St Andrew's Academy

Department of Mathematics

Advanced Higher

Course Notes

Book 2

Contents

FORMULAE LIST

Standard derivatives	
$f(x)$	$f'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\dfrac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax)+c$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)+c$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c$
$\dfrac{1}{x}$	$\ln\lvert x\rvert+c$
e^{ax}	$\dfrac{1}{a}e^{ax}+c$

Summations

(Arithmetic series) $$S_n = \frac{1}{2}n\left[2a+(n-1)d\right]$$

(Geometric series) $$S_n = \frac{a(1-r^n)}{1-r}$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \ldots$$

FORMULAE LIST (continued)

De Moivre's theorem

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Vector product

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ about the origin, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Definitions

an ordered list of terms, usually, but not always, with a clear relationship between terms ..... **Sequence**

a number (or element) in a sequence - the n^{th} term is u_n ..... **Term**

a sequence with an end term $u_1, u_2, u_3, \ldots u_n$ ..... **Finite Sequence**

a sequence which continues indefinitely ..... **Infinite Sequence**

the sum of the terms of a sequence ..... **Series**

Arithmetic Sequences and Series

In an arithmetic sequence, each term differs from the previous term by a constant, d - the common difference

i.e. 1, 4, 7, 10, ...

so $d = u_{n+1} - u_n$

101, 91, 81, ...

If a is the first term in a sequence, then:

$u_1 = a$

$u_2 = a + d$

$u_3 = a + 2d$

$u_4 = a + 3d$

$u_n = a + (n-1)d$ The n^{th} term of an arithmetic sequence

Example 1: Find a formula for the n^{th} term of the sequence: 14, 22, 30, 38, ... and find the value of u_{19}

Sum to n terms of an arithmetic series

$S_n = a + (a + d) + (a + 2d) + ... + (a + (n-2)d) + (a + (n-1)d)$

reversing the order, then:

$S_n = (a + (n-1)d) + (a + (n-2)d) + ... + (a + 2d) + (a + 2d) + a$

adding , gives:

$2S_n = (2a + (n-1)d) + (2a + (n-1)d) + ... + (2a + (n-1)d) + (2a + (n-1)d)$

$$2S_n = n[2a + (n-1)d]$$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{sum to n terms of an arithmetic series}}$$

Example 2: For 4 + 6 + 8 + ..., find S_{11}

Example 3: Find the number of terms in the sequence $7, 11, 15, ..., 147$

Example 4: Find the sum of the series $3 + 9 + 15 + ... + 81$

Example 5: If $u_3 = 15$ and $u_7 = 31$, find the values of a, d and S_{12}

Example 6: If $u_5 = 23$ and $S_6 = 102$, find the values of a, d and S_{10}

Geometric Sequences and Series

In a geometric sequence, the ratio of each term to the previous term is a constant, r - the common ratio.

e.g. 1, 2, 4, 8, ...

10, 1, 0.1, 0.01, ...

so $r = \frac{u_{n+1}}{u_n}$ or $u_{n+1} = ru_n$

If a is the first term in a sequence, then:

$u_1 = a$

$u_2 = ar$

$u_3 = ar^2$

$u_4 = ar^3$

$u_n = ar^{(n-1)}$ The n^{th} term of a geometic sequence

Example 7: Find the first 5 terms of the geometric sequence:

$$3\left(\frac{2}{3}\right)^{n-1}$$

Example 8: Find the formula for the n^{th} term of the sequence:

$4, \ -10, \ 25, \ -62\frac{1}{2}, \ \ldots$ and find u_6

Sum to n terms of a geometric series

$$S_n = a + ar + ar^2 + \ldots + ar^{(n-2)} + ar^{(n-1)} \quad \_\_\_(1)$$

(1) x r gives:

$$rS_n = ar + ar^2 + ar^3 \ldots + ar^{(n-1)} + ar^n \quad \_\_\_(2)$$

(1) - (2) gives: $S_n - rS_n = a - ar^n$

$$(1-r)S_n = a(1-r^n)$$

$$\boxed{S_n = \frac{a(1-r^n)}{(1-r)} \quad \text{sum to n terms of a geometric series, r < 1}}$$

(1) - (2) gives:

$$\boxed{S_n = \frac{a(r^n-1)}{(r-1)} \quad \text{sum to n terms of a geometric series, r > 1}}$$

Example 9: Find S_9 for the geometric series: $6 + 4\frac{1}{2} + 3\frac{3}{8} + \ldots$

Example 10: Given $u_4 = 256$ and $u_7 = 2048$, find a, r and S_{10}

Example 11: Find n if: $5 + 5^2 + 5^3 + \ldots + 5^n = 97655$

Sum to infinity of a geometric series

For the infinite series, 8 + 4 + 2 + 1 + ½ + ...

$S_{10} = 15\tfrac{63}{64}$ $S_{11} = 15\tfrac{127}{128}$ $S_{12} = 15\tfrac{255}{256}$

The sum appears to be approaching 16, as *n* increases.

i.e. $S_n \to 16$ as $n \to \infty$

or $S_\infty = 16$

Similar to recurrence relations, a finite limit can only exist when
-1 < r < 1

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$\boxed{S_\infty = \frac{a}{1-r}}$$ sum to infinity of a geometric series, -1 < r < 1

Example 12: Find S_∞ for the geometric series : 99 + 66 + 44 + ...

Example 13: Find S_∞ for the geometric series : 200 - 120 + 72 - ...

Expansion of $\dfrac{1}{1-r}$ as a geometric series

We know that the binomial coefficient,

$$ {}^n_rC = \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

We also know that

$$ (a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{n}b^n $$

Therefore:

$$ (a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\times 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3 + \ldots + b^n $$

$$ \frac{1}{1-r} = (1+(-r))^{-1} $$

$$ = 1 + \frac{-1}{1}(-r) + \frac{(-1)(-2)}{1\times 2}(-r)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(-r)^3 + \ldots $$

$$ \boxed{\frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots} $$

Expansion of $\frac{1}{1-r}$

Example 14:

Expand $\frac{1}{1-2x}$ in ascending powers of x, giving the first four terms.

Example 15:

Expand $\dfrac{1}{1+\frac{x}{3}}$ in ascending powers of x, giving the first four terms.

Power Series

A power series is a series in the form:

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ..$$

A divergent power series is one whose sum will increase infinitely.

A convergent power series is one whose sum will approach a limit.

Maclaurin Series

If $\quad f(x) = a + bx + cx^2 + dx^3 \qquad\qquad f(0) = a$

$f'(x) = b + 2cx + 3dx^2 \qquad\qquad f'(0) = b$

$f''(x) = 2c + 6dx \qquad\qquad f''(0) = 2c \qquad c = \dfrac{f''(0)}{2}$

$f'''(x) = 6d \qquad\qquad f'''(0) = 6d \qquad d = \dfrac{f'''(0)}{6}$

$f^{iv}(x) = 0$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

Maclaurin's Theorem

For a given function $f(x)$, where $f^{n}(0)$ for all n, then:

$$\boxed{f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^n(0)}{n!}x^n}$$

Example 16:

Use Maclaurin's Theorem to expand as a series of ascending powers of x:

$$f(x) = e^x$$

$f(x) =$ $f(0) =$

$f'(x) =$ $f'(0) =$

$f''(x) =$ $f''(0) =$

$f'''(x) =$ $f'''(0) =$

Example 17:

Use Maclaurin's Theorem to expand as a series of ascending powers of x:

$$f(x) = \sin x$$

$f(x) =$ $f(0) =$

$f'(x) =$ $f'(0) =$

$f''(x) =$ $f''(0) =$

$f'''(x) =$ $f'''(0) =$

Example 18:

Use Maclaurin's Theorem to expand as a series of ascending powers of x:

$$f(x) = \ln(1 + x)$$

$f(x) =$ $\qquad$ $f(0) =$

$f'(x) =$ $\qquad$ $f'(0) =$

$f''(x) =$ $\qquad$ $f''(0) =$

$f'''(x) =$ $\qquad$ $f'''(0) =$

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots} \quad \text{for all } x \in R$$

$$\boxed{\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots} \quad \text{for all } x$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots} \quad \text{for } -1 < x < 1$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots} \quad \text{for all } x$$

$$\boxed{tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots} \quad \text{for all } x$$

Expansions of Composite Functions

Example 19: Write out the expansion of e^{-2x} as far as x^4

Example 20: Write out the expansion of $\ln(2 + x)$ as far as x^4

Combining Expansions

Example 21: Expand $\cos(\sin x)$ as far as x^4

Example 22: Expand $\ln(\sin x)$ as far as x^2

Example 23: Expand $e^{2x}\cos 3x$ as far as x^4

Past Paper Questions

Sequences and Series

2002

Define $S_n(x)$ by

$$S_n(x) = 1 + 2x + 3x^2 + \ldots + nx^{n-1}$$

where n is a positive integer.

Express $S_n(1)$ in terms of n.

By considering $(1-x)S_n(x)$, show that

$$S_n(x) = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{(1-x)}, \quad x \neq 1.$$

Obtain the value of $\lim_{n\to\infty}\left\{\frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \ldots + \frac{n}{3^{n-1}} + \frac{3}{2}\cdot\frac{n}{3^n}\right\}$. **(2, 4, 3 marks)**

2003

Given that $u_k = 11 - 2k, \quad (k \geq 1)$, obtain a formula for $S_n = \sum_{k=1}^{n} u_k$.

Find the values of n for which $S_n = 21$. **(3, 2 marks)**

2004

(a) Obtain the sum of the series $8 + 11 + 14 + \ldots + 56$.

(b) A geometric sequence of positive terms has first term 2, and the sum of the first three terms is 266, Calculate the common ratio.

(c) An arithmetic sequence, A, has first term a and common difference 2, and a geometric sequence, B, has first term a and common ratio 2. The first four terms of each sequence have the same sum. Obtain the value of a.

Obtain the smallest value of n such that the sum to n terms for sequence B is more than twice the sum to n terms for the sequence A. **(2, 3, 3, 2 marks)**

2005

The sum, $S(n)$, of the first n terms of a sequence, $u_1, u_2, u_3, \ldots$ is given by

$$S(n) = 8n - n^2, \quad n \geq 1.$$

Calculate the values of u_1, u_2, u_3 and state what type of sequence it is.

Obtain a formula for u_n in terms of n, simplifying your answer. **(3, 2 marks)**

2006

The first three terms of a geometric sequence are

$$\frac{x(x+1)}{(x-2)}, \frac{x(x+1)^2}{(x-2)^2} \text{ and } \frac{x(x+1)^3}{(x-2)^3}, \text{ where } x<2.$$

(a) Obtain expressions for the common ratio and the nth term of the sequence.

(b) Find an expression for the sum of the first n terms of the sequence.

