consider a vector $\underline{d}$ and a fixed point A on line L.
The point R exists such that AR is parallel to $\underline{d}$.



Example 8: Find the vector equation of the straight line through (2,-1,6) parallel to the vector $\underline{i} + 2\underline{j} - 8\underline{k}$

Parametric form

If $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underset{\sim}{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underset{\sim}{d} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

then $\underset{\sim}{r} = \underset{\sim}{a} + t\,\underset{\sim}{d}$ becomes:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t\begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

$$x = a + tl$$
$$y = b + tm$$
$$z = c + tn$$

These are the equations of the line in **parametric form.**

Example 9: Find the parametric equations of the line through (6, 3, -1) and parallel to vector $2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$

Symmetric (Cartesian) form

Rearranging each of the parametric equations with t as the subject we get:

$$t = \frac{x-a}{l} \qquad t = \frac{y-b}{m} \qquad t = \frac{z-c}{n}$$

As each fraction has a common value then:

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

The **direction ratio** is the ratio $l : m : n$

Example 10: Find the symmetrical form of the equation of the line

a) through the point (4, -1, 3) in direction $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$

b) through the point (-2, 4, 0) parallel to the line $\frac{x+3}{2} = \frac{x}{-1} = \frac{x-1}{6}$

Example 11: Find the equation of the line joining the points

A(0, 2, 3) and B(4, 3, 1)

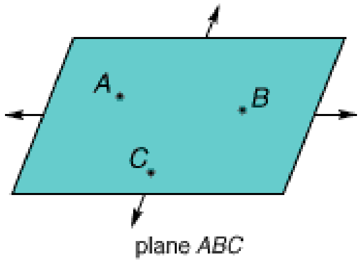
a) in symmetrical form and b) in parametric form

Note: The equation of a line is not unique. Choosing to use the coordinates of point B would result in a different (but equally correct) equation.

The Equation of a Plane

Definition:

Plane : a flat surface on which a straight line joining any two points on it would wholly lie.



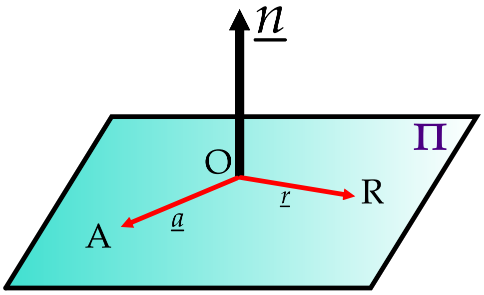
A plane can be represented in several forms:

Scalar product (Vector) form

Symmetrical (Cartesian) form

Parametric form

Scalar product (Vector) form



A and R both lie on plane π.

A is a fixed point and R is a variable point

If $\underline{n}$ is normal to both OA and OR, then $\underline{n}$ is perpendicular to $\overrightarrow{AR}$.

$$\underline{n} \bullet \overrightarrow{AR} = 0$$

$$\underline{n} \bullet (\underline{r} - \underline{a}) = 0$$

$$\underline{n} \bullet \underline{r} - \underline{n} \bullet \underline{a} = 0$$

$$\underline{n} \bullet \underline{r} = \underline{n} \bullet \underline{a}$$

$\underline{n}$ and $\underline{a}$ are both fixed, so scalar product is a constant

Equation of plane: $\underline{n} \bullet \underline{r} = k$ where k is a constant

Symmetrical (Cartesian) form

From scalar product form: $\underline{n} \bullet \underline{r} = k$

If $\underline{n} = a\underline{i} + b\underline{j} = c\underline{k}$ and $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

then $$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = k$$

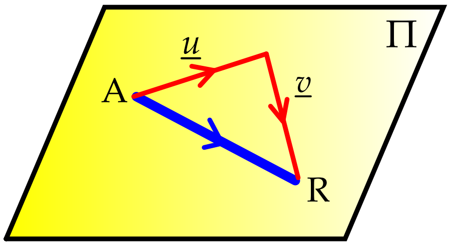
Equation of plane: $\mathbf{a}x + \mathbf{b}y + \mathbf{c}z = \mathbf{k}$

Example 12: Find the vector and Cartesian equations of the plane through (-1, 2, 1) with normal vector $\underline{n} = \underline{i} + 3\underline{j} - 2\underline{k}$

Example 13: Find the Cartesian equation of the plane through A(-1, 3, 1), B(1, -3, -3) and C(3, -1, 5)

Example 14: Find the Cartesian equation of the plane containing the point (3, -2, -7) and the line: $\frac{x - 5}{3} = \frac{y}{1} = \frac{z + 6}{4}$

Parametric form



$\overrightarrow{AR}$ lies on plane Π.

$\underline{u}$ and $\underline{v}$ are parallel to Π.

The point R is variable and
$\overrightarrow{AR} = \lambda\underline{u} + \mu\underline{v}$,
where λ and μ are parameters.

$\overrightarrow{AR} = \underline{r} - \underline{a}$ and $\overrightarrow{AR} = \lambda\underline{u} + \mu\underline{v}$

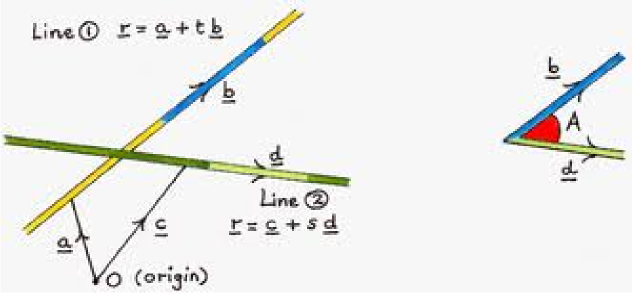
so $\underline{sr} - \underline{a} = \lambda\underline{u} + \mu\underline{v}$

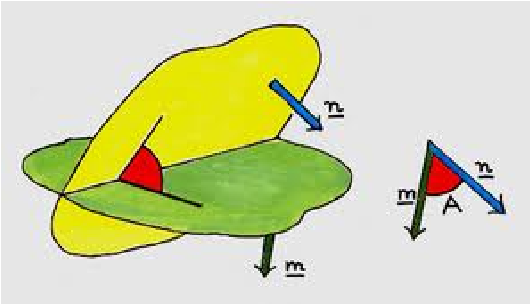
Equation of plane: (containing A and parallel to $\underline{u}$ and $\underline{v}$)	$\underline{r} = \underline{a} + \lambda\underline{u} + \mu\underline{v}$

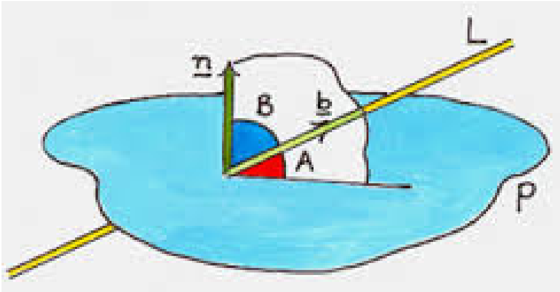
This form is used when 3 points on a plane are known, or 1 point and 2 parallel vectors. (The normal vector is not known, nor required.)

Example 15: Find a parametric equation of the plane containing A(1, 1, -1), B(2, 0, 2) and C(0, -2, 1).

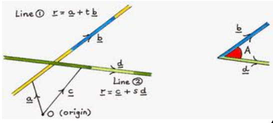
The angle between 2 lines, 2 planes or a line and a plane





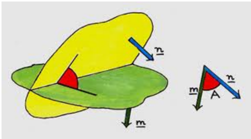


Intersections of 2 lines, 2 planes or a line and a plane



Angle between 2 lines:

Calculate the angle between direction vectors using $\cos\theta = \dfrac{a \bullet b}{|a||b|}$

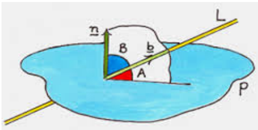


Angle between 2 planes:

Calculate the angle between normal vectors using $\cos\theta = \dfrac{a \bullet b}{|a||b|}$

Angle between a line and a plane:

Calculate the angle between the normal vector of the plane and the direction vector of the line, θ. The angle between the line and plane is 90 - θ.



