St Andrew's Academy

Department of Mathematics

Advanced Higher

Course Notes

Book 1

Contents

FORMULAE LIST

Standard derivatives

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals

$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)+c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
e^{ax}	$\frac{1}{a}e^{ax}+c$

Summations

(Arithmetic series) $$S_n = \frac{1}{2}n\left[2a+(n-1)d\right]$$

(Geometric series) $$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \dots$$

FORMULAE LIST (continued)

De Moivre's theorem

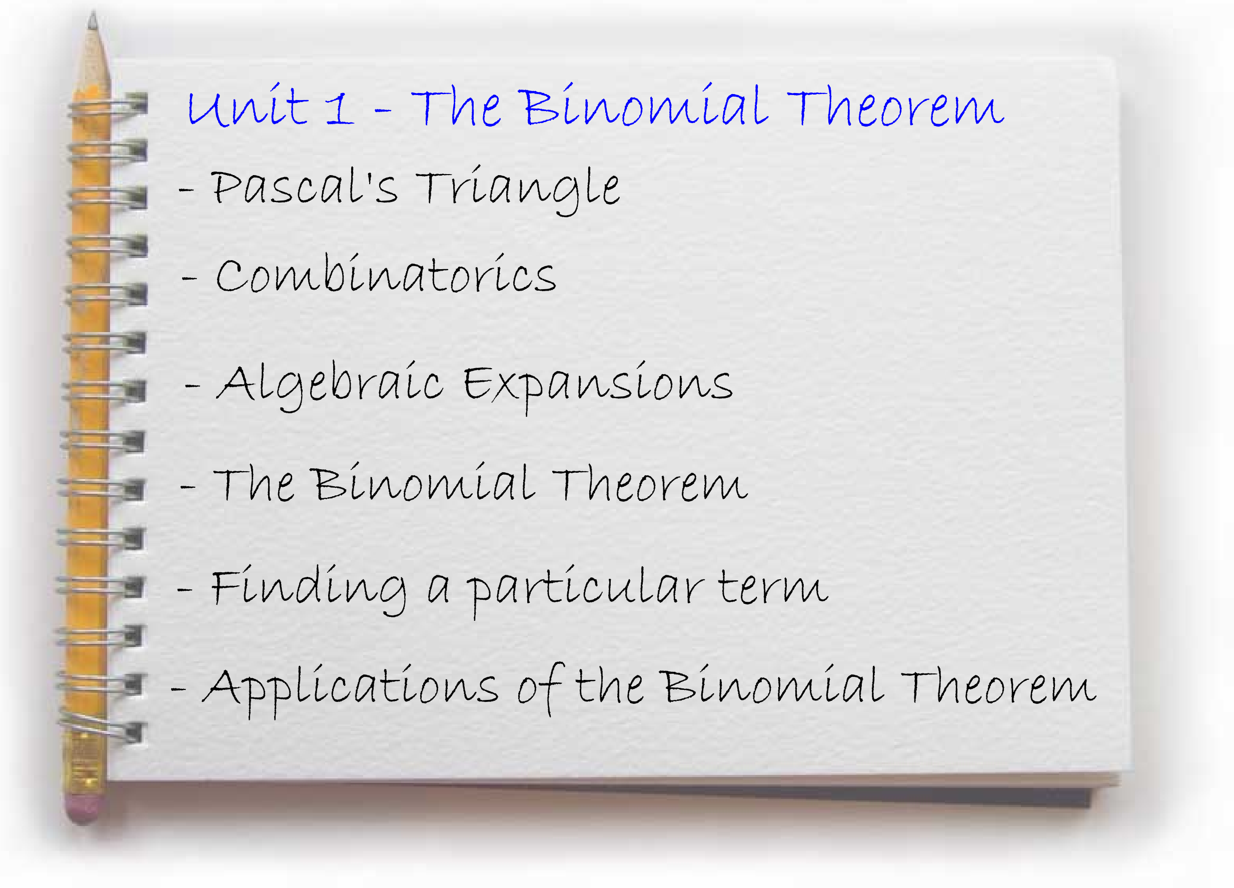
$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Vector product

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ about the origin, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$




Marin Mersenne


Pierre de Fermat

Blaise Pascal
1623 - 1662


Chevalier de Mere

Pascal's Triangle

One of the most interesting number patterns in all of mathematics!

A few properties of Pascal's Triangle:

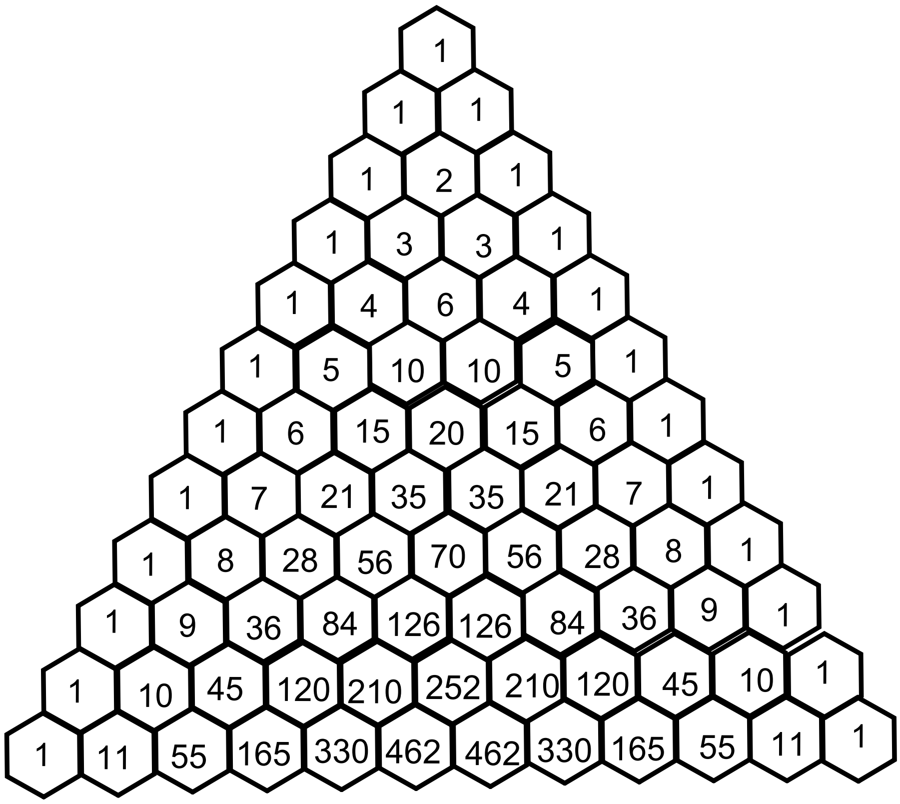
Rows are numbered from Row 0 **1**

The sum of any 2 consecutive numbers is the value of the number below. **2**

The sum of each row is a power of 2, such that the sum of Row n is 2^n. **3**

'Column' 2 is the set of triangular numbers **4**

Values in each row have direct applications in combinatorics and algebraic expansions **5**



Combinatorics

One aspect of Combinatorics (the study of countable discrete numbers), is that of combinations .