(c) Obtain the range of values of x for which the sequence has a sum to infinity and find an expression for the sum to infinity.

(3, 3, 4 marks)

2007

Show that $\sum_{r=1}^{n}(4-6r)=n-3n^2$.

Hence write down a formula for $\sum_{r=1}^{2q}(4-6r)$.

Show that $\sum_{r=q+1}^{2q}(4-6r)=q-9q^2$. **(2, 1, 2 marks)**

2008

The first term of an arithmetic sequence is 2 and the $20th$ term is 97. Obtain the sum of the first 50 terms.

(4 marks)

2009

The first two terms of a geometric sequence are $a_1=p$ and $a_2=p^2$.

Obtain expressions for S_n and S_{2n} in terms of p, where $S_k=\sum_{j=1}^{k}a_j$.

Given that $S_{2n}=65S_n$ show that $p^n=64$.

Given also that $a_3=2p$ and that $p>0$, obtain the exact value of p and hence the value of n.

(1, 1, 2, 1, 1 marks)

2010

The second and third terms of a geometric series are -6 and 3 respectively.
Explain why the series has a sum to infinity and obtain this sum.

(5 marks)

2011

① Write down an expression for $\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2$ and an expression for $\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2$.

(1, 3 marks)

② The first three terms of an arithmetic sequence are $a, \frac{1}{a}, 1$ where $a < 0$.

Obtain the value of a and the common difference.

Obtain the smallest value of n for which the sum of the first n terms is greater than 1000.

(5, 4 marks)

2012

The first and fourth terms of a geometric sequence are 2048 and 256 respectively.
Calculate the value of the common ratio.

Given that the sum of the first n terms is 4088, find the value of n.

(2, 3 marks)

2013

Write down the sums to infinity of the geometric series

$$1 + x + x^2 + x^3 + \ldots\ldots$$

and

$$1 - x + x^2 - x^3 + \ldots\ldots$$

valid for $|x| < 1$.

Assuming that it is permitted to integrate an infinite series term by term, show that, for $|x| < 1$,

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right)$$

Show how this series can be used to evaluate $\ln 2$.
Hence determine the value of $\ln 2$ correct to 3 decimal places. **(7, 3 marks)**

2015

The sum of the first twenty terms of an arithmetic sequence is 320.

The twenty-first term is 37.

What is the sum of the first ten terms? **(5 marks)**

Past Paper Questions

Maclaurin Series

2001

Find the first four terms in the Maclaurin Series for $(2+x)\ln(2+x)$.

(4 marks)

2002

Find the Maclaurin expansion of $f(x)=\ln(\cos x)$, $0 \le x \le \frac{\pi}{2}$, as far as the term in x^4

(5 marks)

2003

Obtain the Maclaurin Series for $f(x)=\sin^2 x$ up to the term in x^4.

Hence write down a series for $\cos^2 x$ up to the term in x^4.

(5 marks)

2004

Obtain the first three non-zero terms in the Maclaurin Series of $f(x)=e^x \sin x$.

(5 marks)

2005

Write down the Maclaurin expansion of e^x as far as the term in x^4.

Deduce the Maclaurin expansion of e^{x^2} as far as the term in x^4.

Hence or otherwise, find the Maclaurin expansion of e^{x+x^2} as far as the term in x^4.

(2, 1, 3 marks)

2007

Find the Maclaurin Series for $\cos x$ as far as the term in x^4.

Deduce the Maclaurin Series for $f(x)=\frac{1}{2}\cos 2x$ as far as the term in x^4.

Hence write down the first three non-zero terms of the series for $f(3x)$.

(2, 2, 1 marks)

2008

Obtain the first three non-zero terms in the Maclaurin expansion of $x\ln(2+x)$.

Hence, or otherwise, deduce the first three non-zero terms in the Maclaurin expansion of $x\ln(2-x)$.

Hence obtain the first two non-zero terms in the Maclaurin expansion of $x\ln(4-x^2)$.

(3, 2, 2 marks)

2009

Express $\dfrac{x^2+6x-4}{(x+2)^2(x-4)}$ in partial fractions. Hence, or otherwise, obtain the first 3 non-zero terms in the Maclaurin expansion of $\dfrac{x^2+6x-4}{(x+2)^2(x-4)}$.

(4, 3, 2 marks)

2010

Obtain the first three non-zero terms in the Maclaurin expansion of $(1+\sin^2 x)$.

(4 marks)

2011

Obtain the first four terms in the Maclaurin series of $\sqrt{1+x}$ and hence write down the first four terms in the Maclaurin series of $\sqrt{1+x^2}$.

Hence obtain the first four terms in the Maclaurin series of $\sqrt{(1+x)(1+x^2)}$.

(4, 2 marks)

2012

Write down the Maclaurin expansion of e^x as far as the term in x^3.
Hence, or otherwise, obtain the Maclaurin expansion of $(1+e^x)^2$ as far as the term in x^3.

(1, 4 marks)

2013

Write down the sums to infinity of the geometric series

$$1+x+x^2+x^3+\dots$$

and

$$1-x+x^2-x^3+\dots$$

valid for $|x|<1$.

Assuming that it is permitted to integrate an infinite series term by term, show that, for $|x|<1$,

$$\ln\left(\frac{1+x}{1-x}\right)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right).$$

Show how this series can be used to evaluate $\ln 2$.
Hence determine the value of $\ln 2$ correct to 3 decimal places. **(7, 3 marks)**

2014

Give the first three non-zero terms of the Maclaurin series for $\cos 3x$.
Write down the first four terms of the Maclaurin series for e^{2x}.
Hence, or otherwise, determine the Maclaurin series for $e^{2x}\cos 3x$ up to, and including, the term in x^3. **(2, 1 ,3 marks)**

Advanced Higher Mathematics
Formal Homework Assignment

Sequences and Series

1. The sum of the first n terms of an arithmetic series is $n(n+5)$.
Find the first 3 terms of the series.

2. (a) In an arithmetic series of 9 terms, the first term is 5 and the last is 23.
Find the sum of the 9 terms.

(b) Find the sum of the first seven terms of a geometric series that has an eighth term of $\frac{2}{3}$ and a fifth term of 18.

3. Find the sum to infinity for the series: $64 - 16 + 4 - 1 + \frac{1}{4} - \ldots$

4. The third term of an arithmetic series is 17 and the 7th term is 33.
Find a formula for the sum of n terms and find the sum of 20 terms.

5. Find the values of x for which $x - 6$, $2x$ and $8x + 20$ are consecutive terms of a geometric series.

6. Expand $\dfrac{1}{1+4x}$ as a geometric series and state the necessary condition on x for the series to be valid.

7. State the Maclaurin series for $f(x) = \ln(1-2x)$ as far as the x^4 term.

8. Find the Maclaurin series for $f(x) = e^{2x}\sin 3x$ as far as the x^3 term.

9. The first two terms of a geometric sequence are $a_1 = p$ and $a_2 = p^2$.
Obtain expressions for S_n and S_{2n} in terms of p, where $S_k = \sum_{j=1}^{k} a_j$.
Given that $S_{2n} = 65S_n$, show that $p^n = 64$.
Given also that $a_3 = 2p$ and that $p > 0$, obtain the exact value of p and hence the value of n.

10. Reflect on your understanding of the new techniques in partial fractions, particularly in relation to:

(a) connections to your existing knowledge and

(b) any elements of this new area which interest you.

Past Paper Answers
Sequences and Series

2002
(i) $S_n(1) = \frac{1}{2}$n(n+1)   (ii) Proof   (iii) $\frac{5}{4}$

2003
(i) $11n - n^2$ (ii) n=3 or n=7

2004
(a) 544   (b) r = 11   (c) $a = \frac{12}{11}$; smallest value of n is 7.

2005
(i) $u_1 = 7, u_2 = 5, u_3 = 3,$ arithmetic sequence   (ii) 9 – 2n

2006
(a) $r = \frac{x+1}{x-2}$, $n^{th}\ term$: $\frac{x(x+1)^n}{(x-2)^n}$ (b) $-\frac{x(x+1)[(x-2)^n-(x+1)^n]}{3(x-2)^n}$ (c) $x < \frac{1}{2}$, $S_\infty = -\frac{1}{3}x(x+1)$

2007
(i) Proof   (ii) $2q - 12q^2$ (iii) Proof

2008
$S_{50} = 6225$

2009
(i) $S_n = \frac{p(1-p^n)}{1-p}$, $S_{2n} = \frac{p(1-p^{2n})}{1-p}$ (ii) Proof   (iii) $p = \sqrt{2}$, $n = 12$

2010
(i) $r = -\frac{1}{2}$; as $-1 < r < 1$, the series has a sum to infinity.   (ii) $S_\infty = 8$

2011
(1) (i) 0   (ii) $\frac{n^2}{2}(n+1)^2$ (2) (i) $a = -2,\ d = \frac{3}{2}$ (ii) n = 39

2012
(i) $r = \frac{1}{2}$ (ii) n = 9

2013
(i) $S_\infty = \frac{1}{1-x}$ (ii) $S_\infty = \frac{1}{1+x}$ (iii) Proof   (iv) $ln2\ when\ x = \frac{1}{3}$ (v) 0.693

2015
$S_{10} = 60$

Past Paper Answers
Maclaurin Series

2001

$f(x) = 2ln2 + (ln2 + 1)x + \frac{x^2}{4} - \frac{x^3}{24} + \cdots$

2002

$f(x) = -\frac{x^2}{2} - \frac{x^4}{12} + \cdots$

2003

$sin^2x = x^2 - \frac{x^4}{3} + \cdots$ $\quad cos^2x = 1 - x^2 + \frac{x^4}{3} + \cdots$

2004

$f(x) = x + x^2 + \frac{x^3}{3} + \cdots$

2005

(i)$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \dots$ (ii) $e^{x^2} = 1 + x^2 + \frac{x^4}{2}$ (iii) $e^{x^2+x} = 1 + x + \frac{3x^2}{2} + \frac{7x^3}{6} + \frac{25x^4}{24}$

2007

(i) $1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$ (ii) $\frac{1}{2} - x^2 + \frac{x^4}{3} - \cdots$ (iii) $\frac{1}{2} - 9x^2 + 27x^4 - \cdots$

2008

(i) $xln2 + \frac{x^2}{2} - \frac{x^3}{8} + \cdots$ (ii) $xln2 - \frac{x^2}{2} - \frac{x^3}{8} + \cdots$ (iii) $xln4 - \frac{x^3}{4} + \cdots$

2009

(i) $\frac{2}{(x+2)^2} + \frac{1}{x-4}$ (ii) $\frac{1}{4} - \frac{9x}{16} + \frac{23x^2}{64} - \cdots$