Example 16: Calculate the angle of intersection of the lines:

$$L_1 : \frac{x+4}{2} = \frac{y-5}{-4} = \frac{z-3}{1} \quad \text{and} \quad L_2 : \frac{x}{-1} = \frac{y-3}{-1} = \frac{z-2}{1}$$

Example 17: Find the acute angle between the planes:

$2x + y - 2z = 5$ and $3x - 6y - 2z = 7$

Example 18: Find the angle between the line $\frac{x-1}{4}=\frac{y-2}{3}=\frac{z+3}{2}$

and the plane $10x - 2y + z = -1$

Example 19: Prove that the lines L_1 and L_2 intersect and find the point of intersection, given:

$$L_1 : \frac{x+4}{2} = \frac{y-5}{-4} = \frac{z-3}{1} \quad \text{and} \quad L_2 : \frac{x}{-1} = \frac{y-3}{-1} = \frac{z-2}{1}$$

Example 20: Find the point of intersection between the plane

$$2x + 3y + 5z = -2$$

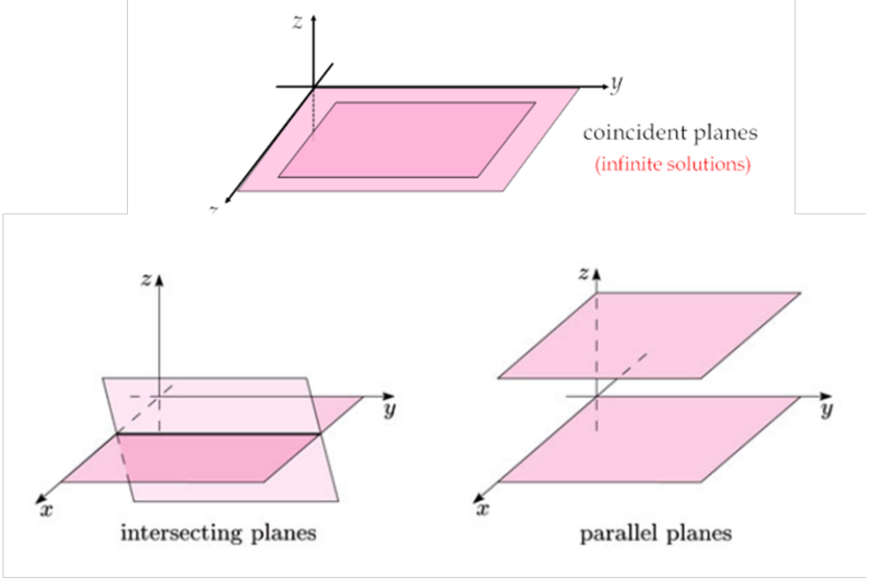
and a perpendicular line passing through the point A(3, 5, 3).

Intersections of 2 planes

Two planes will always: be coincident (overlying)

be parallel

or intersect on a line.



There are two methods of finding the equation of the line formed by the intersection of 2 planes:

Substitution

Direction and Point

Substitution Method

Example 21: Find the equation of the line formed by the intersection of the planes: $4x + y - 2z = 3$ and $x + y - z = 1$

Direction and Point Method

Example 22: Find the equation of the line formed by the intersection of the planes: $4x + y - 2z = 3$ and $x + y - z = 1$

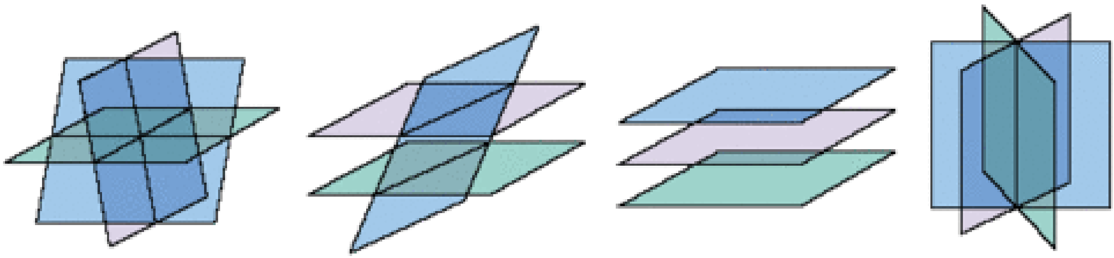
Intersections of 3 planes

Three planes intersect either at a point, along a line, or not at all.
Gaussian elimination can be used in all three cases.

Intersection at a point — Using Gaussian elimination, the solution (x,y,z) is the point of intersection of the 3 planes.

Intersection along a line — When a system is redundant, introduce a parameter and create the equation of the line of intersection.

No intersection — If Gaussian elimination produces an impossible equation (inconsistency), then the planes do not intersect at all.



(a) (b) (c) (d)

Example 23: Find the equation of the line of intersection of the planes:

$x + y - z = 0, \quad 2x - y + 4z = -3$ and $x + 3y - 5z = 2$

Example 24: Find a plane through (2, 1, -1) perpendicular to the line of intersection of the planes $2x + y - z = 3$ and $x + 2y + z = 2$

Past Paper Questions

Vectors

2001

Let L_1 and L_2 be the lines

$L_1: x = 8 - 2t,\ y = -4 + 2t,\ z = 3 + t$

$L_2: \dfrac{x}{-2} = \dfrac{y+2}{-1} = \dfrac{z-9}{2}.$

(a)(i) Show that L_1 and L_2 intersect and find their point of intersection.

(ii) Verify the acute angle between them is $\cos^{-1}\left(\dfrac{4}{9}\right)$.

(b) (i) Obtain an equation of the plane Π that is perpendicular to L_2 and passes through the point $(1,-4,2)$.

(ii) Find the coordinates of the point of intersection of the plane Π and the line L_1.

(4, 2, 3, 2 marks)

2002

(a) Find an equation for the plane π_1 which contains the points $A(1,1,0)$, $B(3,1,-1)$ and $C(2,0,-3)$.

(b) Given that π_2 is the plane whose equation is $x + 2y + z = 3$, calculate the size of the acute angle between the plane π_1 and π_2.

(4, 3 marks)

2003

Find the point of intersection of the line $\dfrac{x-3}{4} = \dfrac{y-2}{-1} = \dfrac{z+1}{2}$

and the plane with equation $2x + y - z = 4$. **(4 marks)**

2004

(a) Find an equation of the plane π_1 containing the points $A(1,0,3)$, $B(0,2,-1)$ and $C(1,1,0)$.

Calculate the size of the acute angle between π_1 and the plane π_2 with equation $x + y - z = 0$.

(b) Find the point of intersection of the plane π_2 and the line $\dfrac{x-11}{4} = \dfrac{y-15}{5} = \dfrac{z-12}{2}$.

(4, 3, 3 marks)

2005

The equations of two planes are $x - 4y + 2z = 1$ and $x - y - z = -5$. By letting $z = t$ or otherwise, obtain parametric equations for the line of intersection of the planes.

Show that this line lies in the plane with equation $x + 2y - 4z = -11$.

(4, 1 marks)

2006

Obtain an equation for the plane passing through the point $P(1,1,0)$ which is perpendicular to the line L given by $\frac{x+1}{2}=\frac{y-2}{1}=\frac{z}{-1}$.

Find the coordinates of the point Q where the plane and L intersect.

Hence, or otherwise, obtain the shortest distance from P to L and explain why this is the shortest distance.

(3, 4, 2, 1 marks)

2007

Lines L_1 and L_2 are given by the parametric equations

$$L_1 : x=2+s,\ y=-s,\ z=2-s \qquad\qquad L_2 : x=-1-2t,\ y=t,\ z=2+3t.$$

(a) Show that L_1 and L_2 do not intersect.

(b) The line L_3 passes through the point $P(1,1,3)$ and its direction is perpendicular to the directions of both L_1 and L_2. Obtain parametric equations for L_3.

(c)Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1.

(d) PQ is the shortest distance between the lines L_1 and L_2. Calculate PQ.

(3, 3, 3, 1 marks)

2008

(a) Find an equation of the plane π_1 through the point $A(1,1,1)$, $B(2,-1,1)$ and $C(0,3,3)$.

(b) The plane π_2 has equation $x+3y-z=2$.

Given that the point $(0,a,b)$ lies on both the planes π_1 and π_2, find the values of a and b. Hence find an equation of the line of intersection of the planes π_1 and π_2.

(c) Find the size of the acute angle between the planes π_1 and π_2.

(3, 4, 3 marks)

2009

(a) Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x+y-z&=6\\ 2x-3y+2z&=2\\ -5x+2y-4z&=1 \end{aligned}$$

(b) Show that the line of intersection, L, of the planes $x+y-z=6$ and $2x-3y+2z=2$ has parametric equations

$$\begin{aligned} x&=\lambda\\ y&=4\lambda-14\\ z&=5\lambda-20. \end{aligned}$$

(c) Find the acute angle between line L and the plane $-5x+2y-4z=1$.

(5, 2, 4 marks)

2010

Given $\underline{u} = -2\underline{i} + 5\underline{k}$, $\underline{v} = 3\underline{i} + 2\underline{j} - \underline{k}$ and $\underline{w} = -\underline{i} + \underline{j} + 4\underline{k}$.

Calculate $\underline{u}.(\underline{v} \times \underline{w})$. **(4 marks)**

2011

The lines L_1 and L_2 are given by the equations $\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1}$ and $\frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ respectively.