The statistical analysis of the selection of r elements from a set of n elements is denoted

$$^{n}C_{r} \quad \text{or} \quad \binom{n}{r} \qquad \text{"n choose r"}$$

i.e. $^{6}C_{2}$ is the number of ways 2 people can be chosen from a group of 6 people.

$$^{6}C_{2} = \binom{6}{2} =$$

Example -1: Calculate the values of:

a) $^{3}C_{2}$ b) $^{5}C_{3}$ c) $^{8}C_{4}$ d) $^{11}C_{3}$

Algebraic Expansions

Multiply out and simplify:

$(x + y)^2$ $(x + y)^3$ $(x + y)^4$ and $(x + y)^5$

Notice that for the expansion of any expression $(x+y)^n$,

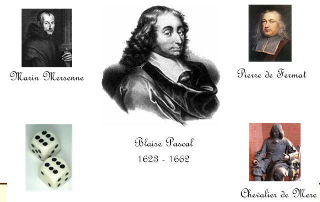
- coefficients of each term are found in Row n of Pascal's Triangle
- powers of x decrease, and powers of y increase

Without the dreary task of multiplying out brackets we can deduce,

$(x + y)^6 =$

$(x + y)^7 =$

$(x + y)^8 =$



The Problem of Points

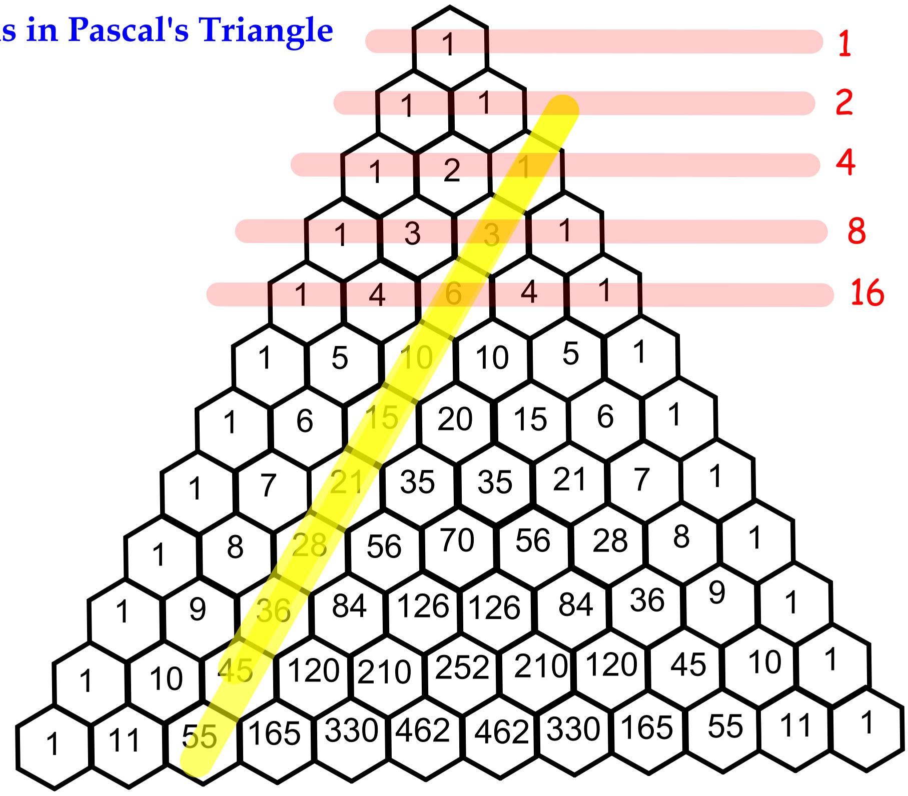
Two men are playing a game of chance; in each round they roll 2 dice and the person with the highest total score wins. The winner is the first person to win 3 games

Each man has put 32 Francs in the prize pot, winner takes all.

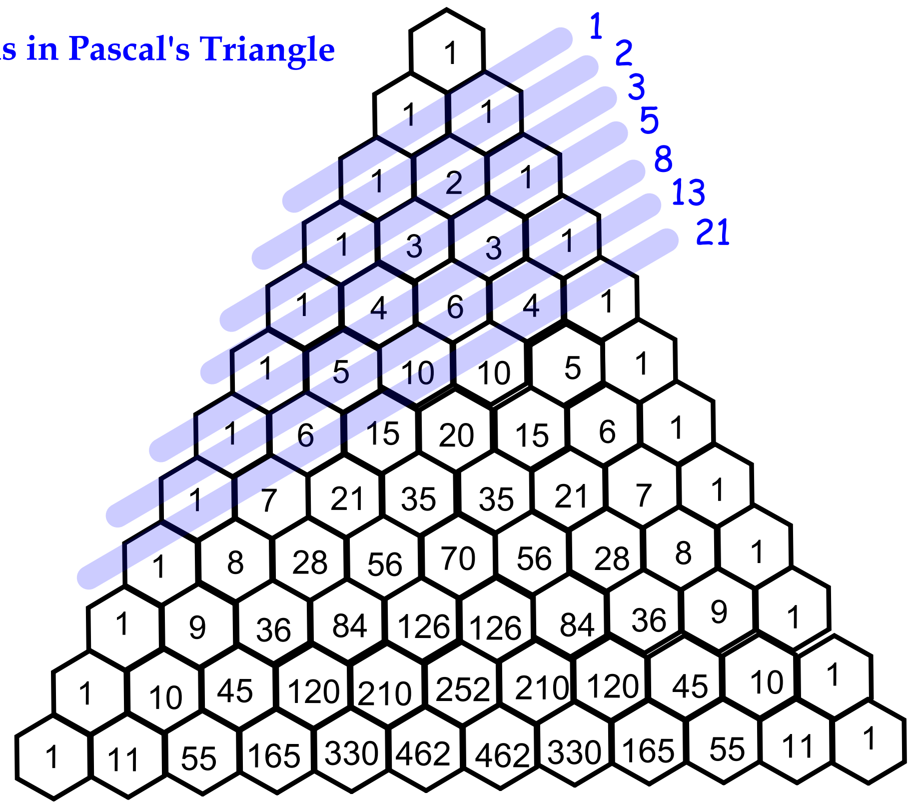
The score is 2-1 when the game is forced to end.

How should the money be distributed to fairly reflect the situation?