2010

$f(x) = 1 + x^2 - \frac{x^4}{3} + \cdots$

2011

(i) $f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \dots$ $f(x^2) = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \cdots$(ii) $f(x) = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} - \cdots$

2012

(i) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ (ii) $(1 + e^x)^2 = 4 + 4x + 3x^2 + \frac{5x^3}{3} + \cdots$

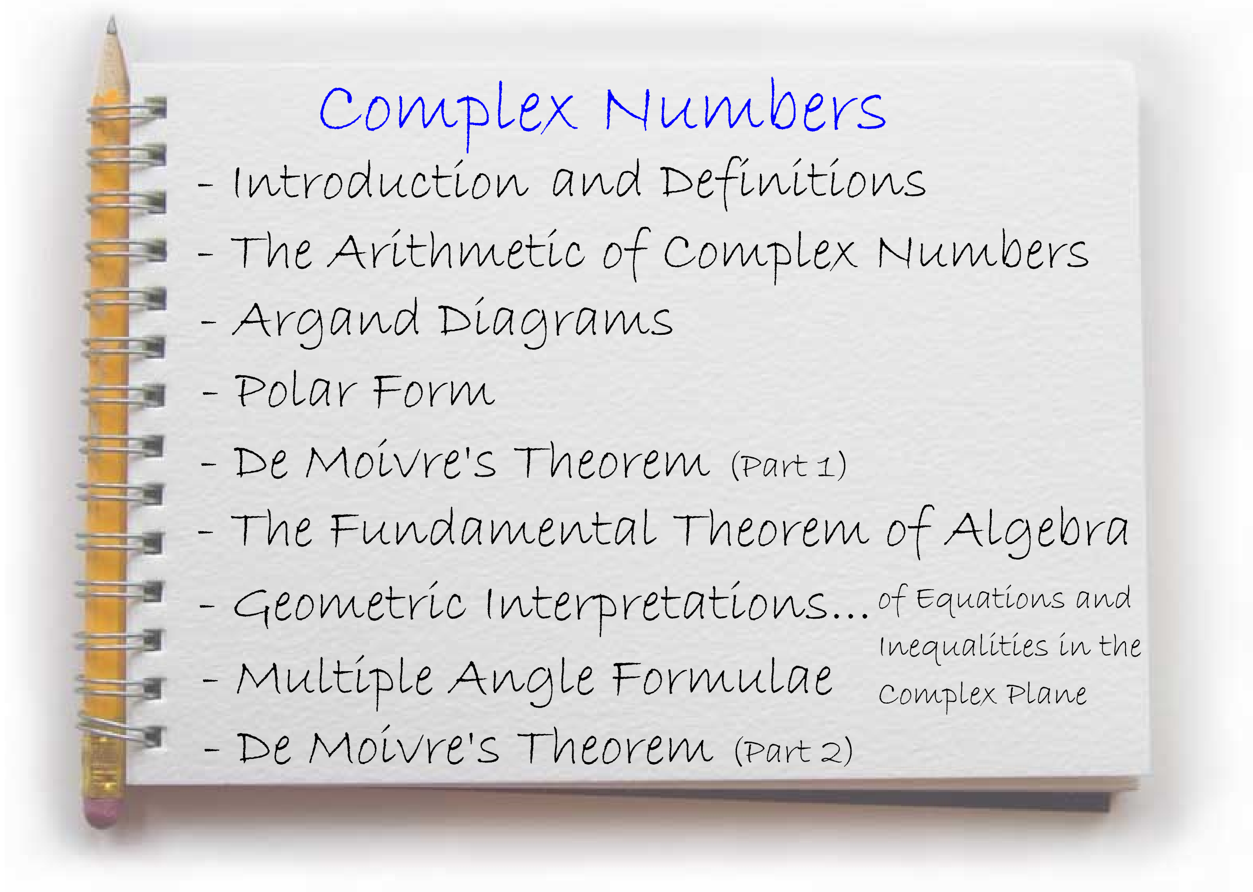
2013

(i) $S_\infty = \frac{1}{1-x}$ (ii) $S_\infty = \frac{1}{1+x}$ (iii) Proof (iv) $ln2\ when\ x = \frac{1}{3}$ (v) 0.693

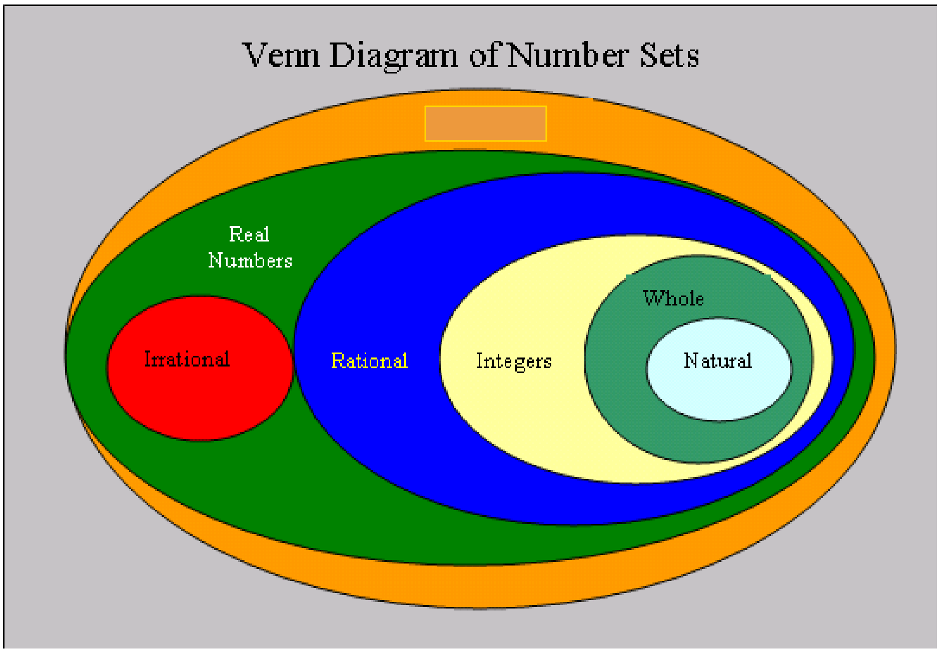
2014

(i) $cos3x = 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \cdots$ (ii) $e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots$

(iii) $e^{2x}cos3x = 1 + 2x - \frac{5x^2}{2} - \frac{22x^3}{3} + \cdots$



The story of numbers so far.....



Introduction

At present we can find the solutions of any quadratic equation:

$$ax^2 + bx + c = 0$$

as long as the discriminant is greater or equal to zero.

In such cases the roots (solutions) of the equation are REAL.

Imaginary Numbers

We define i as a non-zero number, such that:

$$i^2 = -1$$

Note: $\sqrt{-1} = \pm i$

Thus:

$\sqrt{-16}$ $\sqrt{-50}$ $(7i)^2$

We can now use i to solve quadratic equations with no real roots

Example 1: Find the roots of the equation: $x^2 - 4x + 13 = 0$

Definitions

- An imaginary number is a number of the form bi, where $b \in R$
- A complex number is a number of the form $a + bi$, where $a, b \in R$
- A complex number is usually denoted by the letter z.
  If $z = a + bi$, then a is the real part of z - $\text{Re}(z)$
  b is the imaginary part of z - $\text{Im}(z)$
- The complex conjugate of $z = a + bi$ *is* $\overline{z} = a - bi$
- If $z = a + bi$ is a solution of a quadratic equation
  then $\overline{z}$ is also a solution.

The Arithmetic of Complex Numbers

Addition: $(a + bi) + (c + di)$

Subtraction: $(a + bi) - (c + di)$

Example 2: $(2 + 3i) + (1 - i)$

Example 3: $\left(4-\frac{\sqrt{2}}{2}i\right)-\left(\frac{1}{2}+\frac{\sqrt{2}}{6}i\right)$

The Arithmetic of Complex Numbers

Multiplication:

$(a + bi)(c + di)$

Divison: Use complex conjugates

Example 4:

Simplify: $(4 + 3i)(5 - 2i)$

Example 5:

Simplify: $\dfrac{7+i}{3-4i}$

The Arithmetic of Complex Numbers

Square Root: Find 2 equations which can be solved simultaneously.

Example 6: Find $\sqrt{5-12i}$

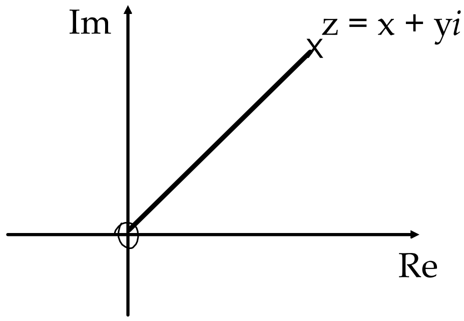
Argand Diagrams

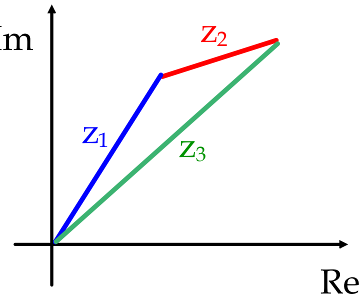
There are two ways to represent complex numbers -
forms of Argand Diagrams.

Cartesian - useful for demonstration of addition and subtraction

Polar - useful for demonstration of multiplication and division

Cartesian



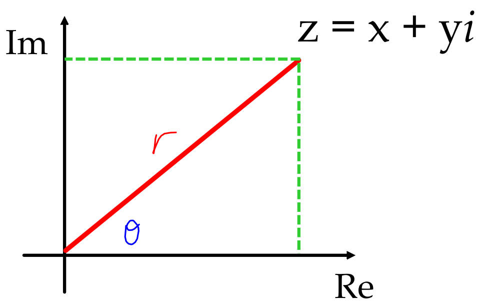


Example 7: Plot these complex numbers on Argand diagrams:

a) $5 + 3i$ and $-1 - 3i$ b) i, -1, $-i$, and 1

Polar Form

In polar form, a coordinate is specified by the distance, r, of the point from the origin and the angle, θ, between the line joining the point to the origin and the positive direction of the x-axis.



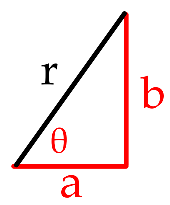
In polar form, r is the modulus of z ⟶ mod(z)

θ is the argument of z ⟶ arg(z)

Conversion from Cartesian to Polar Form

Given z = a + b*i*,

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$



Example 8: Find the modulus and argument of $z = 1 + i$

How do we know which angle to use as the argument?

As there are usually two possible solutions for the argument, sketch the number in Cartesian form to determine in which quadrant it lies.