Find

(a) The value of k for which L_1 and L_2 intersect and the point of intersection.

(b) The acute angle between L_1 and L_2.

(6, 4 marks)

2012

Obtain an equation for the plane passing through the points $P(-2,1,-1)$, $Q(1,2,3)$ and $R(3,0,1)$.

(5 marks)

2013

(a) Find an equation of the plane π_1 through the points $A(0,-1,3)$, $B(1,0,3)$ $C(0,0,5)$.

(b) π_2 is the plane through A with normal in the direction $-\underline{j} + \underline{k}$.

Find an equation of the plane π_2.

(c) Determine the acute angle between the planes π_1 and π_2.

(4, 2, 3 marks)

2014

Three vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ are given by $\underline{u}, \underline{v}$ and $\underline{w}$ where

$\underline{u} = 5\underline{i} + 13\underline{j}$, $\underline{v} = 2\underline{i} + \underline{j} + 3\underline{k}$, $\underline{w} = \underline{i} + 4\underline{j} - \underline{k}$.

Calculate $\underline{u}.(\underline{v} \times \underline{w})$.

Interpret your result geometrically.

(3, 1 marks)

2015

A line L_1, passes through the point P$(2,4,1)$ and is parallel to

$$\underline{u}_1 = \underline{i} + 2\underline{j} - \underline{k}$$

and a second line, L_2, passes through Q$(-5,2,5)$ and is parallel to

$$\underline{u}_2 = -4\underline{i} + 4\underline{j} + \underline{k}.$$

(a) Write down the vector equations for L_1 and L_2.

(b) Show that the lines L_1 and L_2 intersect and find their point of intersection.

(c) Determine the equation of the plane containing L_1 and L_2. **(2, 4, 4 marks)**

Past Paper Answers

Vectors

2001
(a) (i) Proof; PoI (4,0,5) (ii) Proof (b) $2x + y - 2z = -6$; PoI (2,2,6)

2002
(a) $x - 5y + 2z = -4$ (b) 58.6°

2003
(-1, 3, -3)

2004
(a) 2x + 3y + z = 5 ; 51.9° (b) (3, 5, 8)

2005
x = 2t – 7, y = t – 2, z = t; Proof

2006
2x + y – z = 3; $Q(0, \frac{5}{2}, -\frac{1}{2})$; $\frac{\sqrt{14}}{2}$ = PQ, perpendicular to line

2007
(a) Proof (b) x = 1 + 2u, y = 1 + u, z = 3 + u (c) Q(-1, 0, 2) (d) $\sqrt{6}$ units

2008
(a) 2x + y = 3 (b) a = 3, b = 7; $\frac{x}{1} = \frac{y-3}{-2} = \frac{z-7}{-5}$ (c) 47.6°

2009
(a) (3, -2, -5) (b) Proof (c) 23.0°

2010
$\underline{u}.(\underline{v} \times \underline{w}) = 7$

2011
(a) k = 2 (b) 60°

2012
6x + 14y – 8z = 10

2013
(a) 2x - 2y + z = 5 (b) -y + z = 4 (c) 45°

2014
$\underline{u}.(\underline{v} \times \underline{w}) = 0$; as scalar product = 0, $\underline{u}$ and ($\underline{v}$ x $\underline{w}$) are perpendicular.
($\underline{v}$ x $\underline{w}$) is the normal vector to the plane on which $\underline{v}$ and $\underline{w}$ lie.
Therefore $\underline{u}$, $\underline{v}$ and $\underline{w}$ must lie on the same plane (are co-planar).

2015
(a) $L_1: \begin{pmatrix}2\\4\\1\end{pmatrix} + s\begin{pmatrix}1\\2\\-1\end{pmatrix}$ $L_2: \begin{pmatrix}-5\\2\\5\end{pmatrix} + t\begin{pmatrix}-4\\4\\1\end{pmatrix}$ (b) Proof; PoI (-1, -2, 4) (c) 2x + y + 4z = 12

Advanced Higher Mathematics
Formal Homework Assignment

Vectors

1. Find the unit vectors which are perpendicular to both of the vectors $u = j + 4k$ and $v = 3i + 2j + 4k$.

2. (a) Find the equation of the line determined by the points A(1, 9, 5) and B(3, 5, 7) in:

(i) vector form
(ii) parametric form
(iii) symmetric form

(b) Determine the coordinates of the point where the line intersects the XY plane.

3. (a) Obtain the equation of the plane Π on which lie the points A(1 ,2, -2), B(3, 3, -3) and C(2, 4, -1).

(b) If L is the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+2}{3}$ through A, find, in parametric form, the equation of the line M through A, lying in the plane Π and perpendicular to L.

4. Find the point where the line $\frac{x+4}{-2} = \frac{y-1}{0} = \frac{z-9}{4}$ intersects the plane $2x + 2y - z = 5$.

5. A line, L, is the intersection of two planes,

Π_1: $x + y + z = 1$ $\qquad$ Π_2: $x - 2y + 3z = 2$

Find the equation of the plane containing L and passing through the origin.

6. Lines L_1 and L_2 are given by the parametric equations:

L_1 : $x = 2 + s$, $y = -s$, $z = 2 - s$ $\qquad$ L_2 : $x = -1 - 2t$, $y = t$, $z = 2 + 3t$

(a) Show that L_1 and L_2 do not intersect.
(b) The line L_3 passes through the point P(1, 1, 3) and its direction is perpendicular to the directions of both L_1 and L_2. Obtain parametric equations for L_3.
(c) Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1.
(d) PQ is the shortest distance between the lines L_1 and L_2. Calculate the length of PQ.

7. Reflect on your understanding of vectors, particularly in relation to connections to your existing knowledge and any elements of these new topics which interest you.

Review of Integration at Higher Level

Basic Rules:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \sin x dx = -\cos x + c$$

Developments:

$$\int (px+q)^n dx = \frac{(px+q)^{n+1}}{p(n+1)} + c$$

$$\int p\cos(qx+r)dx = \frac{p}{q}\sin(qx+r) + c$$

$$\int p\sin(qx+r)dx = -\frac{p}{q}\cos(qx+r) + c$$

Example 1: Integrate:

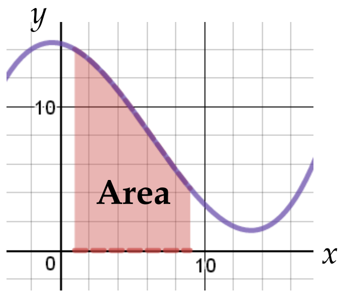
(a) $\int x^5 dx$ (b) $\int (3a-4)^3 dx$

(c) $\int \frac{1}{2}\cos(\frac{x}{4}+1)dx$ (d) $\int \sin^2(\frac{x}{3})dx$

Fundamental Theorem of Calculus

If $f(x) = F'(x)$ then $\int_a^b f(x)dx = F(a) - F(b)$ where $a \le x \le b$

Area between curve and the x-axis

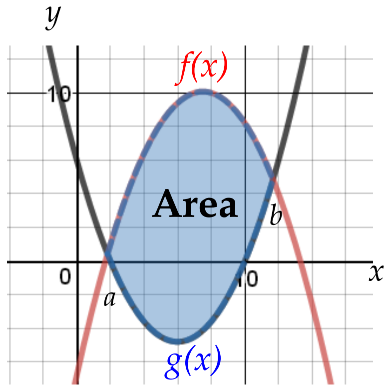


$$A = \int_a^b f(x)dx$$

above x-axis A is positive

below x-axis A is negative

Area between 2 curves

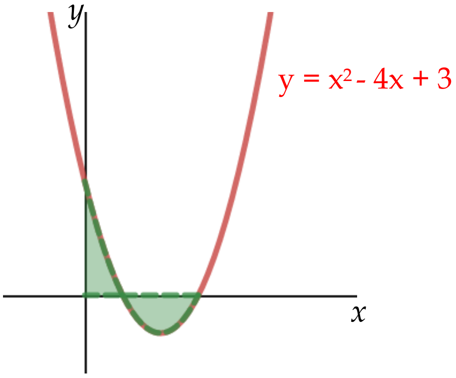


$$A = \int_a^b [f(x) - g(x)]dx$$

f is the 'upper' function

g is the 'lower' function

Example 2: Calculate the area shaded in the graph below:



Example 3: Calculate the area bounded by the line $y = x + 2$ and the curve $y = x^2$.

New Standard Integrals

We know:

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

Thus,

$$\int e^x dx = e^x + c$$

$$\int \frac{1}{x} dx = \ln x + c$$

$$\int \sec^2 x dx = \tan x + c$$

New Standard Integrals

The same developments can be made as for earlier integrals, taking the chain rule into account.