Patterns in Pascal's Triangle

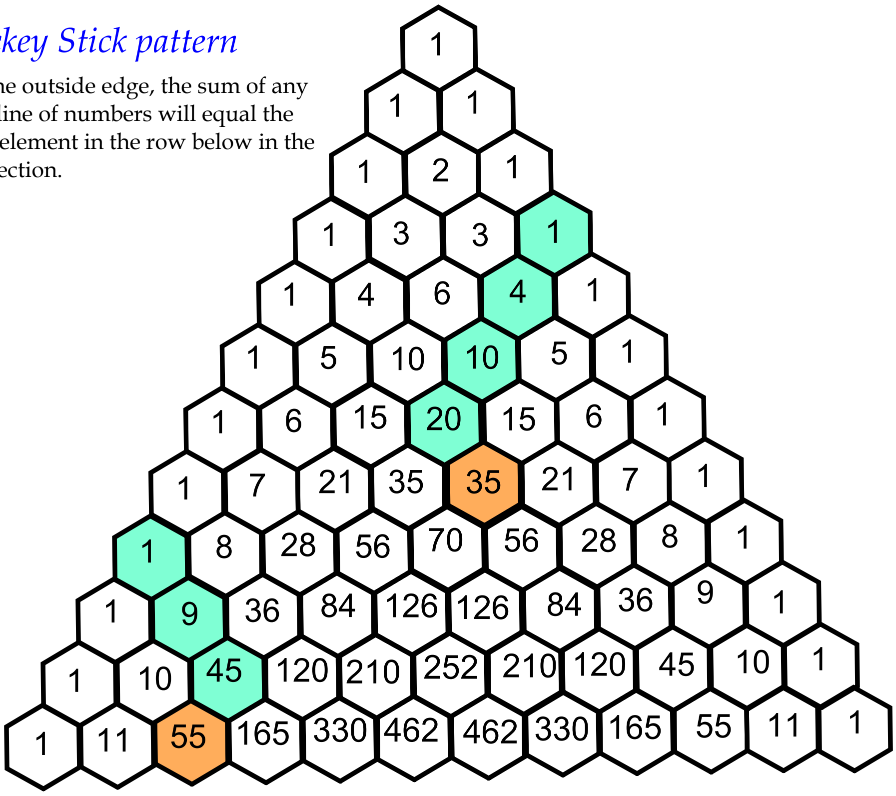


Patterns in Pascal's Triangle



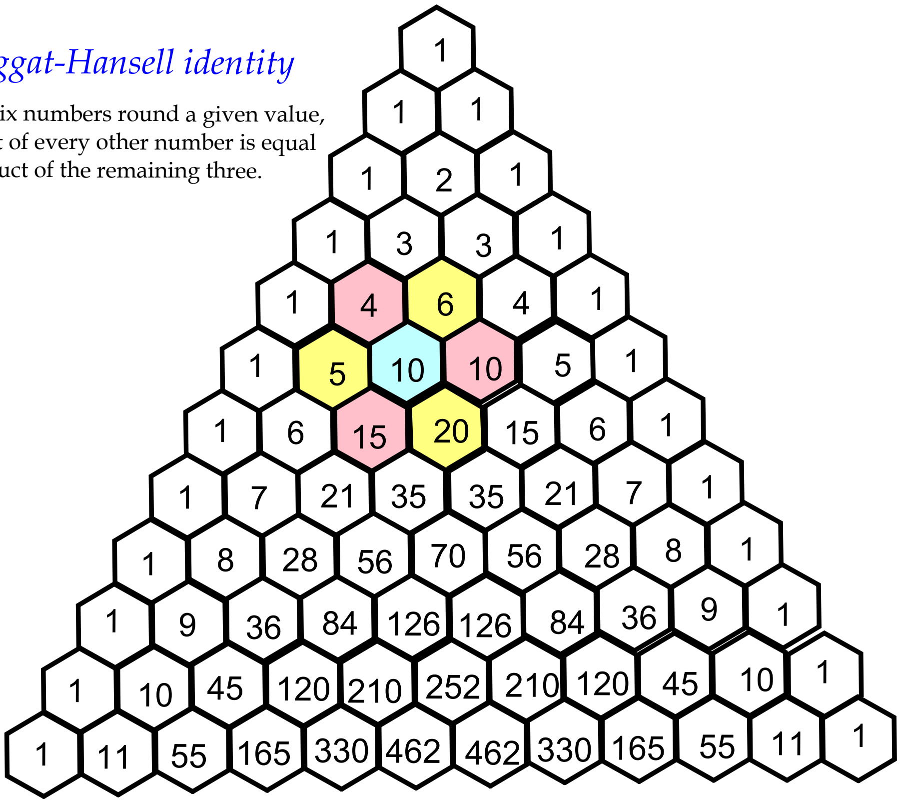
The *Hockey Stick pattern*

Starting at the outside edge, the sum of any descending line of numbers will equal the value of the element in the row below in the opposite direction.



The *Hoggat-Hansell identity*

Given the six numbers round a given value, the product of every other number is equal to the product of the remaining three.



The Factorial Operation

$4! = 4 \times 3 \times 2 \times 1 =$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 =$

In general,

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

Calculate: a) $6!$ b) $10!$ c) $25!$

What is $0!$?

Simplifying Factorial Expressions

Simplify: a) $\dfrac{4!}{3!}$ b) $\dfrac{7!}{4!}$ c) $\dfrac{(n+2)!}{n!}$ d) $\dfrac{5!n!}{2!(n-3)!}$

Binomial Coefficient Rules

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 0: Use the binomial coefficient formula to find:

a) $\binom{5}{2}$ b) $\binom{9}{6}$

Practise: Find values for:

a) $\binom{4}{2}$ b) $\binom{6}{4}$ c) $\binom{10}{3}$

Binomial Coefficient Rules

Two other rules:

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

Working back

Example 1: Find n when $\binom{n}{2} = 6$

The Binomial Theorem

Earlier we saw that for the expansion of $(x+y)^n$,
the coefficients for each term in the expansion are the elements of row n in Pascal's Triangle.