Example 9: Find the modulus and argument of $z = 1 - \sqrt{3}i$

The argument can be expressed
a) in degree or radians
b) between 0 and 2π or $-\pi$ and π

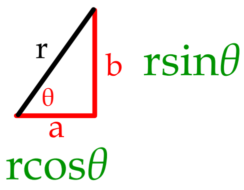
Subtract 2π from angles greater than π radians to get the 'negative' value.

Example 10: Find 4 ways to express the argument of $z = -1 - i$

Polar Form Notation

Given modulus, r, and argument, θ,

$$z = a + bi$$



can be written in the form

$$z = r\cos\theta + r\sin\theta i$$

or $$\mathbf{z = r(\cos\theta + i\sin\theta)}$$

Example 11: Express $z = 2 - 2\sqrt{3}i$ in polar form $[-\pi,\pi)$

As **$\cos(-\theta) = \cos\theta$** and **$\sin(-\theta) = -\sin\theta$**

then **$\cos(-\theta) + i\sin(-\theta) = \cos\theta - \sin\theta$**

Properties of Polar Form

Multiplication and Division

Given that $z_1 = -3 + 3\sqrt{3}i$ and $z_2 = \sqrt{3} + i$,

a) Calculate: $|z_1|$, $|z_2|$, $|z_1z_2|$ and $\left|\dfrac{z_1}{z_2}\right|$

b) Calculate: $\arg(z_1)$, $\arg(z_2)$ $\arg(z_1z_2)$ and $\arg\left(\dfrac{z_1}{z_2}\right)$

c) State a rule for $|z_1z_2|$ and $\arg(z_1z_2)$ in terms of z_1 and z_2

d) State a rule for $\left|\dfrac{z_1}{z_2}\right|$ and $\arg\left(\dfrac{z_1}{z_2}\right)$ in terms of z_1 and z_2

In summary:

For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Multiplication $$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i(sin(\theta_1+\theta_2)]$$

Division $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i(sin(\theta_1-\theta_2)]$$

Properties of Polar Form

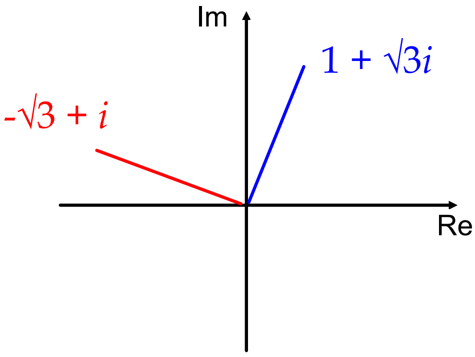
Other Rules

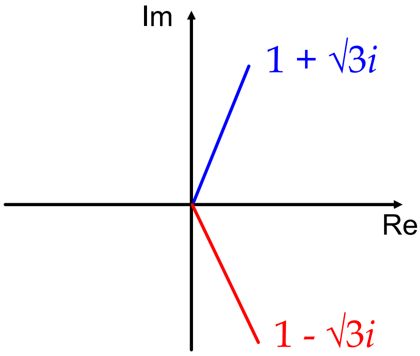
Given that , $z = 1 + \sqrt{3}i$

a) Calculate: $|z|$, $|iz|$, $|\bar{z}|$ and $|z\bar{z}|$

b) Calculate: $\arg(z)$, $\arg(iz)$, $\arg(\bar{z})$ and $\arg(z\bar{z})$

c) Summarise the results





Properties of Polar Form

Summary

$\arg(\bar{z}) = -\arg(z)$ $\bar{z}$ is the reflection of z in the real axis.

$|z\bar{z}| = |z|^2$ $z\bar{z}$ is always real

$\arg(iz) = \arg(z) + \frac{\pi}{2}$ Multiplying by i rotates z by $\frac{\pi}{2}$ radians about the origin.

$|\bar{z}| = |z|$

$|iz| = |z|$

$\arg(z\bar{z}) = 0$

Multiplication and Division in Polar Form

Practise:

a) $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ x $4(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

b) $3(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ x $\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

c) $\frac{1}{2}(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$ x $5(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$

d) $\dfrac{8(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})}{2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})}$

e) $\dfrac{3(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})}{9(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3})}$

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$

then $z^2 =$

De Moivre's Theorem

For any real number, n,

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Example 12: Simplify $[\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})]^4$

Example 13: Given $z = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$, calculate z^5.

Example 14: Show that if $z = \cos\theta + i\sin\theta$, then $z^3 + \frac{1}{z^3} = 2\cos 3\theta$

The Fundamental Theorem of Algebra

A polynomial of degree n will have n roots in the set of complex numbers.
Thus every quadratic equation can be solved and factorised in this set.

We already know that if z = a + bi is a root of a polynomial equation, then $\bar{z}$ = a - bi is also a root.

Both will be factors of the polynomial.

Also, $(x + z)(x + \bar{z})$ is a quadratic factor which can be used as a divisor to find remaining factors.

Example 15: Find all the solutions of the equation $z^3 - z^2 - z - 2 = 0$

Example 16: Prove that $z = 1 + \sqrt{6}i$ is a root of the equation $z^4 - 2z^3 + 8z^2 - 2z + 7 = 0$ and find the other roots.

Geometric Interpretations of Equations and Inequalities in the Complex Plane

To solve equations and inequalities of the form:

$$|z - c| = d$$

substitute a + b*i* for z and solve.

Example 17: Explain the geometric representation of $|z + 2| = 5$

Example 18: Give a geometric interpretation of $|z + 2i| = |z + 3|$,
and find its equation using x and y coordinates.

Multiple Angle Formulae

De Moivre's theorem allows us to generate multiple angle formulae for $\sin ax$ and $\cos ax$, where $a \geq 3$.

Example 19 : Find an expression for a) $\cos 3\theta$ in terms of $\cos \theta$
b) $\sin 3\theta$ in terms of $\sin \theta$

Cube Roots of a Complex Number

De Moivre's Theorem also applies to **fractional indices**.

To find roots of a complex number we have to consider multiple revolutions (arg + 2π, 4π etc) in the same way that we find additional solutions for some trig equations.

If $z = r(\cos\theta + i\sin\theta)$,

Example 20: Find the cube roots of $1+\sqrt{3}i$

Practise: Find the cube roots of: a) $\sqrt{3}+i$ b) $i+1$

Roots of Unity

n^{th} roots of unity satisfy the equation

$$z^n = 1.$$

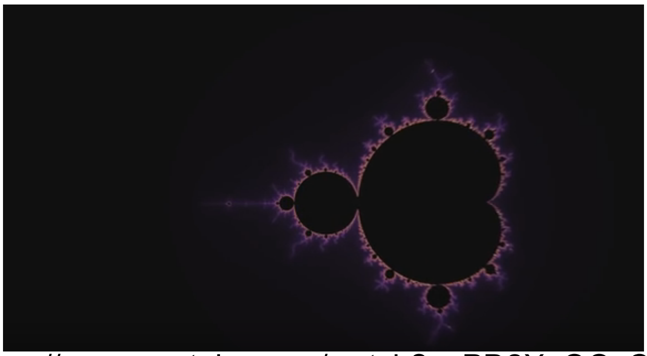
As $1 = 1 + 0i$, we can rewrite the equation as:

$$z^n = \cos 2\pi + i\sin 2\pi$$

Thus, for the cube roots of unity:

The Mandelbrot Set

Deep Zoom

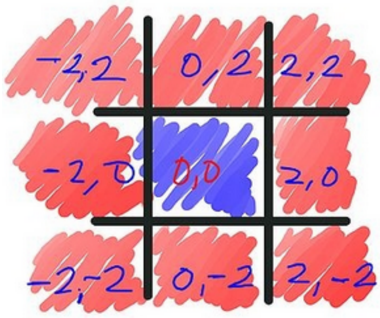


https://www.youtube.com/watch?v=PD2XgQOyCCk

Explanation

https://www.youtube.com/watch?v=9gk_8mQuerg

Plot the Mandelbrot Set by hand



http://www.wikihow.com/Plot-the-Mandelbrot-Set-By-Hand

Past Paper Questions

Complex Numbers

2001

(a) Given that $-1=\cos\theta+i\sin\theta,\ -\pi<\theta<\pi,$ state the value of θ.

(b) Use de Moivre's Theorem to find the non-real solutions, z_1 and z_2, of the equation $z^3+1=0$.

Hence show that $z_1^2=-z_2$ and $z_2^2=-z_1$.

(c) Plot all the solution of $z^3+1=0$ on an Argand diagram and state their geometrical significance.

(1,5,2,3 marks)

2002

Verify that i is a solution of $z^4+4z^3+3z^2+4z+2=0$.
Hence find all the solutions **(5 marks)**

2003

① Identify the locus in the complex plane given by $|z+i|=2$ **(3 marks)**

② Given that $w=\cos\theta+i\sin\theta$, show that $\dfrac{1}{w}=\cos\theta-i\sin\theta$.

Use de Moivre's Theorem to prove $w^k+w^{-k}=2\cos k\theta$, where k is a natural number.

Expand $\left(w+w^{-1}\right)^4$ by the binomial theorem and hence show that

$$\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}.$$

(1, 3, 5 marks)

2004

Given $z=1+2i$, express $z^2(z+3)$ in the form $a+ib$.
Hence, or otherwise, verify that $1+2i$ is a root of the equation

$$z^3+3z^2-5z+25=0.$$

Obtain the other roots of this equation. **(2, 2, 2 marks)**

2005

① Given the equation $z+2i\bar{z}=8+7i$, express z in the form $a+ib$. **(4 marks)**

② Let $z=\cos\theta+i\sin\theta$.

(a) Use the binomial expansion to express z^4 in the form $u+iv$, where u and v are expression involving $\sin\theta$ and $\cos\theta$.

(b) Use de Moivre's theorem to write down a second expression for z^4.

(c) Using the results of (a) and (b), show that

$$\frac{\cos 4\theta}{\cos^2\theta}=p\cos^2\theta+q\sec^2\theta+r, \text{ where } -\frac{\pi}{2}<\theta<\frac{\pi}{2},$$

stating the values of p, q and r.

(3, 1, 6 marks)

2006

Express the complex number $z=-i+\dfrac{1}{1-i}$ in the form $z=x+iy$, stating the values of x and y.

Find the modulus and argument of z and plot z and $\bar{z}$ on an Argand diagram.

(3, 4 marks)

2007

① Show that $z=3+3i$ is a root of the equation $z^3-18z+108=0$ and obtain the remaining roots of the equation.

(4 marks)

② Given that $|z-2|=|z+i|$, where $z=x+iy$, show that $ax+by+c=0$ for suitable values of a, b and c.

Indicate on an Argand diagram the locus of complex numbers z which satisfy $|z-2|=|z+i|$.

(3, 1 marks)

2008

Given $z=\cos\theta+i\sin\theta$, use de Moivre's theorem to write down an expression for z^k in terms of θ, where k is a positive integer.