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + c$$

$$\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + c$$

Example 4: Find:

(a) $\int e^{-3x}dx$ (b) $\int \frac{3}{4x+1}dx$ (c) $\int 5\sec^2(\frac{\pi}{2}-3x)dx$

Integration by Substitution

In order to integrate a composite function, it may be possible to substitute one variable for another in order to make the integration easier, in the same way that the chain rule is used in differentiation.

Where the integral is of the form:

$$\int g(f(x)) \cdot f'(x)dx$$

the simpler expression is all or part of the derivative of the other function.

Example 5: Find: $\int x(x^2+3)^3 dx$

Example 6: Find: $\int 12\cos x \sin^2 x dx$

Example 7: Find: $\int 12x^2\sqrt{(x^3-5)}\,dx$

Example 8: Find: $\int \frac{6e^x}{3e^x+1}\,dx$

Integration by Substitution II

Where the simpler expression is not found by differentiating the more complex expression, some extra substitution is required.
Trig expression are often substituted, taking advantage of known identities.

Example 9: Find: $\int x(2x-1)^4\,dx$

Example 10: Find: $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ given $x = 2\sin\theta$

Definite Integrals

When substituting u for x during integration, it is possible to calculate definite integrals in terms of u, making it unnecessary to express the integrand in terms of x again (as long as the function is continuous).

Note: New limits (in terms of u) need to be calculated and used.

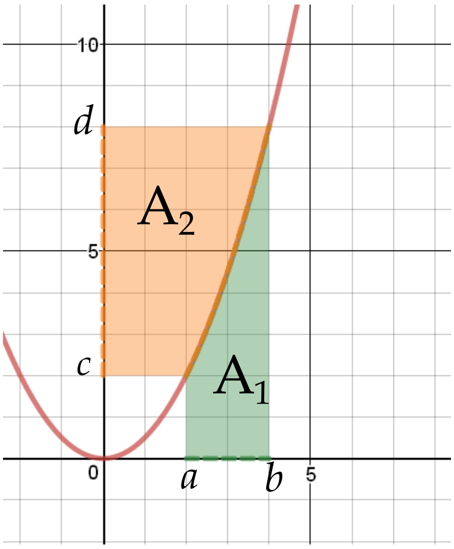
Example 11: Evaluate: $\int_0^1 6x^2(x^3-2)^4\,dx$

Example 12: Evaluate: $\int_{0}^{\frac{\pi}{2}} \cos x e^{\sin x} dx$

Example 13: Evaluate: $\int_0^{\frac{\pi}{4}} \cos^2 x \sin^3 x dx$ using the substitution $u = \cos x$

Example 14: Evaluate: $\displaystyle\int_{\frac{1}{2}}^{1} \frac{1-x}{\sqrt{1-x^2}}\,dx$ using the substitution $x = \sin u$

Area between a curve and the y-axis



Interactive Link

We know that:

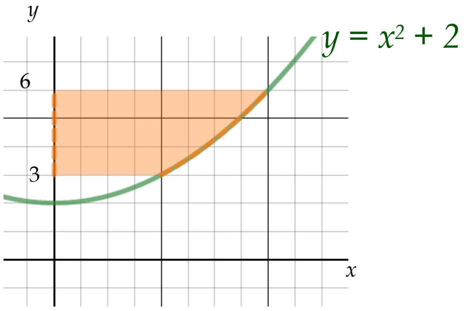
$$A_1 = \int_a^b f(x)dx$$

In the same way:

$$A_2 = \int_c^d f(y)dy$$

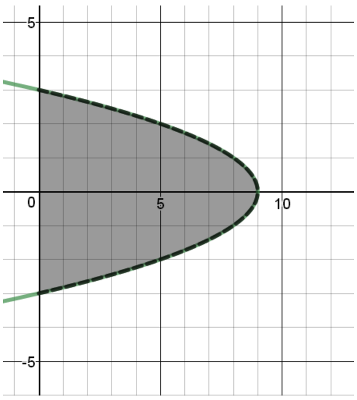
In order to calculate such an integral, a given function *f(x)* must be represented as a function of *y* - this is not always possible.

Example 15: Calculate the area between the curve $y = x^2 + 2$ and the lines $y = 3$ and $y = 6$.



Example 16:

Calculate the area enclosed by the y-axis and the function $y^2 = 9 - x$.



Inverse Trig Functions

We know that:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

therefore:

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + c \qquad \int \frac{1}{1+x^2}dx = \tan^{-1} x + c$$

Also:

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2-x^2}} \qquad \frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2}$$

therefore:

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

It may be necessary to rearrange the integrand into one of the above standard forms:

Example 17: Find $\int \frac{1}{\sqrt{4-x^2}}dx$

Example 18: Find $\int \frac{1}{1+9x^2}dx$

Example 19: Find: $\int \frac{1}{\sqrt{25 - 9x^2}} dx$

Example 20: Evaluate: $\int_1^2 \frac{3}{\sqrt{4 - x^2}} dx$

Example 21: Find: $\int \frac{4}{5x^2+9}dx$

Integration Using Partial Fractions

We use this method to evaluate integrals of the form:

$\int \frac{f(x)}{g(x)} dx$ where f(x) and g(x) are polynomials

Consider the 3 types of denominators we encounter in partial fractions:

1) Distinct linear factors
2) Repeated linear factors
3) Irreducible quadratic factors

3 TYPES

Type 1 : Distinct Linear Factors:

Express the integrand in terms of partial fractions and then use the standard integral:

$$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + c$$

Example 22:

Find the indefinite integrals :

(a) $\int \frac{x+16}{2x^2+x-6} dx$ (b) $\int \frac{2x^3+7x^2-2x-2}{2x^2+x-6} dx$

Type 2 : Repeated Linear Factors:

Express the integrand in terms of partial fractions and then use the standard integral:

$$\int \frac{1}{(ax+b)^2}dx = -\frac{1}{a}\left(\frac{1}{ax+b)}\right)+c$$

Example 23:

Find the indefinite integral : $\int \frac{5x+2}{(x-2)^2(x+1)}dx$

Type 3 : Irreducible Quadratic Factors:

Express the integrand in terms of partial fractions
and then use the standard integrals:

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c \qquad \int \frac{x}{x^2+b}dx = \frac{1}{2}\ln\left|x^2+b\right|+c$$

Example 24:

Find the indefinite integral : $\int \frac{x^2+6x+1}{2x^3+2x}dx$

Example 25:

Find the indefinite integral : $\int \frac{3x^2 + 92x}{(x+6)(x^2+1)} dx$

Integration by Parts

Integration by Parts allows us to integrate products of functions.
It is a technique derived from the Product Rule.

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$ Rule

If we integrate both sides then:

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

So:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$\int uv'dx = uv - \int u'vdx$$

We nominate a function, u, which is (easily) differentiable
and a function, v, which can be (easily) integrated.

Example 26: Find: $\int x\cos 2x dx$

Example 27: Find: $\int \frac{\ln x}{x^2}\,dx$

Repeated Integration by Parts

Terms where *f(x)* is a polynomial of degree greater than one will require 2 or more steps of integration by parts.

Example 28: Find: $\int x^2 e^x dx$

Example 29: Find: $\int x^2 \sin 3x dx$

For a polynomial of degree n, repeat process n times........**RULE**

Additional Techniques I - Tabular Method

To avoid multiple applications, it is possible to use a tabular method of Integration by Parts.

Example 30: Find: $\int x^3 e^x dx$

Practise: **1)** $\int x^3 \sin x dx$ **2)** $\int (x-5)^4 e^x dx$

Additional Techniques II - The Dummy Function

Integration by parts can also be used to integrate a function which has no standard integral, but has a standard derivative.
In these cases, the dummy function "1" is introduced.

Example 31: Find: $\int \ln x^2 dx$

Example 32: Find: $\int \cos^{-1} 3x dx$

Additional Techniques III - Round in Circles

When a resultant integral is the same as the original, any further integration is pointless. Instead, rearrange algebraically and solve.

Example 33: Find: $\int e^{2x}\cos x\,dx$

Past Paper Questions

Integration

2001

(a) Obtain partial fractions for

$$\frac{x}{x^2-1},\ x>1.$$

(b) Use the result of (a) to find

$$\int \frac{x^3}{x^2-1}\,dx,\ x>1.$$ **(2, 4 marks)**

2002

Use the substitution $x+2=2\tan\theta$ to obtain $\int \frac{1}{x^2+4x+8}\,dx$. **(5 marks)**

2003

Use the substitution $x=1+\sin\theta$ to evaluate $\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)^3}\,d\theta$. **(5 marks)**

2004

① Express $\frac{1}{x^2-x-6}$ in partial fractions.