$(x + y)^4 =$

From this we can derive the general form of anyterm in the expansion:

for $(x+y)^n$, then $T_{r+1} = \binom{n}{r} x^{n-r} y^r$

The Binomial Theorem

The Binomial Theorem states that if $x, y \in R$ and $n \in N$, then

$$\boxed{(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r}$$

Example 2:

Use the Binomial Theorem to expand:

a) $(x+y)^6$

b) $(3a+1)^4$

c) $\left(\frac{1}{2}x-3y\right)^5$

Finding a Particular Term

It is not always necessary to find all the terms in an expansion. In order to find the coefficient of a particular term we use the general term:

$$T_{r+1} = \binom{n}{r} x^{n-r} y^r$$

Example 3:

a) Find the coefficient of x^2y^3 in the expansion of $(x+y)^5$

b) Find the coefficient of x^3y in the expansion of $(2x-y)^4$

Example 4:

a) Find the coefficient of x^6 in the expansion of $(x+\frac{3}{x})^8$

b) Find the coefficient of t^4 in the expansion of $(1-2t^2)^5$

More Practise

1) Find the coefficient of x^6y^3 in the expansion of $(x+y)^9$
2) Find the coefficient of xy^4 in the expansion of $(\frac{1}{2}x-y)^5$
3) Find the coefficient of x^5 in the expansion of $(2a-\frac{1}{a})^7$
4) Find the coefficient of p^{11} in the expansion of $(p^2+2p^3)^4$

Development - Constant Terms

We can use this method to find the constant term in an expansion such as $(2x+\frac{1}{x})^n$

Example 5:
Find the constant term in the expansions of:

a) $(3x+\frac{1}{x})^{12}$

Find the constant term in the expansions of:

b) $\left(\frac{1}{2}a - \frac{1}{3a}\right)^4$

More Complex Expansions

Example 6: Find the coefficient of the x^3 term in the expansion of:

$$(x+2)(2x-3)^3$$

Example 7: Find the coefficient of the x^5 term in the expansion of :

$$(1+x)^4(1-2x)^3$$

More Complex Expansions

Practise:
Find the coefficients of :

a) the x^4 term in the expansion of $(x-3)(x - 2)^4$

a) the x^2 term in the expansion of $(2x+1)(4-3x)^3$

c) the x^3 term in the expansion of $(3x^2-2)(x + 7)^5$

d) the x^4 term in the expansion of $(1+x)^2(1+2x)^3$

e) the x^7 term in the expansion of $(1+2x)^4(1-2x)^6$

f) the x^5 term in the expansion of $(1-x)^3(2+x)^4$

Applications of the Binomial Theorem

1) Evaluation of the powers of real numbers

Example 8:

a) Evaluate 0.98^4

b) Evaluate 2.3^5

Practise:
Evaluate c) 1.9^3 d) 1.03^4

Applications of the Binomial Theorem

Statistical Probability

The probability of single or multiple events within a time-specific period can be calculated using the Binomial Theorem.

Example:
In taking the bus to school each morning,
let t = probability of bus being on time = 0.9
l = probability of bus being late = 0.1

Then $(t+l)^5$ gives the condition of probability for 1 week (Mon-Fri).

We can use the general term formula to avoid expanding the whole expression if just one outcome is required.

Example 9:
The probability of rain each day next week is 0.4. I need 5 dry days (out of 7) to paint the exterior of my house. What is the probability that I will be able to accomplish this task?

Practise Questions

1) You are taking a 10 question multiple choice test. If each question has four choices and you guess on each question, what is the probability of getting exactly 7 questions correct?

2) Find the probability of rolling at least 2 sixes with 4 dice.

3) A certain species of dog produces more female pups than males, with the probability of a given pup being female being $^2/_3$.
What is the probability that, in a litter of 8 pups, there are more males than females?

4) In recent years, the probability of an AH Maths pupil being male is 0.72. What is the probability that an AH class of 11 pupils will have 6 female students?

5) Experience has shown that 1/200 of all CDs produced by a certain machine are defective. If a quality control technician randomly tests twenty CDs, compute each of the following probabilities:
   - P(exactly one is defective)
   - P(half are defective)
   - P(no more than two are defective)

6) After studying a couple's family history, a doctor determines that the probability of any child born to this couple having a gene for disease X is 1 out of 4.
If the couple has three children, what is the probability that exactly two of the children have the gene for disease X?

7) When Joe bowls, he can get a strike (knock down all of the pins) 60% of the time. How many times more likely is it for Joe to bowl at least three strikes out of four times as it is for him to bowl zero strikes out of four tries? Round your answer to the nearest whole number.

8) A board game has a spinner on a circle that has five equal sectors, numbered 1, 2, 3, 4, and 5, respectively. If a player has four spins, find the probability that the player spins an even number no more than two times on those four spins.

Past Paper Questions

Binomial Theorem

2001

Expand $\left(x^2 - \frac{2}{x}\right)^4$, $x \neq 0$ and simplify as far as possible. **(5 marks)**

2004

Obtain the binomial expansion of $\left(a^2 - 3\right)^4$. **(3 marks)**

2007

Express the binomial expansion of $\left(x - \frac{2}{x}\right)^4$ in the form $ax^4 + bx^2 + c + \frac{d}{x^2} + \frac{e}{x^4}$

for integers a, b, c, d and e. **(4 marks)**

2008

Write down and simplify the general term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$.

Hence or otherwise, obtain the term in x^{14}. **(3, 2 marks)**

2009

(a) Write down the binomial expansion of $(1+x)^5$.

(b) Hence show that $0 \cdot 9^5$ is $0 \cdot 59049$. **(1, 2 marks)**

2010

Show that

$$\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$$

where the integer n is greater than or equal to 3. **(4 marks)**

2011

Use the binomial theorem to expand $\left(\frac{1}{2}x - 3\right)^4$ and simplify your answer. **(3 marks)**

2012

Write down and simplify the general term in the expansion of $\left(2x - \frac{1}{x^2}\right)^9$.

Hence, or otherwise, obtain the term independent of x. **(3, 2 marks)**

2013

Write down the binomial expansion of $\left(3x-\frac{2}{x^2}\right)^4$ and simplify your answer. **(4 marks)**

2014

Write down and simplify the general term in the expression $\left(\frac{2}{x}+\frac{1}{4x^2}\right)^{10}$.

Hence, or otherwise, obtain the term in $\frac{1}{x^{13}}$. **(5 marks)**

2015

① Use the binomial theorem to expand and simplify

$$\left(\frac{x^2}{3}-\frac{2}{x}\right)^5.$$ **(4 marks)**

② Show that $\binom{n+2}{3}-\binom{n}{3}=n^2$, for all integers, n, where $n \geq 3$. **(4 marks)**

Past Paper Answers
Binomial Theorem

2001 (Q.A6)
$x^8 - 8x^5 + 24x^2 - \frac{32}{x} + \frac{16}{x^4}$

2004 (Q.2)
$a^8 - 12a^6 + 54a^4 - 108a^2 + 81$

2007 (Q.1)
$x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$

2008 (Q.8)
$\binom{10}{r} x^{20-3r}$; $45x^{14}$

2009 (Q.8)
$1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$; proof

2010 (Q.5)
proof

2011 (Q.2)
$\frac{1}{16}x^4 - \frac{3}{2}x^3 + \frac{27}{2}x^2 - 54x + 81$

2012 (Q.4)
$\binom{9}{r} 2^{9-r}(-1)^r x^{9-3r}$; -5376

2013 (Q.1)
$81x^4 - 216x + \frac{216}{x^2} - \frac{96}{x^5} + \frac{16}{x^8}$

2014 (Q.4)
$\binom{10}{r} 2^{10-r}(\frac{1}{4})^r x^{-10-r}$ or $\binom{10}{r} 2^{10-3r} x^{-10-r}$; $\frac{240}{x^{13}}$