Hence show that $\dfrac{1}{z^k}=\cos k\theta-i\sin k\theta$.

Deduce expression for $\cos k\theta$ and $\sin k\theta$ in terms of z.

Show that $\cos^2\theta\sin^2\theta=-\dfrac{1}{16}\left(z^2-\dfrac{1}{z^2}\right)^2$.

Hence show that $\cos^2\theta\sin^2\theta=a+b\cos 4\theta$, for suitable constants a and b.

(3, 2, 3, 2 marks)

2009

Express $z=\dfrac{(1+2i)^2}{7-1}$ in the form $a+ib$ where a and b are real numbers.

Show z on an Argand diagram and evaluate $|z|$ and $\arg(z)$.

(6 marks)

2010

Given $z=r(\cos\theta+i\sin\theta)$, use de Moivre's theorem to express z^3 in polar form.

Hence obtain $\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)^3$ in the form $a+ib$.

Hence, or otherwise, obtain the roots of the equation $z^3=8$ in Cartesian form.

Denoting the roots of $z^3=8$ by $z_1,\ z_2,\ z_3$:

(a) state the value $z_1+z_2+z_3$;

(b) obtain the value of $z_1^6+z_2^6+z_3^6$.

(1, 2, 4, 3 marks)

2011

Identify the locus in the complex plane given by

$$|z-1|=3.$$

Show in a diagram the region given by $|z-1|\le 3$. **(5 marks)**

2012

① Given that $(-1+2i)$ is a root of the equation

$$z^3+5z^2+11z+15=0,$$

obtain all roots.

Plot all the roots on an Argand diagram. **(4, 2 marks)**

② (a) Prove by induction that

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

for all integers $n\ge 1$.

(b) Show that the real part of $\dfrac{\left(\cos\dfrac{\pi}{18}+i\sin\dfrac{\pi}{18}\right)^{11}}{\left(\cos\dfrac{\pi}{36}+i\sin\dfrac{\pi}{36}\right)^{4}}$ is zero.

(6, 4 marks)

2013

① Given that $z = 1 - \sqrt{3}i$, write down $\bar{z}$ and express $\bar{z}^{2}$ in polar form. **(4 marks)**

② Describe the loci in the complex plane given by:

(a) $|z+i| = 1$;

(b) $|z-1| = |z+5|$. **(2, 3 marks)**

2014

(a) Express -1 as a complex number in polar form and hence determine the solutions to the equation $z^4 + 1 = 0$.

(b) Write down the four solutions to the equation $z^4 - 1 = 0$.

(c) Plot the solutions of both equations on an Argand diagram.

(d) Show that the solutions of $z^4 + 1 = 0$ and the solutions of $z^4 - 1 = 0$ are also solutions of the equation $z^8 - 1 = 0$.

(e) Hence identify all the solutions to the equation

$$z^6 + z^4 + z^2 + 1 = 0.$$

(3, 2, 1, 2, 2 marks)

2015

By writing z in the form $x + iy$:

(a) solve the equation $z^2 = |z|^2 - 4$;

(b) find the solutions to the equation $z^2 = i\left(|z|^2 - 4\right)$. **(3, 4 marks)**

Advanced Higher Mathematics
Formal Homework Assignment
Complex Numbers

1. Express in the form $z = x + iy$ the complex number $\dfrac{1+i}{3-i}$.

Find the modulus and argument of z.
Illustrate $\bar{z}$ on an Argand diagram.

2. Solve: $z^3 = 125(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$, leaving your answer in polar form.

3. Verify that $z = 1+i$ is a complex solution of the equation:

$$z^4 - z^3 + 2z^2 - 2z + 4 = 0.$$

Write down another solution of the equation.

Show that $z^4 - z^3 + 2z^2 - 2z + 4$ has an irreducible factor of the form $z^2 + kz + c$ stating the values of k and c.

Hence write down all the solutions of $z^4 - z^3 + 2z^2 - 2z + 4 = 0$.

4. Use de Moivre's Theorem to simplify $(-1+\sqrt{3}i)^8$ giving your answer in both polar and Cartesian form.

5. Find the 5th roots of unity. (i.e. solve the equation: $z^5 = 1$)
Illustrate the solutions on an Argand diagram.

6. Given $z = x + iy$, find the equation of the locus where:

(a) $|z| = 2$ (b) $|z+i| = |z-2|$

7. Given that $w = \cos\theta + i\sin\theta$, show that $\dfrac{1}{w} = \cos\theta - i\sin\theta$.

Use de Moivre's Theorem to prove $w^k + w^{-k} = 2\cos k\theta$, where k $\in$ N.

Expand $(w + w^{-1})^4$ by the binomial theorem and hence show that

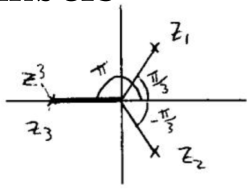
$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$

8. Reflect on your learning and understanding of complex numbers particularly in relation to: (a) connections to your existing knowledge and (b) any elements of this new topic which interest you.

Past Paper Answers
Complex Numbers

2001

(a) $\theta = \pi$ (b) $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ (c) (i)



(c) (ii) The solutions are equally spaced around the origin $\left(\frac{2}{3}\pi\right)$ and equidistant from the origin (1 unit).

2002

(i) Proof (ii) Other solutions: $z = -2 \pm \sqrt{2}$

2003

(1) $x^2 + (y+1)^2 = 4$, which is the equation of the circle centre (0,–1) and radius 2 units
(2) Proofs

2004

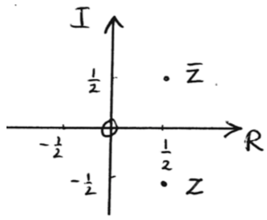
(i) -20 + 10i (ii) Proof (iii) Other roots: z=1-2i and z= -5

2005

(1) z = 2 + 3i
(2) (a) $z^4 = cos^4\Theta - 6cos^2\Theta sin^2\Theta + sin^4\Theta + i(4cos^3\Theta sin\Theta - 4sin^3\Theta cos\Theta)$
(b) $z^4 = cos4\Theta + i\sin4\Theta$ (c) p = 8, q = -1, r = -8

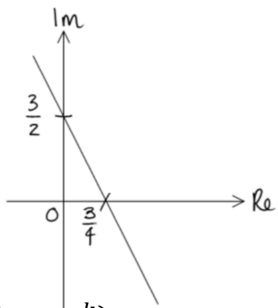
2006

(i) $x = \frac{1}{2}, y = -\frac{1}{2}$ (ii) $|z| = \frac{1}{\sqrt{2}}, \arg(z) = -\frac{\pi}{4}$ (iii)



2007

(1) (i) Proof (ii) Other roots: z = 3 – 3i, z = -6 (2) (i) 4x + 2y - 3 = 0 (ii)

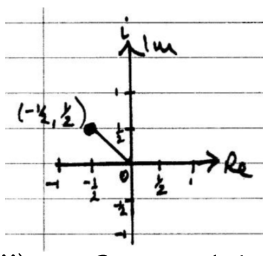


2008

(i) $z^k = cosk\Theta + isink\Theta$ (ii) Proof (iii) $cosk\Theta = \frac{1}{2}(z^k + z^{-k})$, $sink\Theta = \frac{1}{2i}(z^k - z^{-k})$
(iv) Proof (v) $a = \frac{1}{8}, b = -\frac{1}{8}$

2009

(i) $z = -\frac{1}{2} + \frac{1}{2}i$ (ii) $|z| = \frac{1}{\sqrt{2}}, \arg(z) = \frac{3\pi}{4}$,

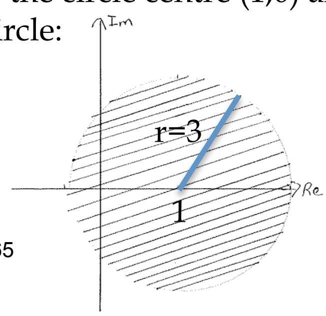


2010

(i) $z^3 = r^3(\cos3\Theta + i\sin3\Theta)$ (ii) z = 1 + 0 (iii) $z = 2, z = -1 \pm \sqrt{3}i$ (iv) (a) 0 (b) 192

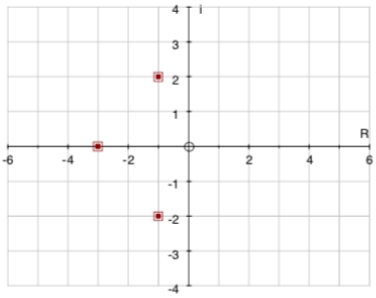
2011

(i) $(x-1)^2 + y^2 = 9$ which is the equation of the circle centre (1,0) and radius 3 units
(ii) The set of points on or inside the above circle:



2012

(1) Other roots: z = -3, z = -1 – 2



(2) Proofs

2013

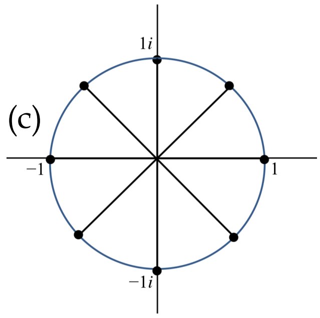
(1) $\bar{z} = 1 + \sqrt{3}i, \bar{z}^2 = 4\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$

(2) (a) $x^2 + (y+1)^2 = 1$, which is the equation of the circle centre (0,-1) and radius 1 unit

(b) x= -2, which is a vertical line intersecting the x-axis at (0,-2)