Evaluate $\int_0^1 \frac{1}{x^2-x-6}\,dx$. **(2, 4 marks)**

② A solid is formed by rotating the curve $y=e^{-2x}$ between $x=0$ and $x=1$ through $360°$ about the x-axis. Calculate the volume of the solid that is formed. **(5 marks)**

2005

① Use the substitution $u=1+x$ to evaluate $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$. **(5 marks)**

② Express $\frac{1}{x^3+x}$ in partial fractions.

Obtain a formula for $I(k)$, where $I(k)=\int_1^k \frac{1}{x^3+x}\,dx$, expressing it in the form $\ln\frac{a}{b}$, where a and b depend on k.

Write down an expression for $e^{I(k)}$ and obtain the value of $\lim_{k\to\infty} e^{I(k)}$. **(4, 4, 2 marks)**

③ (a) Given $f(x)=\sqrt{\sin x}$, where $0<x<\pi$, obtain $f'(x)$.

(b) If, in general, $f(x)=\sqrt{g(x)}$, where $g(x)>0$, show that $f'(x)=\dfrac{g'(x)}{k\sqrt{g(x)}}$,

stating the value of k.

Hence, or otherwise, find $\int \frac{x}{\sqrt{1-x^2}}\,dx$. **(1, 2, 3 marks)**

2006

Find $\int \frac{12x^3-6x}{x^4-x^2+1}\,dx$. **(3 marks)**

2007

① Express $\dfrac{2x^2-9x-6}{x(x^2-x-6)}$ in partial fractions.

Given that $\int_4^6 \frac{2x^2-9x-6}{x(x^2-x-6)}\,dx=\ln\frac{m}{n}$,

determine values for the integers m and n. **(3, 3 marks)**

② Use the substitution $u=1+x^2$ to obtain $\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$.

A solid is formed by rotating the curve $y=\dfrac{x^{3/2}}{(1+x^2)^2}$ between $x=0$ and $x=1$ through

$360°$ about the x-axis. Write down the volume of this solid. **(5, 1 marks)**

2008

① Express $\dfrac{12x^2+20}{x(x^2+5)}$ in partial fractions.

Hence evaluate $\int_1^2 \frac{12x^2+20}{x(x^2+5)}\,dx$. **(3, 3 marks)**

② Write down the derivative of $\tan x$.

Show that $1+\tan^2 x=\sec^2 x$.

Hence obtain $\int \tan^2 x\,dx$. **(1, 1, 2 marks)**

2009

① Show that $\int_{\ln \frac{3}{2}}^{\ln 2} \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\, dx = \ln \frac{9}{5}$. **(4 marks)**

② Use the substitution $x = 2\sin\theta$ to obtain the exact value of $\int_{0}^{\sqrt{2}} \frac{x^{2}}{\sqrt{4-x^{2}}}\, dx$.

(Note that $\cos 2A = 1 - 2\sin^{2} A$.) **(6 marks)**

2010

① Evaluate $$\int_{1}^{2} \frac{3x+5}{(x+1)(x+2)(x+3)}\, dx$$

expressing your answer in the form $\ln \frac{a}{b}$, where a and b are integers. **(6 marks)**

② A new board game has been invented and the symmetrical design on the board is made from 4 identical "petal" shapes. One of these petals is the region enclosed between the curves $y = x^{2}$ and $y^{2} = 8x$ as shown shaded in diagram 1 below.

Calculate the area of the complete design, as shown in diagram 2.

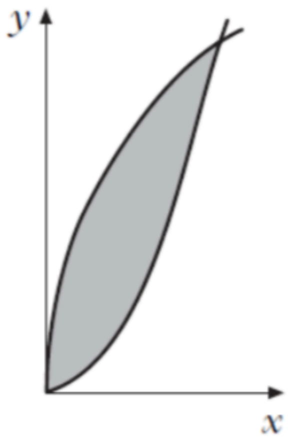


Diagram 1

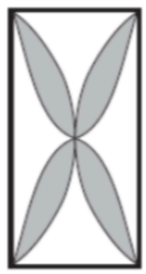


Diagram 2

The counter used in the game is formed by rotating the shaded area shown in diagram 1 above, through $360°$ about the y-axis.
Find the volume of plastic required to make one counter. **(5, 5 marks)**

2011

① Express $\frac{13-x}{x^2+4x-5}$ in partial fractions and hence obtain $\int \frac{13-x}{x^2+4x-5}\,dx$. **(5 marks)**

② Obtain the exact value of $\int_0^{\pi/4} (\sec x - x)(\sec x + x)\ dx$. **(3 marks)**

2012

Use the substitution $x = 4\sin\theta$ to evaluate $\int_0^2 \sqrt{16-x^2}\ dx$. **(6 marks)**

2013

① The velocity, v, of a particle P at time t is given by

$$v = e^{3t} + 2e^{t}$$

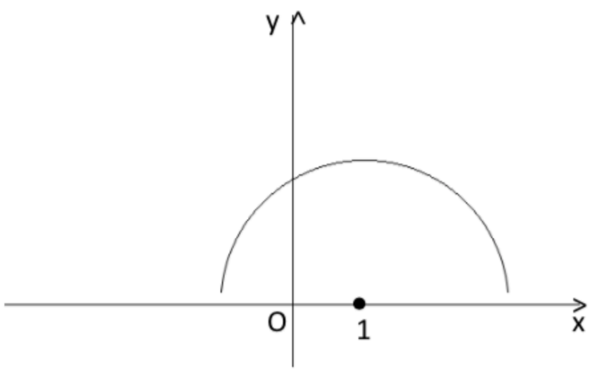
(a) Find the acceleration of P at time t.

(b) Find the distance covered by P between $t=0$ and $t=\ln 3$. **(2, 3 marks)**

② Integrate $\frac{\sec^2 3x}{1+\tan 3x}$ with respect to x. **(4 marks)**

2014

① A semi-circle with centre $(1,0)$ and radius 2, lies on the x-axis as shown.
Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis.



(5 marks)

② Use the substitution $x = \tan\theta$ to determine the exact value of

$$\int_0^1 \frac{dx}{\left(1+x^2\right)^{3/2}}.$$

(6 marks)

<u>2015</u>

Find $\int \frac{2x^3 - x - 1}{(x-3)\left(x^2+1\right)}\,dx,\ x > 3.$ **(9 marks)**

2008 Q7

7. Use integration by parts to obtain $\int 8x^2 \sin 4x \, \mathrm{d}x$.

2009 Q9

9. Use integration by parts to obtain the exact value of $\int_0^1 x \tan^{-1} x^2 \, dx$.

2014 Q15

15. (*a*) Use integration by parts to obtain an expression for

$$\int e^x \cos x \, dx.$$

(*b*) Similarly, given $I_n = \int e^x \cos nx \, dx$ where $n \neq 0$,

obtain an expression for I_n.

(*c*) Hence evaluate $\int_0^{\frac{\pi}{2}} e^x \cos 8x \, dx$.

2015 Q10

10. Obtain the exact value of $\int_0^2 x^2 e^{4x} dx$.

Past Paper Answers
Integration

2001

(a) $x + \frac{1}{2(x+1)} + \frac{1}{2(x-1)}$ (b) $\frac{1}{2}(x + \ln|x + 1| + \ln|x - 1|) + c$

2002

$\frac{1}{2}tan^{-1}\left(\frac{x+2}{2}\right)$+c

2003

$\frac{3}{8}$

2004

(1) $\frac{1}{5(x-3)} - \frac{1}{5(x+2)}$; $\frac{1}{5}ln\left(\frac{4}{9}\right)$ (2) $\frac{\pi}{4} - \frac{\pi}{4e^4} = 0.7710\ to\ 4sf$

2005

(1) $\frac{8}{3}$ (2) $\frac{1}{x} - \frac{x}{x^2+1}$; $ln\left(\frac{k\sqrt{2}}{\sqrt{k^2+1}}\right)$; $\frac{k\sqrt{2}}{\sqrt{k^2+1}}$; lim-> $\sqrt{2}$

(3)(a) $\frac{cosx}{2(\sqrt{sinx})}$ (b) proof, k=2 ; $-\sqrt{1-x^2} + c$

2006

$3ln|x^4 - x^2 + 1| + c$

2007

(1) $\frac{1}{x} + \frac{2}{x+2} - \frac{1}{x-3}$; m=8, n=9 (2) $\frac{1}{24}$; $V = \frac{\pi}{24}$ cubic units

2008

(1) $\frac{4}{x} + \frac{8x}{x^2+5}$; $4ln3$ (2) sec^2x; proof; $tanx - x + c$

2009

(1) Proof (2) $\frac{\pi}{2} - 1$

2010

(1) $ln\left(\frac{32}{25}\right)$ (2) $A = \frac{32}{3}$ sq units; $V = \frac{24\pi}{5}$ cubic units