2015 (Q.1)
$\frac{x^{10}}{243} - \frac{10}{81}x^7 + \frac{40}{27}x^4 - \frac{80}{9}x + \frac{80}{3x^2} - \frac{32}{x^5}$; proof

Advanced Higher Mathematics
Formal Homework Assignment

Binomial Theorem

1. Express as factorials: a) 10 x 9 x 8 b) 11 x 10 x 9 x 8 x 7

2. Simplify: a) $\dfrac{n!}{(n-1)!}$ b) $n! - (n-1)!$

3. Prove that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

4. Use the Binomial Theorem to expand: $\left(2a + \dfrac{b}{3}\right)^5$.

5. Expand $\left(\dfrac{x}{2} - \dfrac{2}{x}\right)^6$, expressing your answers with positive powers of x.

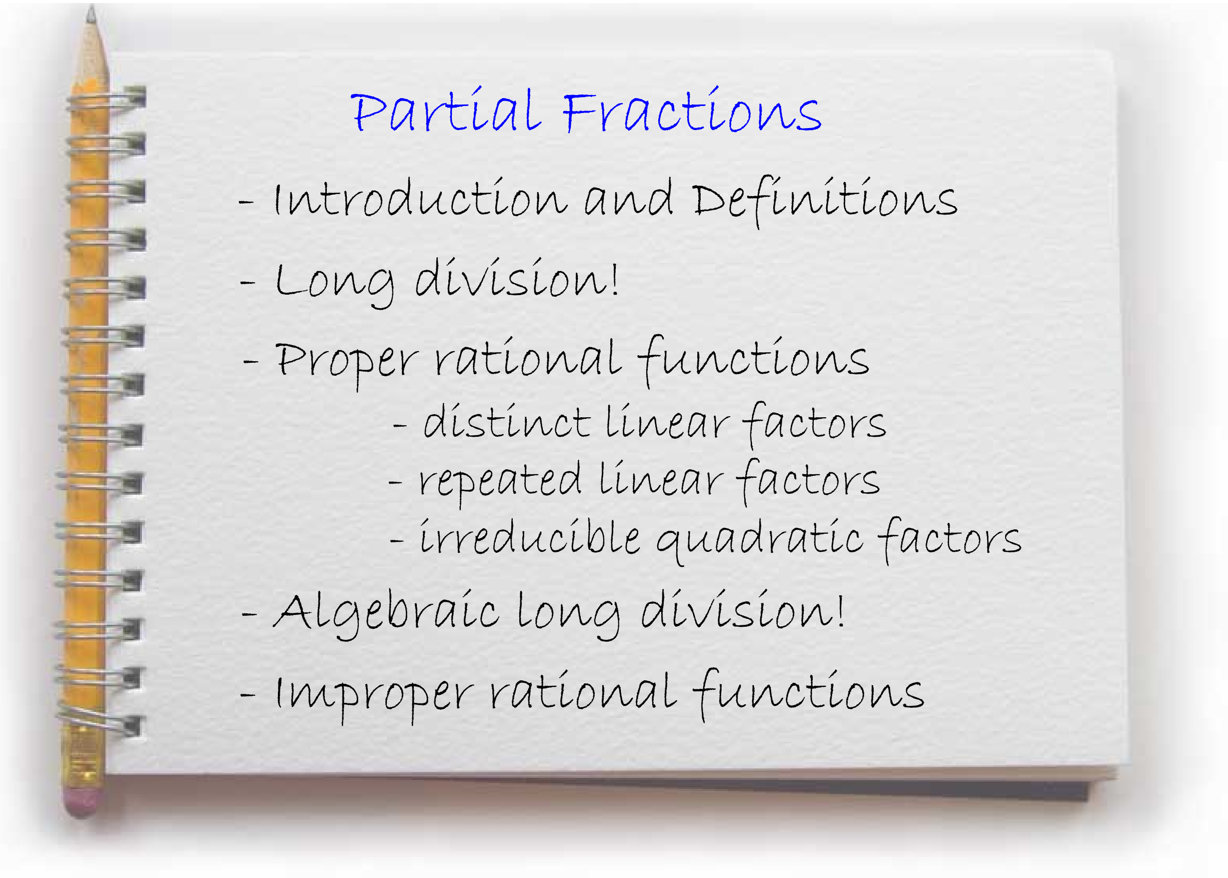
6. Calculate the coefficient of t^3 in the expansion of $(3+2t)^5$.

7. Use the Binomial Theorem to evaluate 0.97^4.

8. Find the term independent of a in $\left(\dfrac{3a^2}{2} - \dfrac{1}{3a}\right)^9$.

9. The probability of being stopped at any set of traffic lights on a particular journey is 0.7. Using the Binomial Theorem, calculate the probability of being stopped at 2 out of the 7 sets of traffic lights on this route.

10. Reflect on your understanding of Pascal's Triangle, factorial notation and the theory and applications of the Binomial Theorem, particularly in relation to:
(a) connections to your existing knowledge and
(b) any elements of these new topics which interest you.


Partial Fractions
- Introduction and Definitions
- Long division!
- Proper rational functions
- distinct linear factors
- repeated linear factors
- irreducible quadratic factors
- Algebraic long division!
- Improper rational functions

Introduction and Definitions

Most processes or algorithms in mathematics are reversible.

Addition	Subtraction
Multiplying out brackets	Factorising
Differentiation	Integration
n^{th} power	n^{th} root

We have already encountered the addition of algebraic fractions:

$$\frac{2}{x-1} + \frac{1}{x+3}$$

For some purposes, such as integration, it helps to reduce complex fractions into their smaller, separate components - **partial fractions**.

A **rational** function must be **proper**
in order to express it in partial fractions.

Definitions

The **degree**, n, of any polynomial is the value of its highest power, so long as $n \in \mathbb{N}$

A **rational function** is a fraction whose numerator and denominator are both polynomials.

A **proper rational function** exists when the degree of the polynomial in the denominator is greater than in the numerator, otherwise the function is **improper**.

Proper rational functions

Improper rational functions

Long Division

We usually perform a mental algorithm to divide one number by another. Long division is simply the written form of this process, which allows for more complex division to take place.