2014

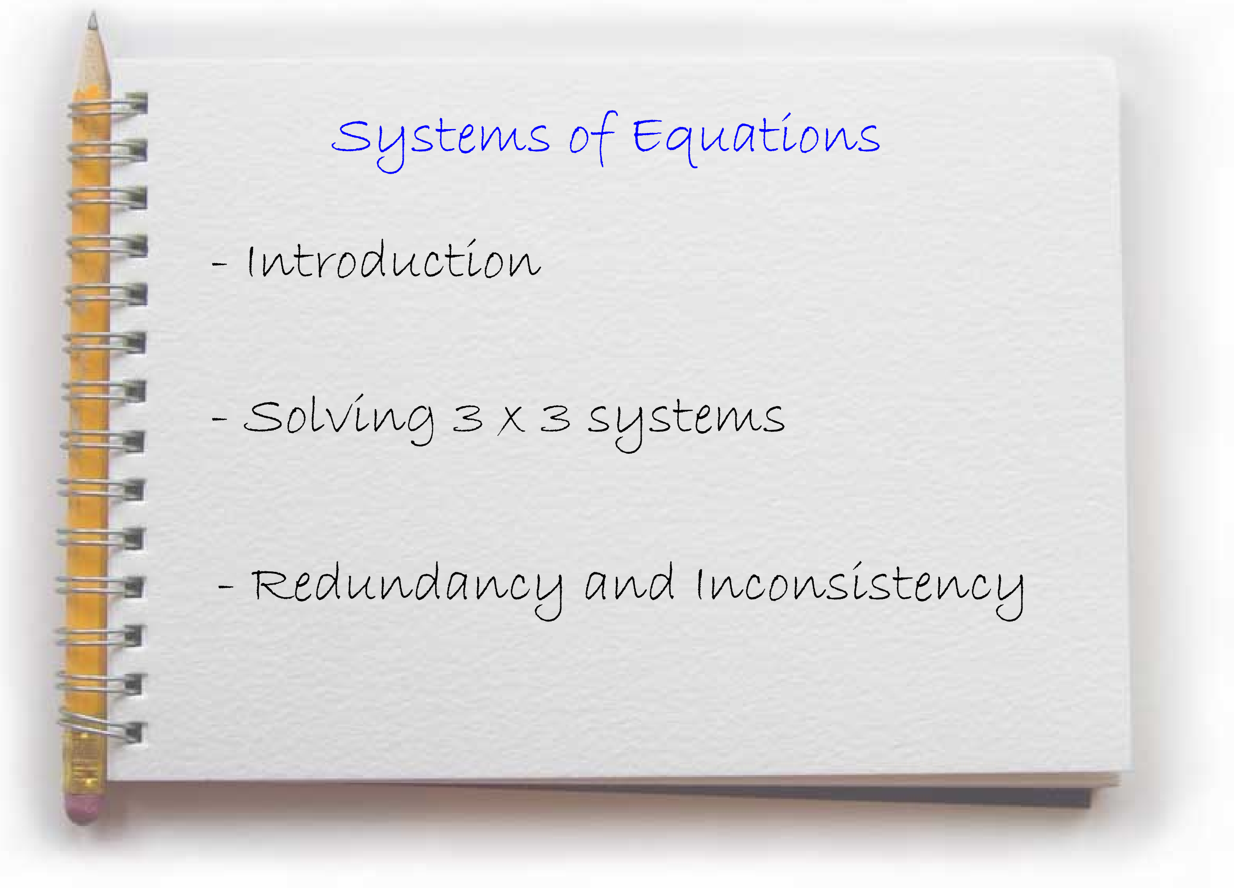
(a) $-1 = cos\pi + isin\pi$, solutions: $\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$ (b) solutions: $\pm1, \pm i$ (c)



(d) Proof (e) $z = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i,\ z = -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, z = \pm i$

2015

(a) Solutions: $z = \pm\sqrt{2}i$ (b) Solutions: $z = \pm(1-i)$



Introduction

A system of equations is defined by:

a) the number of equations it contains

b) the number of variables within the equations

$$\begin{aligned} x + 4y &= 6 \\ 2x + 6y &= 10 \end{aligned}$$

is a 2 x 2 system of equations

We can solve this system 'simultaneously', but we need a more flexible method for larger systems, such as 3 x 3 systems.

This flexible method uses **matrices**.

Matrices

A matrix is a table of information (usually numerical values) arranged in rows and columns.

$$A = \begin{pmatrix} 2 & 5 \\ 3 & 1 \\ 3 & 4 \end{pmatrix}$$

is a $\underbrace{3}_{\text{rows}} \times \underbrace{2}_{\text{columns}}$ matrix. It has 6 **elements**.

Matrix names are always capital letters.

Each element can be described by a_{ij} — i^{th} row, j^{th} column

A row matrix is a $1 \times n$ matrix

A column matrix is an $n \times 1$ matrix

A square matrix is an $n \times n$ matrix

Solving Systems of Equations

$x + 4y = 6$
$2x + 6y = 10$ is rewritten as $AX = B$

where

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 6 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$$

coefficient matrix

constants matrix

For simplicity the system can be written in an augmented matrix:

$$\left(\begin{array}{cc:c} 1 & 4 & 6 \\ 2 & 6 & 10 \end{array}\right)$$ which contains both coefficients and constants.

Solving such systems is done by way of creating an upper triangular matrix using elementary row operations (E.R.O.s)

$$\left(\begin{array}{cc:c} a & b & d \\ 0 & c & e \end{array}\right) \qquad \left(\begin{array}{ccc:c} a & b & c & g \\ 0 & d & e & h \\ 0 & 0 & f & i \end{array}\right)$$

Elementary Row Operations:

- rows in a matrix can be interchanged (first element of column one should be the smallest)
- a row can be multiplied by a (non-zero) constant
- a row can be changed by adding a multiple of another row

Solving 2 x 2 systems

Example 1: Solve: $3x - 2y = -7$

$$x + 6y = 1$$

Solving 3 x 3 systems

Example 2: Solve:

$$\begin{aligned} -2x - 3y + z &= 1 \\ 4x - y + z &= 17 \\ x + 3y - z &= -4 \end{aligned}$$

Example 3: Solve:

$$\begin{aligned} x + 2y + z &= 8 \\ 3x + y - 2z &= -1 \\ x + 5y - z &= 8 \end{aligned}$$

Example 4: A parabola has an equation of the form:

$$y = ax^2 + bx + c$$

The points (1,0), (2,7) and (3,18) lie on the curve.
By constructing a 3 x 3 system of equations,
find the equation of the parabola.

Redundancy

Not all systems of equations have a unique solution.

$$\begin{aligned} x + 2y + 2z &= 11 \\ x - y + 3z &= 8 \\ 4x - y + 11z &= 35 \end{aligned} \quad \text{leads to} \quad \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Here, $0 = 0$, a redundant equation.

Where redundancy occurs, the system has an infinite number of solutions.

Inconsistency

$$\begin{aligned} x + 2y + 2z &= 11 \\ 2x - y + z &= 8 \\ 3x + y + 3z &= 18 \end{aligned} \quad \text{leads to} \quad \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

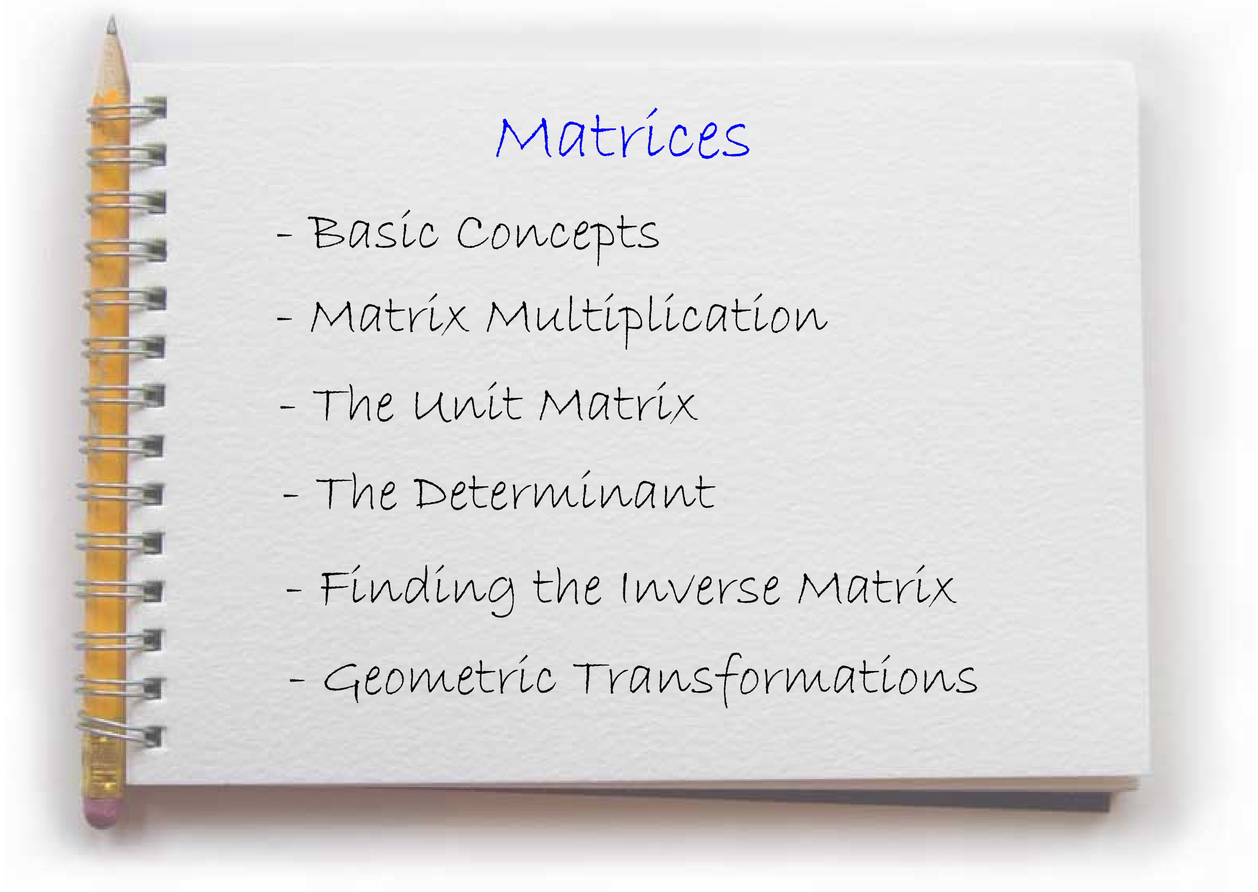
Here, $0 = 1$, an inconsistent equation.

Where inconsistency occurs, the system has no solutions.

Example 5: For the system of equations:

$$\begin{aligned} 2x + y + z &= 1 \\ x + 2y + 2z &= 1 \\ 3x + y + pz &= q \end{aligned}$$

find the values of p and q for which the system has:
a) no solutions and b) infinite solutions



Basic Concepts

1) **Order:** no. of rows x no. of columns

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & 3 \end{pmatrix}$$

2) **Transpose:** row n becomes column n

$$A^T = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ -1 & 3 \end{pmatrix} \qquad (M^T)^T = M$$

3) **Square Matrix:** has n^2 elements

Basic Concepts

4) Adding Matrices

Only matrices of the same order can be added or subtracted.

$$P=\begin{pmatrix}3 & -1 & -2\\ 5 & 0 & 1\end{pmatrix} \qquad Q=\begin{pmatrix}4 & 0 & 3\\ 1 & 1 & -3\end{pmatrix}$$

$$\boxed{A+B=B+A \qquad \text{Associative Law}}$$

Also, $$\boxed{(A+B)^T = A^T + B^T}$$

$$P=\begin{pmatrix}3 & -1 & -2\\ 5 & 0 & 1\end{pmatrix} \qquad Q=\begin{pmatrix}4 & 0 & 3\\ 1 & 1 & -3\end{pmatrix}$$

Basic Concepts

5) Multiplying by a scalar

If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $kA=\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

Example 1: Solve: $4X-\begin{pmatrix} 3 & 1 \\ 4 & 7 \end{pmatrix}=\begin{pmatrix} 5 & 3 \\ 0 & 13 \end{pmatrix}$

Matrix Multiplication

Matrices must be **conformable** in order to be multiplied together.