2011

(1) $\frac{2}{x-1} - \frac{3}{x+5}$; $ln\left(\frac{(x-1)^2}{(x+5^3)}\right) + c$ (2) $1 - \frac{\pi^3}{192}$

2012

$\frac{4\pi+6\sqrt{3}}{3}$

2013

(1) (a) $a = 3e^{3t} + 2e^{t}$ (b) $\frac{38}{3}$ units (2) $\frac{1}{3}\ln(1 + tan3x)$+c

2014

(1) $V = 9\pi$ cubic units (2) $\frac{1}{\sqrt{2}}$

2015

$2x - 5ln|x - 3| + \frac{1}{2}ln|x^2 + 1| + c$

2008 q7

$\left(\frac{1}{4} - 2x^2\right)cos4x + xsin4x + c$

2009 q9

$\frac{\pi}{8} - \frac{1}{4}ln2$

2014 q15

(a) $\frac{1}{2}e^{x}(cosx + sinx) + c$ (b) $I_n = \frac{1}{n^2+1}e^{x}(cosnx + nsinnx) + c$

(c) $\frac{1}{65}(e^{\frac{\pi}{2}}$-1) = 0.05862 to 4 sf

2015 q10

$\frac{25e^8-1}{32}$

Advanced Higher Mathematics
Formal Homework Assignment

Integration

1. Integrate the following with respect to x.

(a) $\int \sec^2 4x\,dx$ (b) $\int 9x^2 e^{6x^3}\,dx$

2. Evaluate the following integrals

(a) $\int_1^2 \frac{x^2+x}{x}\,dx$ (b) $\int_0^4 \frac{x+2}{x+1}\,dx$

3. Integrate the following with respect to x.

(a) $\int \frac{4x}{x^2+5}\,dx$ (b) $\int 10\sin^4 x\cos x\,dx$

4. Find the following integrals

(a) $\int \frac{3}{(1+x^2)}\,dx$ (b) $\int \frac{dx}{2\sqrt{1-x^2}}$

(c) $\int \frac{dx}{\sqrt{4-x^2}}$ (d) $\int \frac{dx}{(9+x^2)}$

5. (a) Express in partial fractions $\frac{x^2+2x}{(x-2)(x^2+4)}$,

(b) Hence find $\int \frac{x^2+2x}{(x-2)(x^2+4)}\,dx$

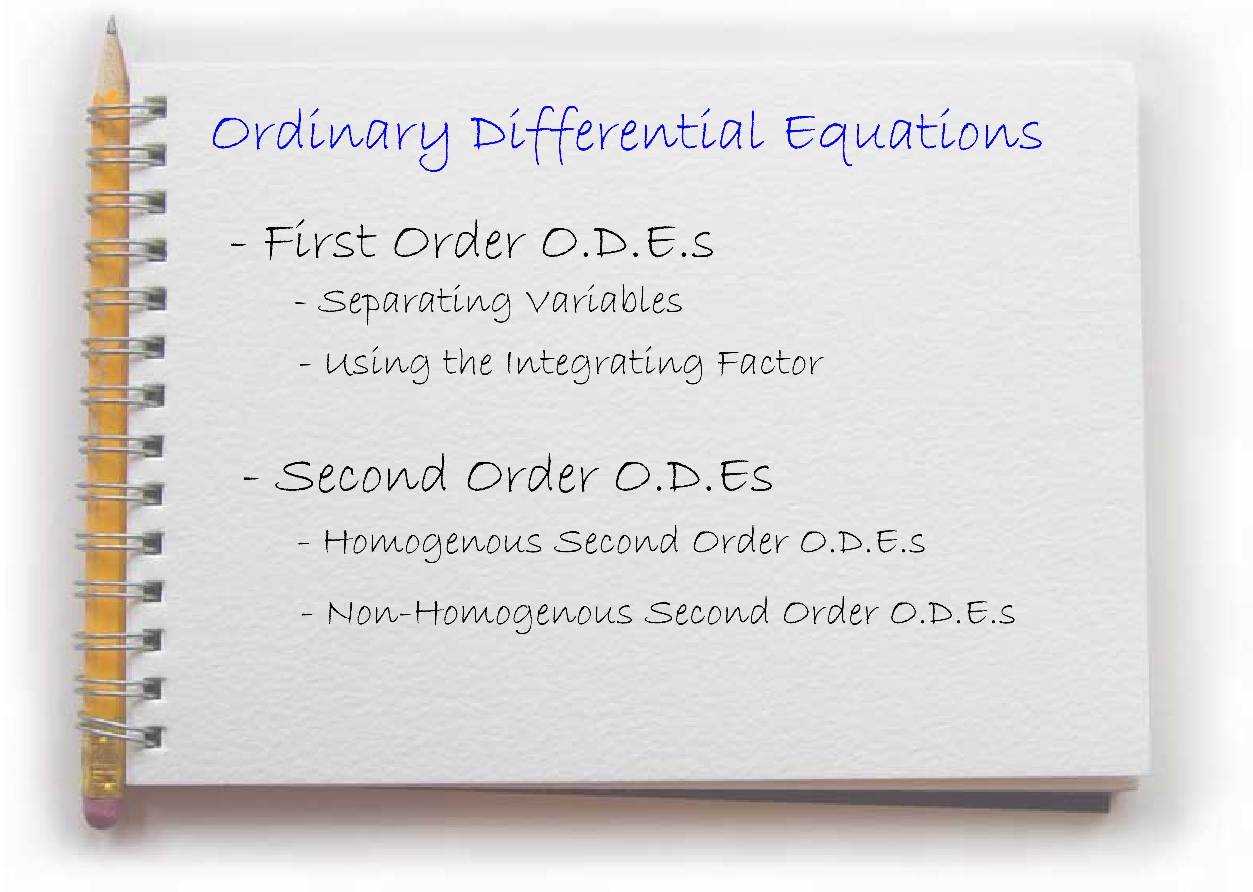
6. Use integration by parts to find

(a) $\int 2x\cos 3x\,dx$ (b) $\int x^2 e^{4x}\,dx$

7. Use the substitution $x = \frac{3}{3}\cos u$ to evaluate

$$\int_0^{\frac{3}{4}} \frac{x}{\sqrt{9-4x^2}}\,dx$$

8. Reflect on your understanding of integration, particularly in relation to connections to your existing knowledge and any elements of these new topics which interest you.



Differential Equations

A differential equation is an equation involving an unknown function and its derivatives. Its order is determined by the highest-order derivative in the equation.

The solution of a differential equation is a function $y = f(x)$ which satisfies the equation.

A general solution contains a constant of integration, c.

A particular solution uses additional information (e.g. initial conditions) to find a unique value for c.

1st order differential equation	$x^2 \frac{dy}{dx} = y + 3$	example 1
2nd order differential equation	$\frac{d^2y}{dx^2} - \frac{dy}{dx} + \sin x = y$	example 2

Example 1: (a) Solve: $\frac{dy}{dx} = x$

(b) Solve: $\frac{dy}{dx} = x$ when x = 2 and y = 5

Separating Variables

For any first order differential equation that can be expressed in the form:

$$f(y)\frac{dy}{dx} = g(x)$$

we separate the variables and integrate both sides.

$$\int f(y)\frac{dy}{dx}dx = \int g(x)dx$$

$$\int f(y)dy = \int g(x)dx$$

Example 2: Find the general solution of: $\frac{dy}{dx} = \frac{1}{x^2 y}$

Example 3: Find the general solution of: $\frac{dy}{dx} = 4x(1+y^2)$

Example 4: Find the general solution of: $x\frac{dy}{dx}+\frac{dy}{dx}-2=y$

Example 5: Find the general solution of: $(1+x)\dfrac{dy}{dx} = xy$

Particular Solutions

To find a particular solution to an ODE, substitute in given values of x and y to find c, then present the solution in a simplified form.

Example 6: Find the particular solution to: $\frac{dy}{dx} = e^{\left(\frac{1}{2}x - y\right)}$

given the initial conditions, x=0 and y=0.

Example 7: Find the particular solution to: $\sec y + e^x \frac{dy}{dx} = 0$

given that $y = \frac{\pi}{6}$ when $x = 0$

Example 8: a) use the technique of integrating by parts to find the integral:

$$\int xe^{-x}dx$$

b) Hence find the particular solution to the equation:

$$e^x\frac{dy}{dx} = xy^2 \qquad \text{given that } y = 1 \text{ when } x = 0$$

Applications of O.D.E.s

Problems involving rate of change can be solved by constructing a differential equation.

Example 9:

There are 500 football stickers to collect to fill an album. The rate at which the number of stickers, N, in the album increases is directionally proportional to the number of stickers still to collect.