Example 1:

a) $144 \div 8$

b) $4788 \div 14$

Practice: Use the method of long division to calculate:

1) $1638 \div 6$
2) $54936 \div 9$
3) $364 \div 13$
4) $3584 \div 16$
5) $6426 \div 21$
6) $4374 \div 27$
7) $62149 \div 19$
8) $12555 \div 31$
9) $127088 \div 47$

There are three different forms of obtaining partial fractions, determined by the nature of the denominator:

1 Distinct linear factors *e.g.* $(x + 1)(x - 2)$

2 Repeated linear factors *e.g.* $(a - 1)^3$ *or* $(y + 2)(y + 1)^2$

3 Irreducible quadratic factors *e.g.* $x^2 + 2x + 2$

Partial Fractions
Type 1 - Distinct Linear Factors

Example 2:

Write in partial fractions: $\dfrac{4x + 1}{(x + 1)(x - 2)}$

[split into fractions with constants on top]

[multiply both sides by the denominator]

[select values of x which will eliminate one unknown constant]

Example 3:

Express in partial fractions: $\dfrac{-12x - 30}{(x + 1)(x - 2)(x + 4)}$

Partial Fractions
Type 2 - Repeated Linear Factors

Example 4:

Express in partial fractions: $\dfrac{5x - 16}{(x - 3)^2}$

Example 5:

Write in partial fractions: $\dfrac{3x^2 - 11x + 5}{(x - 2)(x - 1)^2}$

Example 6:

Write in partial fractions: $\dfrac{2x^2 + 7x + 3}{x^3 + 2x^2 + x}$

Partial Fractions
Type 3 - Irreducible Quadratic Factors

A quadratic factor is irreducible if it has no real roots
i.e. its discriminant is less than zero.

Where a rational function has an irreducible quadratic factor, the resultant partial fraction will have a linear term in the numerator.

Example 7:

Express in partial fractions: $\dfrac{3x^2 + 2x + 1}{(x+1)(x^2 + 2x + 2)}$

Example 8:

Express in partial fractions: $\dfrac{x^2 - 2x + 2}{(x^3 - 1)}$

Algebraic Long Division

Improper rational functions cannot be expressed in partial fraction form.
First, the function must be expressed as a quotient and remainder.

Example 9: Simplify: $x^2 + 3x + 5 \div (x + 2)$

Example 10: Express as a quotient and remainder:

$$\frac{x^4 + 3x^3 + 2x^2 - 3}{x^2 + 2x}$$

Example 11: Simplify: $3x^3 + 7x - 1 \div (x^2 + 3)$

Improper Rational Functions

We see that improper rational functions can be simplified to 'quotient and remainder' form using algebraic long division. The remainder is always a proper rational function, which can then be expressed in partial fractions.

Improper Rational Functions

Example 12: Express $\dfrac{x^3 - x^2 - 5x - 7}{x^2 - 2x - 3}$

as a polynomial function plus partial fractions.

Example 13: Express $\dfrac{(x + 2)(x - 2)}{(x + 1)(x - 1)}$

as a polynomial function plus partial fractions.

Past Paper Questions

Partial Fractions

2001

A5 (a): Obtain partial fractions for $\frac{x}{x^2-1}$, $x>1$. **(2 marks)**

2002

Ist part of A8: Express $\frac{x^2}{(x+1)^2}$ in the form $A+\frac{B}{x+1}+\frac{C}{(x+1)^2}$, $(x\neq -1)$ stating the values of the constants A, B and C. **(3 marks)**

2004

Ist part of Q5: Express $\frac{1}{x^2-x-6}$ in partial fractions. **(2 marks)**

2005

Ist part of Q13: Express $\frac{1}{x^3+x}$ in partial fractions. **(4 marks)**

2007

Ist part of Q4: Express $\frac{2x^2-9x-6}{x(x^2-x-6)}$ in partial fractions. **(3 marks)**

2008

Ist part of Q4: Express $\frac{12x^2+20}{x(x^2+5)}$ in partial fractions. **(3 marks)**

2009

Ist part of Q14: Express $\frac{x^2+6x-4}{(x+2)^2(x-4)}$ in partial fractions. **(4 marks)**

2011

Ist part of Q1: Express $\frac{13-x}{x^2+4x-5}$ in partial fractions. **(3 marks)**

2012

Q15(a): Express $\frac{1}{(x-1)(x+2)^2}$ in partial fractions. **(4 marks)**

2014

Ist part of Q14(b): Express $\frac{1}{3r^2-5r+2}$ in partial fractions. **(2 marks)**

Past Paper Answers
Partial Fractions

2001 (Q.A5a)

$\frac{1}{2(x+1)} + \frac{1}{2(x-1)}$

2002 (Q.A8)

$1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$; so A=1, B=-2, C=1

2004 (Q.5)

$\frac{1}{5(x-3)} - \frac{1}{5(x+2)}$

2005 (Q.13)

$\frac{1}{x} - \frac{x}{x^2+1}$

2007 (Q.4)

$\frac{1}{x} + \frac{2}{x+2} - \frac{1}{x-3}$

2008 (Q.4)

$\frac{4}{x} + \frac{8x}{x^2+5}$

2009 (Q.14)

$\frac{2}{(x+2)^2} + \frac{1}{x-4}$

2011 (Q.1)

$\frac{2}{x-1} - \frac{3}{x+5}$

2012 (Q.15)

$\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$

2014 (Q.14)