The number of columns of the first matrix must equal the number of rows of the second matrix.

i.e. **a** x **b** multiplied by **b** x **c** results in an **a** x **c** matrix.

Example 2: Calculate AB when $A=\begin{pmatrix}3 & -1\\ 2 & 1\end{pmatrix}$ and $B=\begin{pmatrix}4\\ 3\end{pmatrix}$

Example 3: Calculate CD given: $C=\begin{pmatrix}1 & 0\\ 2 & -3\\ 5 & 1\end{pmatrix}$ and $D=\begin{pmatrix}1 & -4\\ -3 & 2\end{pmatrix}$

Multiplication Rules

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix}$$

AB =

BA =

(AB)C =

A(BC) =

Multiplication Rules

$$A=\begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix} \qquad B=\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix} \qquad C=\begin{pmatrix}-1 & 1\\ -1 & 2\end{pmatrix}$$

A(B + C) =

AB + AC =

$$\left(AB\right)^{T} =$$

$$A^{T}B^{T} =$$

$$B^{T}A^{T} =$$

The Unit Matrix

A unit matrix, **I**, is a matrix such that A**I** = A and **I**A = A.
The elements of its principal diagonal are 1
and all other elements are 0.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 x 2 unit matrix

3 x 3 unit matrix

Example 4: Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, show that $A^2 = pA + qI$,

where I is the unit matrix, and p and q are integers.

Determinants

The determinant of a square matrix is a scalar quantity.

2 x 2 Matrix

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then det A or |A| = ad - bc

If det A = 0, A is called a singular matrix - it has no inverse.

If det A $\neq$ 0, A is a non-singular matrix - it has an inverse.

Example 5: Find **det A** when $A = \begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix}$

3 x 3 Matrix

Given $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

then det A = (ei - fh)a - (di - fg)b + (dh - eg)c

Example 6: Find det M when $M = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 5 & 2 \\ -1 & -3 & 0 \end{pmatrix}$

Note: For any matrices A and B of the same order, det(AB) = detA x detB

Example 7: Calculate $\begin{vmatrix} \ln 9 & \ln 5 \\ \ln 3 & \ln 2 \end{vmatrix}$

Finding the Inverse of a Matrix

For any matrix A, its inverse A^{-1} is such that:

$$AA^{-1} = I$$

2 x 2 Matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 8: Calculate the inverse matrix given $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and show that $AA^{-1} = I$.

3 x 3 Matrix

The method of finding the inverse of a 3 x 3 matrix is performed by elementary row operations.

Example 9: Calculate the inverse matrix given $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 5 & 2 & 3 \end{pmatrix}$

Using the Inverse Matrix to Solve Equations

Consider the calculation: $\frac{3}{8} \div \frac{7}{4}$

We can rearrange to give: $\frac{3}{8} \times \frac{4}{7}$

(Division is replaced by multiplication by an inverse.)

In the same way, we can introduce an inverse matrix to solve basic matrix equations.

Given $AX = B$

Example 10: Solve: $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

Example 11: Solve the system of equations using the inverse matrix:

$$3x - y = 5$$
$$2x + y = 15$$

Additional Matrix Properties

Matrix M is **symmetrical** if $M^T = M$

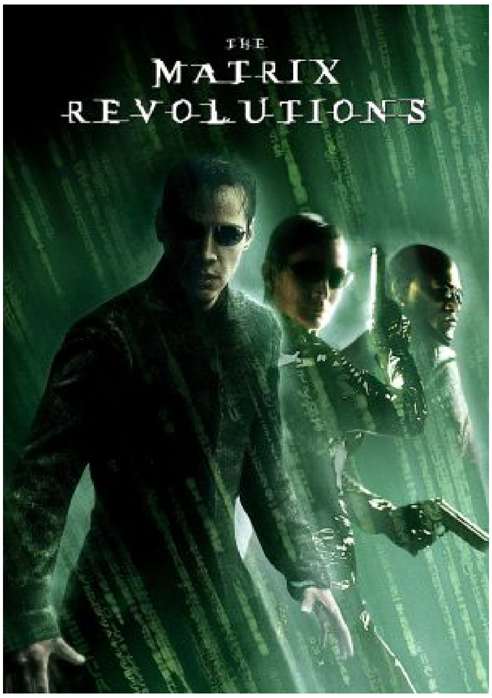
Matrix M is **skew-symmetrical** if $M^T = -M$

Matrix M is **orthogonal** if $\det M = \pm 1$

Example 12: a) Find a, b and c if $M = \begin{pmatrix} 1 & a & -1 \\ 4 & 2 & 0 \\ b & c & 3 \end{pmatrix}$ is symmetrical.

Example 12: b) Complete M so that $M = \begin{pmatrix} \# & \# & 1 \\ 5 & \# & -2 \\ \# & \# & \# \end{pmatrix}$ is skew-symmetrical.

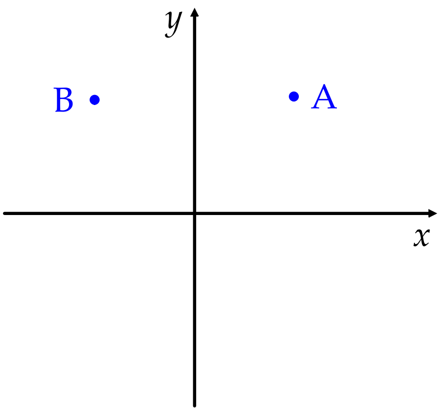
Example 12: c) Find a given that $M = \begin{pmatrix} 4 & 7 \\ -1 & a \end{pmatrix}$ is orthogonal.



Geometric Transformations

Reflections or rotations of a point (x,y) in a cartesian plane can be represented by 2 x 2 matrices.

$$B = MA \quad \text{where} \quad A = \begin{pmatrix} x \\ y \end{pmatrix}$$ original point



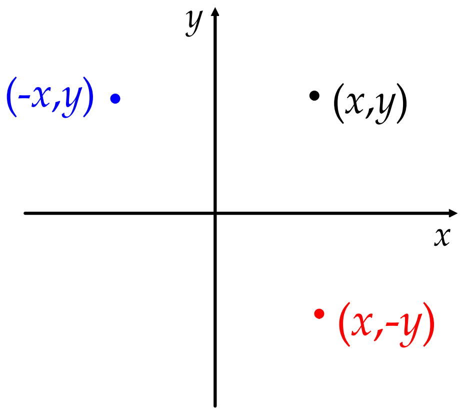
$$B = \begin{pmatrix} x' \\ y' \end{pmatrix}$$ new point

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

M is the transformation matrix

We can deduce the elements of M by considering the effects of **reflection** or **rotation** on each of the original x and y values.

Reflection



Reflection in the y-axis

x coordinate changes sign

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x+0y \\ 0x+y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ matrix associated with reflection in y-axis

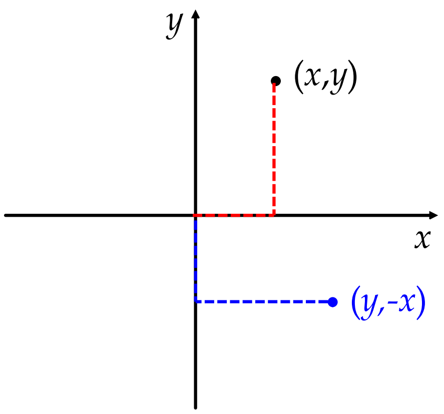
Reflection in the x-axis

y coordinate changes sign

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x+0y \\ 0x-y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ matrix associated with reflection in x-axis

Rotation



$90^0\left(\dfrac{\pi}{2}\right)$ rotation clockwise about O

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0x+1y \\ -x+0y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ matrix associated with 90^0 clockwise rotation

Derive the matrices associate with a) clockwise rotation of π radians.

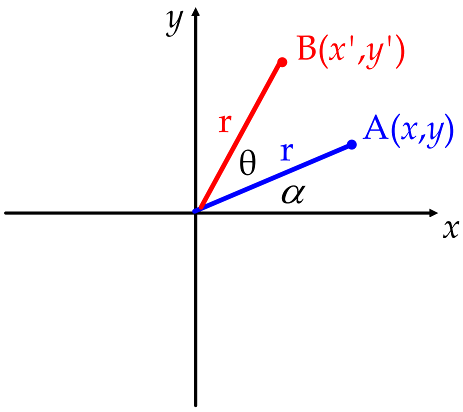
b) anticlockwise rotation of $\dfrac{\pi}{2}$ radians.

Example 13: Derive the matrices associated with:

a) A clockwise rotation of π radians

b) An anticlockwise rotation of $\frac{\pi}{2}$ radians

General Rotation



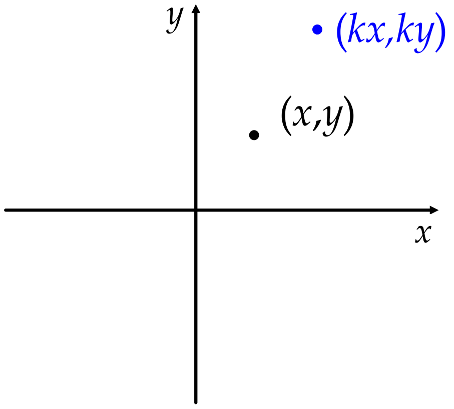
Example 14: Proof of general rotation matrix

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

matrix associated with rotation of θ radians anticlockwise.

Example 15: Find the matrices associated with a rotation of $\frac{\pi}{3}$ radians :
a) anticlockwise and b) clockwise about the origin.

Dilation



Dilation about O will multiply both coordinates by a **dilation factor** so that

$$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

where k is the dilation factor.

Independent scaling occurs when $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, a ≠ b.

Example 16: Find the dilation matrix associated with the transformation of points A(6, 8) and B(-2, -12) to A'(15,20) and B'(-5, -30)

Compound Transformations

Where more than one transformation is applied to a point (or set of points), the transformation matrices can be multiplied **in reverse order** to create one compound transformation matrix.

Example 17: Find the matrix associated with a transformation involving a reflection in the x-axis then a rotation of $\frac{\pi}{6}$ radians about the origin.