A boy buys stickers for his album every day. Let t be the number of days since he started his collection.

a) At $t = 0$ the boy has no stickers. After 4 days he has 50 stickers. Find the particular solution to the differential equation to express N in terms of t.

b) When he needs only 50 more stickers to complete his collection the boy can send away for them. After how many days can he do this?

Using the Integrating Factor

First order linear differential equations can be expressed in the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

To solve these equations, we start by multiplying both sides by an integrating factor, $\mu(x)$.

Example 10: Find the general solution of:

$$\frac{dy}{dx} + \frac{y}{x} = 1$$

Example 11: Find the general solution of:

$$x\frac{dy}{dx} + (x-2)y = x^3$$

Example 12: Find the general solution of:

$$(1+x^2)\frac{dy}{dx} - xy = x(1+x^2)$$

Particular Solutions of 1st Order O.D.E.s

As before, particular solutions can be found for given values of x and y.

Example 13: Find the particular solution of:

$$x\frac{dy}{dx} + 2y = x^3 \qquad \text{when } x = 1 \text{ and } y = 2$$

Example 14:

In the first few weeks after birth a baby gains weight at a rate proportional to its weight. A baby weighing 3.5 kg at birth weighs 4.2 kg after two weeks. How much did it weigh after 5 days?
(Give your answer to two decimal places)?

Second Order O.D.E.s

These are expressed in the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b and c are constants, $a \neq 0$

If f(x) = 0, the equation is HOMOGENOUS

If $f(x) \neq 0$, the equation is NON-HOMOGENOUS

Homogeneous Second Order Differential Equations

These have the general form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad ....(1)$$

where a, b and c are constants, $a \neq 0$

We wish to find values of m such that $y = e^{mx}$ is a solution.

As $y = e^{mx}$ $\frac{dy}{dx} = me^{mx}$ $\frac{d^2y}{dx^2} = m^2e^{mx}$

Substituting into (1) gives:

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

Thus $y = e^{mx}$ is a solution if $am^2 + bm + c = 0$ **Auxiliary Equation**

General Solution of a 2nd Order ODE

Thus $y = e^{mx}$ is a solution if $am^2 + bm + c = 0$ **Auxiliary Equation**

Solving the A.E. will produce two values for m. (m_1 and m_2).

Hence there are 2 solutions - y_1 and y_2

If y_1 and y_2 are independent then :

1) $y_1 + y_2$ is also a solution

2) Ay_1 and By_2 (where A and B are constants) are also solutions

Thus the general solution of equation (1) is:

$$y = Ay_1 + By_2 \quad \text{or} \quad y = Ae^{m_1 x} + Be^{m_2 x}$$

The Nature of a General Solution

The A.E. $am^2 + bm + c = 0$ has 3 possible solutions.

1) REAL, DISTINCT ROOTS ($b^2 - 4ac > 0$)

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

2) EQUAL ROOTS ($b^2 - 4ac = 0$) $m_1 = m_2$ so

$$y = Ae^{mx} + Be^{mx}$$

$$y = (A + Bx)e^{mx}$$

3) COMPLEX ROOTS ($b^2 - 4ac < 0$) $m_1, m_2 = p \pm qi$ so

$$y = e^{px}(A\cos qx + B\sin qx)$$

Example 15: Find the general solution of:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

Example 16: Find the general solution of:

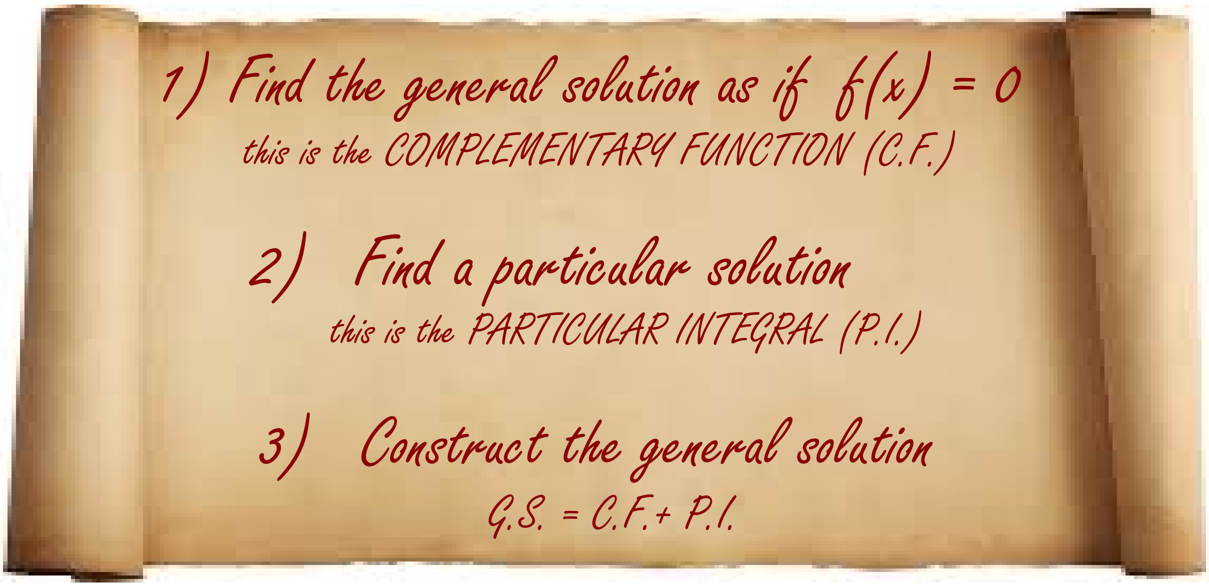
$$9\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 4y = 0$$

Example 17: Find the general solution of:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6y = 0$$

Non-Homogeneous Second Order Linear Differential Equations

Here, $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x) \quad , \quad f(x) \neq 0$



The nature of the P.I. depends on whether *f(x)* is:

a) polynomial
b) trigonometric
c) exponential

(a) Where *f(x)* is a polynomial function

then $y_P = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ (of the same order as $f(x)$)

Example 18: Find the general solution of: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 8x^2 + 3$

(b) Where *f(x)* is a trig function (*a*sin *nx* or *a*cos *nx*)

then $y_P = p\sin nx + q\cos nx$ (p and q are constants)

Example 19: Find the general solution of: $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 25\sin x$

(c) Where *f*(*x*) is an exponential function (e^{nx})

then $y_P = ke^{nx}$ (k is a constant)

Example 20: Find the general solution of: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{4x}$

Additional Notes on 2nd Order ODEs

1) Where a term in the P.I. already appears in the C.F., add an extra x term in the P.I.

e.g. for a CF: $y = Ae^{2x} + Be^{-x}$

where PI should be: $y_P = Ce^{2x}$

instead use: $y_P = Cxe^{2x}$

(Product rule required for differentiation)

2) Where *f(x)* is a sum of different forms, create a PI which combines the required 'templates'.

e.g. for: $f(x) = 3x + \sin 2x$

try: $y_P = Cx + D + E\sin 2x + F\cos 2x$

3) Particular solutions can be derived, given specific values for *x*, *f(x)* and *f '(x)*. Only substitute these in when the complete general solution has been found.

Example 21: Find the particular solution of:

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0 \qquad \text{given } y = 3 \text{ and } \frac{dy}{dx} = 20 \text{ when } x = 0$$

Example 22: Find the particular solution of:

$$\frac{d^2y}{dx^2} - y = 2e^x \qquad \text{given } y = 5 \text{ and } \frac{dy}{dx} = 0 \text{ when } x = 0$$

Past Paper Questions

Differential Equations 1

2001

A chemical plant food loses effectiveness at a rate proportional to the amount present in the soil. The amount M grams of plant food effective after t days satisfies the differential equation

$\frac{dM}{dt} = kM$, where k is a constant.

(a) Find the general solution for M in terms of t where the initial amount of food is M_0 grams.

(b) Find the value of k if, after 30 days, only half the initial amount of plant food is effective.

(c) What percentage of the original amount of plant food is effective after 35 days?

(d) The plant food has to be renewed when its effectiveness falls below 25%.
Is the manufacturer of the plant food justified in calling its product "sixty day super food"?

(3, 3, 2, 2 marks)

2003

The volume $V(t)$ of a cell at time t changes according to the law $\frac{dV}{dt} = V(10 - V)$ for $0 < V < 10$

Show that $\frac{1}{10}\ln V - \frac{1}{10}\ln(10 - V) = t + C$ for some constant C.

Given that $V(0) = 5$, show that $V(t) = \frac{10e^{10t}}{1 + e^{10t}}$.