$\frac{1}{r-1} - \frac{3}{3r-2}$

Advanced Higher Mathematics
Formal Homework Assignment

Partial Fractions

1. Express in partial fractions: $\dfrac{1+x-3x^2}{(x-2)(x+1)^2}$

2. Differentiate: $f(x)=(3-2x)^3\sin\left(\frac{x}{2}\right)$

3. Express in partial fractions: $\dfrac{x^3}{x^2-3x+2}$

4. Express the following as a sum of a polynomial function and partial fractions:

$$\frac{x^4+2x^2-2x+1}{x^3+x}$$

5. A curve is defined by the equation $x^2+xy=5$.
Find the equation of the tangent at $x=1$.

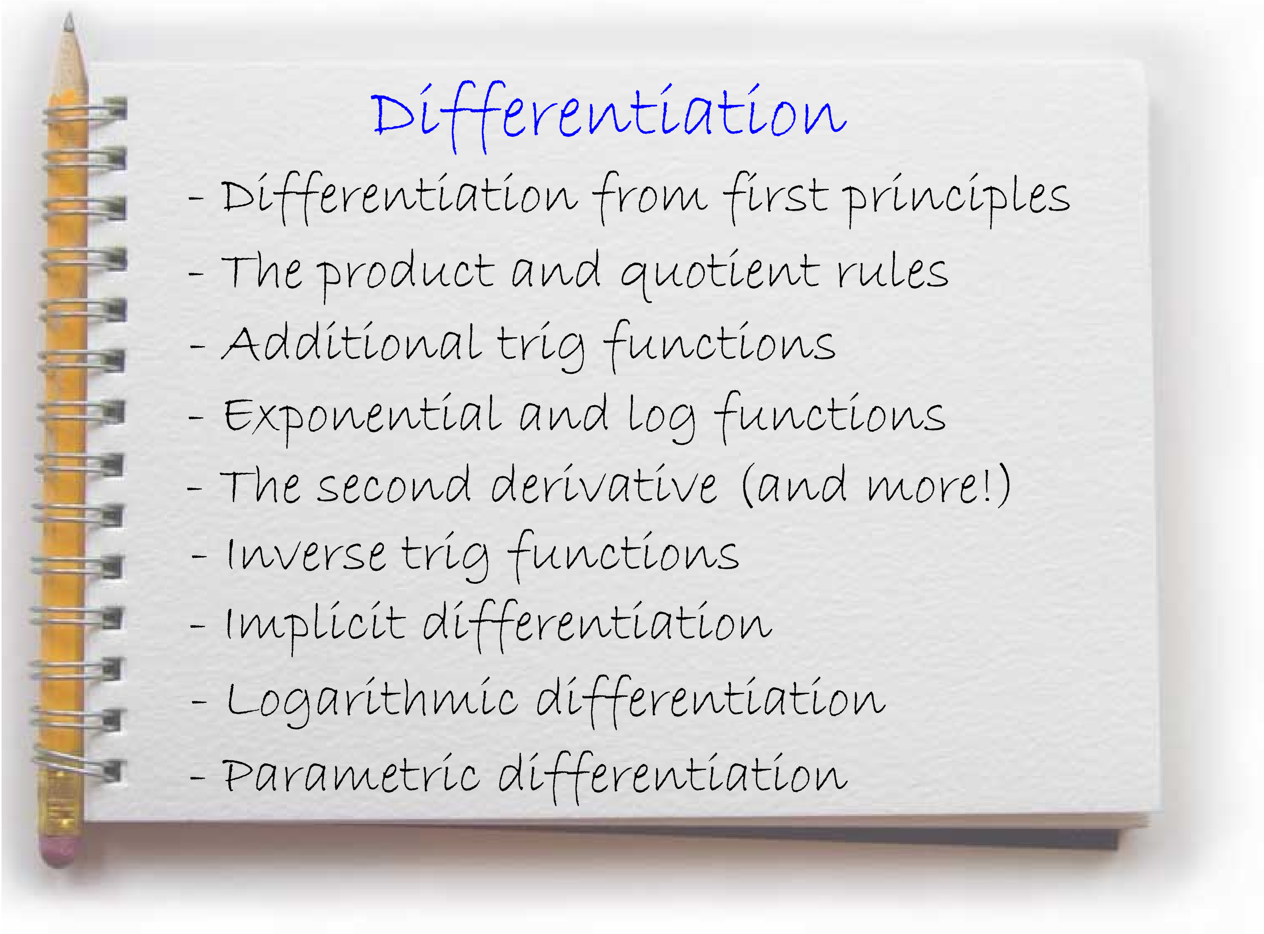
6. Find the term independent of x in: $\left(3x^3+\dfrac{1}{2x^2}\right)^5$

7. If $y=\sin(\cos x)\cos(\cos x)$, show that $\dfrac{dy}{dx}=-\sin x\cos(2\cos x)$

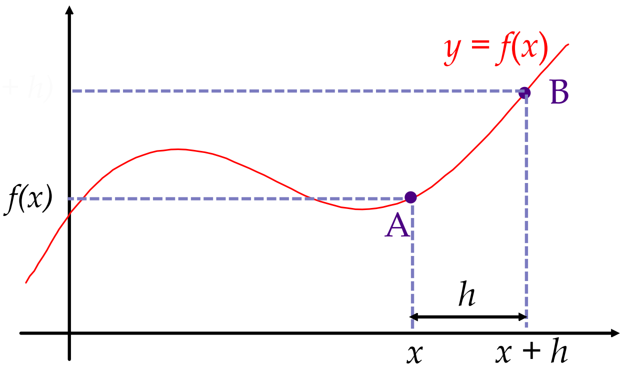
8. Differentiate: $\sin\left(\dfrac{\ln x}{2x}\right)$

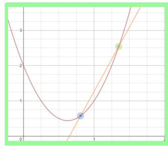
9. If $x^2-2y^2=2x$, find the value of both $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point (4,2) .

10. Reflect on your understanding of the new techniques in partial fractions, particularly in relation to:
(a) connections to your existing knowledge and
(b) any elements of this new area which interest you.


Differentiation
- Differentiation from first principles
- The product and quotient rules
- Additional trig functions
- Exponential and log functions
- The second derivative (and more!)
- Inverse trig functions
- Implicit differentiation
- Logarithmic differentiation
- Parametric differentiation

Differentiation from First Principles





$$m_{AB} = \frac{f(x+h) - f(x)}{h}$$

To estimate the gradient of a tangent at A (x, $f(x)$), we consider the gradient of the chord between A and a point B ($x + h$, $f(x+h)$).

As the value of h tends to zero, the gradient of AB becomes closer to the gradient of the tangent at A, and is denoted by $f\,'(x)$.

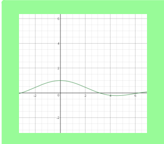
$$\boxed{f'(x) = \lim_{B \to A} m_{AB} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

We use this notation to 'differentiate from first principles'.
i.e. without using any 'known' derivatives or shortcuts.

Example 1: Differentiate $f(x) = 3x$ from first principles.

Example 2: Differentiate $f(x) = x^2 + \dfrac{3}{x}$ from first principles.

Example 3: Differentiate $f(x) = \sin x$ from first principles.



The Product Rule

Given that functions u and v are differentiable, and $f(x) = u(x)v(x)$, then:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Proof:

In practise, it is often easier to use a short form of Lagrange notation:

Given $f = uv$

then $f' = u'v + uv'$

Example 4: Differentiate the function $f(x) = (x + 1)^2(x - 3)^4$

Example 5: Find $f'(x)$ given $f(x) = \dfrac{1}{\sqrt{x}}\sin(3x - 1)$

Sometime we can use existing knowledge to simplify before differentiating.

Example 6: Differentiate: $f(x) = \frac{1}{2}\cos 2x \cos x + \frac{1}{2}\sin 2x \sin x$

Example 7: Find the equation of the tangent to the curve $y = x^2 \cos 2\pi x$ at $x = 4$

The Quotient Rule

Given that functions u and v are differentiable, and $f(x) = \dfrac{u(x)}{v(x)}$,
then:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Proof:

In shortened form:

Given $f = \dfrac{u}{v}$

then $f' = \dfrac{u'v - uv'}{v^2}$

Example 8: Differentiate $\dfrac{x+1}{2x^3-3}$

Example 9: Given $f(x)=\dfrac{\sin^3 x}{x}$, find $f'(x)$

Example 10: Given $f(x)=\dfrac{\cos x}{\sin^2 x}$, find $f'(\frac{\pi}{4})$

Additional Trig Functions

The secant of x,	$\sec x = \dfrac{1}{\cos x}$
The cosecant of x,	$\text{cosec } x = \dfrac{1}{\sin x}$
The cotangent of x,	$\cot x = \dfrac{1}{\tan x}$

These definitions help us to determine the derivatives of more complex trig functions.