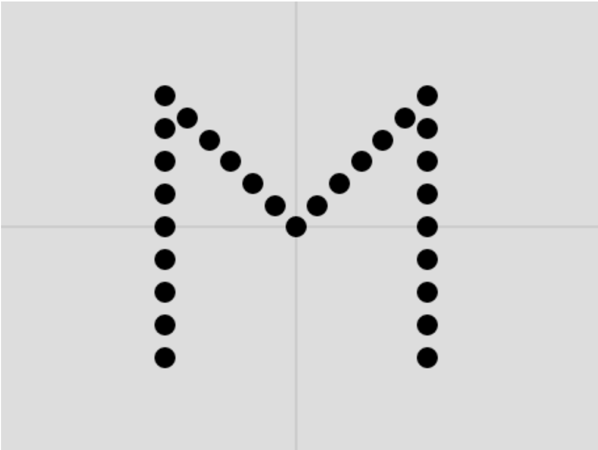
Calculating Points Under Transformations

Knowing the transformation matrix allows us to calculate image points under one or more transformation.

Example 18: Find the image point of A(4,-1) to 1 d.p after a clockwise rotation of $\frac{\pi}{4}$ radians.

Example 19: A point P is reflected in the y axis then rotated clockwise by 60 degrees. The image point of P is (3.6, -0.2) to 1 d.p. Find the composite transformation matrix and hence calculate the coordinates of P.

Transformation Matrix Demonstration



Past Paper Questions

Systems of Equations

2001

Use Gaussian Elimination to solve the following system of equations.

$$x+y+z=10$$
$$2x-y+3z=4$$
$$x+2z=20$$

(5 marks)

2002

Use Gaussian Elimination to solve the following system of equations.

$$x+y+3z=2$$
$$2x+y+z=2$$
$$3x+2y+5z=5$$

(5 marks)

2003

Use elementary row operations to reduce the following system of equations to upper triangular form.

$$x+y+3z=1$$
$$3x+ay+z=1$$
$$x+y+z=-1$$

Hence express x, y and z in terms of the parameter a.

Explain what happens when $a=3$.

(2, 2, 2 marks)

2005

Use Gaussian Elimination to solve the system of equations below when $\lambda \neq 2$.

$$x+y+2z=1$$
$$2x+\lambda y+z=0$$
$$3x+3y+9z=5$$

Explain what happens when $\lambda = 2$.

(4, 2 marks)

2006

Use Gaussian elimination to obtain solutions of the equations

$$2x-y+2z=1$$
$$x+y-2z=2$$
$$x-2y+4z=-1$$

(5 marks)

2009

Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x+y-z&=6 \\ 2x-3y+2z&=2 \\ -5x+2y-4z&=1 \end{aligned}$$

(5 marks)

2010

Use Gaussian elimination to show that the set of equations

$$\begin{aligned} x-y+z&=1 \\ x+y+2z&=0 \\ 2x-y+az&=2 \end{aligned}$$

has a unique solution when $a \neq 2\cdot5$.

Explain what happens when $a = 2\cdot5$.

Obtain the solution when $a = 3$. **(5, 1, 1 marks)**

2012

Use Gaussian elimination to obtain the solution of the following system of equations in terms of the parameter λ.

$$\begin{aligned} 4x+6z&=1 \\ 2x-2y+4z&=-1 \\ -x+y+\lambda z&=2 \end{aligned}$$

Describe what happens when $\lambda = -2$.

When $\lambda = -1\cdot9$ the solution is $x = -22\cdot5$, $y = 8\cdot25$, $z = 15$.

Find the solution when $\lambda = -2\cdot1$.

Comment on these solutions. **(5, 1, 2, 1 marks)**

2014

Use Gaussian elimination on the system of equations below to give an expression for z in terms of λ.

$$\begin{aligned} x+y+z&=2 \\ 4x+3y-\lambda z&=4 \\ 5x+6y+8z&=11 \end{aligned}$$

For what values of λ does this system have a solution?

Determine the solution to this system of equations when $\lambda = 2$. **(6 marks)**

Past Paper Questions

Matrices

2001

Let $A=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & -1 & -1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 0 & 1\\ 4 & -2 & -2\\ -3 & 2 & 1\end{pmatrix}$

Show that $AB = kI$ for some constant k, where I is the identity matrix.
Hence obtain (i) the inverse matrix A^{-1} and (ii) the matrix A^2B. **(4 marks)**

2002

① A matrix $A=\begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix}$. Prove by induction that $A^n=\begin{pmatrix}n+1 & n\\ -n & 1-n\end{pmatrix}$, where n is a positive integer. **(6 marks)**

② Write down the 2×2 matrix A representing a reflection in the x-axis and the 2×2 matrix B representing an anti-clockwise rotation of $30°$ about the origin.
Hence show that the image of a point (x, y) under the transformation A followed by the transformation B is $\left(\frac{kx+y}{2}, \frac{x-ky}{2}\right)$, stating the value of k. **(4 marks)**

2003

The matrix A is such that $A^2 = 4A - 3I$ where I is the corresponding identity matrix.
Find integers p and q such that $A^4 = pA + qI$. **(4 marks)**

2004

Write down the 2×2 matrix M_1 associated with an anti-clockwise rotation of $\frac{\pi}{2}$ radians about the origin.

Write down the matrix M_2 associated with reflection in the x-axis.

Evaluate M_2M_1 and describe geometrically the effect of the transformation represented by M_2M_1.

(2, 1, 2 marks)

2005

Given the matrix $A=\begin{pmatrix}0 & 4 & 2\\ 1 & 0 & 1\\ -1 & -2 & -3\end{pmatrix}$, show that $A^2 + A = kI$ for some constant k, where I is the 3×3 unit matrix.

Obtain the values of p and q for which $A^{-1} = pA + qI$. **(4, 2 marks)**

2006

① Calculate the inverse of the matrix $\begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}$.

For what value of x is this matrix singular? **(4 marks)**

② The square matrices A and B are such that $AB = BA$. Prove by induction that $A^n B = BA^n$ for all integers $n \geq 1$. **(5 marks)**

2007

Matrices A and B are defined by $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}$.

(a) Find the product AB.

(b) Obtain the determinants of A and AB.

Hence or otherwise, obtain an expression for $\det B$. **(2, 2, 1 marks)**

2008

Let the matrix $A = \begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix}$.

(a) Obtain the value(s) of x for which A is singular.

(b) When $x = 2$, show that $A^2 = pA$ for some constant p.

Determine the value of q such that $A^4 = qA$. **(2, 3 marks)**

2009

Given the matrix $A = \begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix}$.

(a) Find A^{-1} in terms of t when A is non-singular.

(b) Write down the value of t such that A is singular.

(c) Given that the transpose of A is $\begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix}$, find t. **(3, 1, 1 marks)**

2010

① Obtain the 2×2 matrix M associated with an enlargement, scale factor 2, followed by a clockwise rotation of 60° about the origin.

(4 marks)

② Use Gaussian elimination to show that the set of equations

$$x-y+z=1$$
$$x+y+2z=0$$
$$2x-y+az=2$$

has a unique solution when $a\neq 2\cdot 5$.

Explain what happens when $a=2\cdot 5$.

Obtain the solution when $a=3$.

Given $A=\begin{pmatrix}5 & 2 & -3\\ 1 & 1 & -1\\ -3 & -1 & 2\end{pmatrix}$ and $B=\begin{pmatrix}1\\ 0\\ 2\end{pmatrix}$, calculate AB.

Hence or otherwise state the relationship between A and the matrix $C=\begin{pmatrix}1 & -1 & 1\\ 1 & 1 & 2\\ 2 & -1 & 3\end{pmatrix}$.

(5, 1, 1, 1, 2 marks)

2011

(a) For what value of λ is $\begin{pmatrix}1 & 2 & -1\\ 3 & 0 & 2\\ -1 & \lambda & 6\end{pmatrix}$ singular?

(b) For $A=\begin{pmatrix}2 & 2\alpha+\beta & -1\\ 3\alpha+2\beta & 4 & 3\\ -1 & 3 & 2\end{pmatrix}$ obtain values of α and β such that $A'=\begin{pmatrix}2 & -5 & -1\\ -1 & 4 & 3\\ -1 & 3 & 2\end{pmatrix}$.

(3, 3 marks)

2012

A non-singular $n\times n$ matrix A satisfies the equation $A+A^{-1}=I$ where I is the $n\times n$ identity matrix. Show that $A^3=kI$ and state the value of k.

(4 marks)

2013

Matrices A and B are defined by $A = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}$

(a) Find A^2.

(b) Find the value of p for which A^2 is singular.

(c) Find the values of p and x if $B = 3A'$. **(1, 2, 2 marks)**

2014

Given A is the matrix $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$,

Prove by induction that $A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}$, $n \geq 1$. **(4 marks)**

2015

① Obtain the value(s) of p for which the matrix $A = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$ is singular. **(4 marks)**

② Write down the 2×2 matrix, M_1, associated with a reflection in the y-axis.

Write down a second 2×2 matrix, M_2, associated with an anticlockwise rotation through an angle of $\frac{\pi}{2}$ radians about the origin.

Find the 2×2 matrix, M_3, associated with an anticlockwise rotation through $\frac{\pi}{2}$ radians about the origin followed by a reflection in the y-axis.

What single transformation is associated with M_3? **(4 marks)**

Advanced Higher Mathematics
Formal Homework Assignment

Systems of Equations and Matrices

1. Use Gaussian elimination to solve the following system of equations:

$$\begin{aligned} x + y + z &= 10 \\ 2x + y + z &= 9 \\ 3x + y + 2z &= 0 \end{aligned}$$

2. Use Gaussian elimination to find the value of λ for which the following system of equations has no solution:

$$\begin{aligned} 2x + y + z &= 0 \\ x + 2y - z &= -1 \\ x \quad - \lambda z &= 0 \end{aligned}$$

3. A parabola with equation y = ax^2 + bx + c passes through the points (-4,5), (2,11) and (1,5). Create a system of equations and solve find the equation of the parabola.

4. Calculate the determinants of: a) $A = \begin{pmatrix} 3 & 5 \\ 2 & -1 \end{pmatrix}$ and b) $B = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix}$

5. Given that the matrix $A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & x & -1 \\ 1 & 0 & x \end{pmatrix}$ is singular, find the value(s) of x.

6. Given that A is an invertible matrix, such that $A^2 = 3A - I$, show that:

a) $A^3 = 8A - 3I$ b) $A^{-1} = 3I - A$

7. Solve for X, where X is a matrix: $\begin{pmatrix} 3 & 1 \\ -2 & -5 \end{pmatrix} + X\begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 4 & 6 \end{pmatrix}$

8. If $G = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$ and $a^2 + b^2 + c^2 = 1$, prove that $G^2 = G$.

9. Calculate the transformation matrix M associated with a rotation of 135 degrees anticlockwise followed by a reflection in the x-axis. Given A(5,2), calculate the coordinates of A′, the image of A under this transformation (to 1 d.p.)

10. Reflect on your understanding of the systems of equations and matrices, particularly in relation to:

(a) connections to your existing knowledge and

(b) any elements of this new area which interest you.