Obtain the limiting value of $V(t)$ as $t \to \infty$. **(4, 3, 2 marks)**

2007

A garden centre advertises young plants to be used as hedging.
After planting, the growth, G metres (ie the increase in height) after t years is modelled by the differential equation

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

where k is a constant and $G = 0$ when $t = 0$.
(a) Express G in terms of t and k.

(b) Given that a plant grows $0 \cdot 6$ metres by the end of 5 years, find the value of k correct to 3 decimal places.

(c) On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified?

(d) Given that the initial height of the plants was $0 \cdot 3$ m, what is the likely long-term height of the plants? **(4, 2, 2, 2 marks)**

2009

Given that $x^2 e^y \frac{dy}{dx} = 1$ and $y = 0$ when $x = 1$, find y in terms of x. **(4 marks)**

2011

Given that $y > -1$ and $x > -1$, obtain the general solution of the differential equation

$$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$$

expressing your answer in the form $y = f(x)$. **(5 marks)**

2013

In an environment without enough resources to support a population greater than 1000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{dP}{dt} = P(1000 - P)$$

Show that $\ln \frac{P}{1000 - P} = 1000t + C$ for some constant C.

Hence show that $P(t) = \frac{1000K}{K + e^{-1000t}}$ for some constant K.

Given that $P(0) = 200$, determine at what time t , $P(t) = 900$. **(4, 3, 3 marks)**

2015

Vegetation can be irrigated by putting a small hole in the bottom of a cylindrical tank, so that the water leaks out slowly. Torricelli's Law states that the rate of change of volume, V, of the water in the tank is proportional to the square root of the height, h, of the water above the hole.
This is given by the differential equation:

$$\frac{dV}{dt} = -k\sqrt{h},\ k > 0$$

(a) For a cylindrical tank with constant cross-sectional area, A, show that the rate of change of the height of the water in the tank is given by

$$\frac{dh}{dt} = \frac{-k}{A}\sqrt{h}\ .$$

(b) Initially, when the height of the water is 144cm, the rate at which the height is changing is -3 cm/hr.

By solving the differential equation in part (a), show that $h = \left(12 - \frac{1}{80}t\right)^2$.

(c) How many days will it take for the tank to empty?

(d) Given that the tank has radius 20cm, find the rate at which the water was being delivered to the vegetation (in cm^3/hr) at the end of the fourth day.

(2, 4, 2, 3 marks)

Past Paper Questions

Differential Equations 2

2001

① Find the solution of the following differential equation:

$$\frac{dy}{dx}+\frac{y}{x}=x, \qquad x>0$$ **(4 marks)**

② Find the general solution of the following differential equation:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=6x-1$$ **(5 marks)**

2002

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=4\cos x$$

Hence determine the solution which satisfies $y(0)=0$ and $y'(0)=1$. **(6, 4 marks)**

2003

Solve the differential equation

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^x$$

given that $y=2$ and $\frac{dy}{dx}=1$, when $x=0$. **(10 marks)**

2004

(a) A mathematical biologist believes that the differential equation $x\frac{dy}{dx}-3y=x^4$ models a process. Find the general solution of the differential equation.

Given that $y=2$ when $x=1$, find the particular solution , expressing y in terms of x.

(5, 2 marks)

(b) The biologist subsequently decides that a better model is given by the equation

$y\frac{dy}{dx}-3x=x^4$.

Given that $y=2$ when $x=1$, obtain y in terms of x. **(4 marks)**

2005

Obtain the general solution of the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=20\sin x$

Hence find the particular solution for which $y=0$ and $\frac{dy}{dx}=0$ when $x=0$. **(7, 3 marks)**

2006

Solve the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0$$

given that when $x=0$, $y=0$ and $\frac{dy}{dx}=2$. **(6 marks)**

2007

Obtain the general solution of the equation $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=e^{2x}$ **(6 marks)**

2008

Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=2x^2$$

Given that $y=\frac{1}{2}$ and $\frac{dy}{dx}=1$, when $x=0$, find the particular solution. **(7, 3 marks)**

2009

(a) Solve the differential equation

$$\left(x+1\right)\frac{dy}{dx}-3y=\left(x+1\right)^4$$

given that $y=16$ when $x=1$, expressing the answer in the form $y=f\left(x\right)$.

(b) Hence find the area enclosed by the graphs of $y=f\left(x\right)$, $y=\left(1-x\right)^4$ and the x-axis.

(6, 4 marks)

2010

Obtain the general solution of the equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0.$$

Hence obtain the solution for which $y=3$ when $x=0$ and $y=e^{-\pi}$ when $x=\frac{\pi}{2}$.

(4, 3 marks)

2011

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=e^x+12$$

Find the particular solution for which $y=\frac{-3}{2}$ and $\frac{dy}{dx}=\frac{1}{2}$ when $x=0$. **(7, 3 marks)**

2012

(a) Express $\frac{1}{(x-1)(x+2)^2}$ in partial fractions.

(b) Obtain the general solution of the differential equation

$$(x-1)\frac{dy}{dx}-y=\frac{x-1}{(x+2)^2}$$

expressing your answer in the form $y=f(x)$. **(4, 7 marks)**

2013

Solve the differential equation

$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=4e^{3x}$, given that $y=1$ and $\frac{dy}{dx}=-1$ when $x=0$. **(11 marks)**

2014

Find the solution $y=f(x)$ to the differential equation

$4\frac{d^2y}{dx^2}-4\frac{dy}{dx}+y=0,$ given that $y=4$ and $\frac{dy}{dx}=3$ when $x=0$. **(6 marks)**

2015

Solve the second order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x}$$

given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 0$. **(10 marks)**

Past Paper Answers

Differential Equations 1

2001 (q.A10)
(a) $M = M_0e^{kt}$ (b) k = -0.023 (c) 44.7% (d) Yes, after 60 days M = 25.1%, >25%

2003 (q.A11)
Proof; Proof; V -> 10

2007 (q.14)
(a) $G = 25k(1 - e^{-\frac{t}{25}})$ (b) k = 0.132 (c) growth = 1.088m, so claim justified (d) 3.60m

2009 (q.3)
$y = ln\left|2 - \frac{1}{x}\right|$

2011 (q.9)
$y = Ae^{2(1+x)^{\frac{3}{2}}} - 1$

2013 (q.16)
Proof; Proof; t = 0.00358

2015 (q.18)
(a) Proof (b) Proof (c) 40 days (d) $-108\pi\ cm^3/hr$

Past Paper Answers

Differential Equations 2

2001 (q.B2 & B5)
(1) $y = \frac{x^2}{3} + \frac{c}{x}$ (2) $y = Ae^x + Be^{-3x} - 2x - 1$

2002 (q.B5)
$y = e^{-x}(Acos2x + Bsin2x) + \frac{2}{5}(2cosx + sinx)$; $y = -\frac{e^{-x}}{10}(8cos2x + 2sin2x) + \frac{2}{5}(2cosx + sinx)$

2003 (q.B6)
$y = (1 - 2x)e^{2x} + e^x$

2004 (q.15)
(a) $y = x^4 + x^3c$; $y = x^4 + x^3$ (b) $y = \left(\frac{2}{5}x^5 + 3x^2 + \frac{3}{5}\right)^{\frac{1}{2}}$

2005 (q.14)
$y = Ae^{2x} + Be^x + 2sinx + 6cosx$; $y = 4e^{2x} - 10e^x + 2sinx + 6cosx$

2006 (q.8)
$y = 2e^{-x}sinx$

2007 (q.8)

$y = Ae^{-3x} + Bxe^{-3x} + \frac{e^{2x}}{25}$

2008 (q.13)

$y = Ae^{x} + Be^{2x} + x^2 + 3x + \frac{7}{2}$; $y = -4e^{x} + e^{2x} + x^2 + 3x + \frac{7}{2}$

2009 (q.3)

(a) $y = (x+1)^4$ (b) $A = \frac{2}{5}$

2010 (q.11)

$y = e^{-2x}(Acosx + Bsinx)$; $y = e^{-2x}(3cosx + sinx)$

2011 (q.14)

$y = Ae^{2x} + Be^{-x} - \frac{e^x}{2} - 6$; $y = 2e^{2x} + 3e^{-x} - \frac{e^x}{2} - 6$

2012 (q.15)

(a) $\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$; (b) $y = (x-1)\left[\frac{1}{9}ln\left|\frac{x-1}{x+2}\right| + \frac{1}{3(x+2)} + c\right]$

2013 (q.14)

$y = (2x^2 - 4x + 1)e^{3x}$

2014 (q.8)

$y = (4+x)e^{\frac{1}{2}x}$

2015 (q.16)

$y = e^{-x}\left(\frac{5}{6}cos3x + \frac{1}{6}sin3x\right) + \frac{1}{6}e^{2x}$