First it is necessary to find some new standard derivatives...

$\tan x$

$f(x) = \tan x$

sec x

$f(x) = \sec x$

cosec x

$f(x) = \operatorname{cosec} x$

$\cot x$

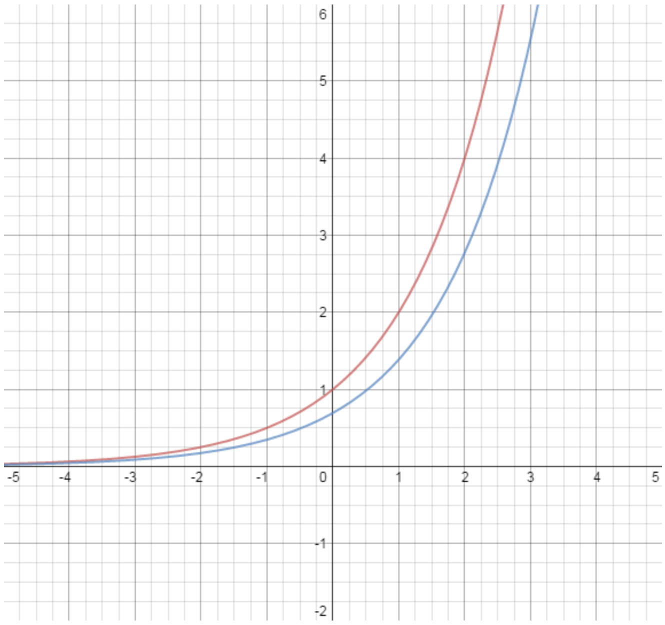
$f(x) = \cot x$

Example 11: Differentiate:

(a) $\sec(3x - 2)$ (b) $\text{cosec}(\cos x)$

Exponential Functions

Is there a function, $f(x) = a^x$
for which the gradient is 1 at $x = 0$?



Consider the functions $f(x) = 2^x$ and $f(x) = 3^x$

$$f'(x) = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} \qquad f'(x) = \lim_{h \to 0} \frac{3^{x+h} - 3^x}{h}$$

at $x = 0$, $$f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \qquad f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h}$$

when h is very small,

$$\frac{2^h - 1}{h} < 1 < \frac{3^h - 1}{h}$$

For what value of a does $\dfrac{a^h - 1}{h} = 1$?

$\dfrac{2.718281281^h - 1}{h} = 1$ We define this exact value as e

From first principles we can find the derivative of e^x

Important fact!

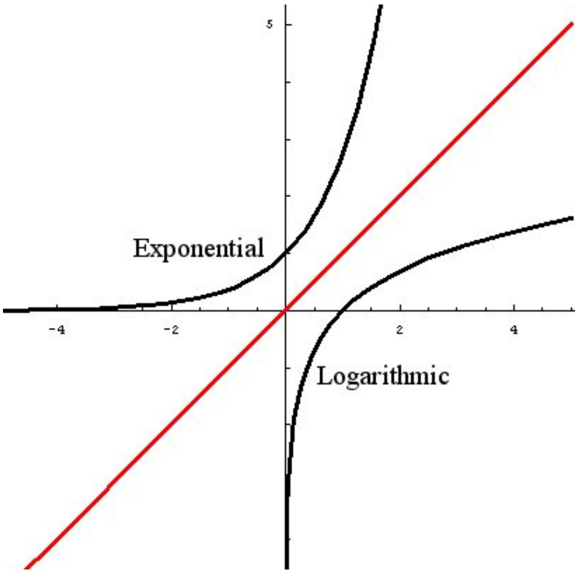
The gradient of the function $y = e^x$ is always equal to the value of the function (the y co-ordinate) at that point. Check it out!

Example 12: Differentiate:

(a) $f(x) = e^{2x+1}$ (b) $f(x) = x^3 e^{\sin x}$

Logarithmic Functions

The natural logarithm function, $g(x) = \ln x$ is the inverse of the exponential function $f(x) = e^x$.



If $y = \ln x$

Given $f(x) = \ln x$, $f'(x) = \dfrac{1}{x}$

Example 13: Differentiate:

(a) $f(x) = \ln 3x$ (b) $f(x) = \ln (4x^2 - 3)$

Example 14: If $f(x) = \sin x \ln x$, find $f'(x)$

Example 15: Differentiate: $\dfrac{\ln(\cos x)}{x^2}$

Higher Derivatives

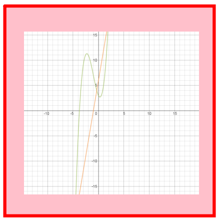
For many functions, it is possible to find second and third derivatives, and more.

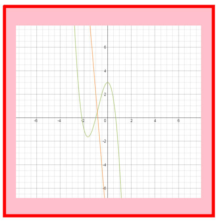
The second derivative has importance in:

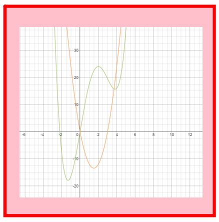
- determining the rate of change of the rate of change
  (e.g. the rate of change of velocity is acceleration)

- determining the nature of stationary points
  (instead of using a table of values)

Determining the Nature of Stationary Points







We have previously used a table of signs to determine the nature of stationary points. We can use the second derivative as an alternative method.

For any function, f(x), at x = a,

if $f''(x)$ is **negative**, $f(x)$ has a **maximum** stationary value.

if $f''(x)$ is **positive**, $f(x)$ has a **minimum** stationary value.

if $f''(x)$ is 0 , $f(x)$ may have a **point of inflexion**
(verify with a table of signs)

Example 16:
Investigate the nature of the stationary points of the function:

$$f(x) = x^3 + \tfrac{3}{2}x^2 - 6x + 1$$

Differentiating Inverse Functions

Finding the derivative of an inverse function depends on the statement:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Where $y = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \frac{dx}{dy}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{\frac{dx}{dy}}$$

$$\boxed{\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f^{-1}(y)}}$$

Example 17 :
Find the derivative of the inverse of the function $f(x) = x^2 - 2$

This algorithm is particularly useful in determining the derivatives of inverse trig functions.

Differentiating Inverse Trig Functions

Example 18: Differentiate f(x) = $\sin^{-1} x$

In the same way differentiate $\cos^{-1} x$

In the same way differentiate $\tan^{-1} x$

We can develop these derivatives using the chain rule.

Example 19: Find the derivative of the function $y = \sin^{-1}\left(\frac{x}{2}\right)$

Summary

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = -\frac{1}{\sqrt{a^2-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2}$$

Example 20: Find the derivative of the function $y = \tan^{-1}(3\cos x)$

Example 21: Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{x-1}{x+1}\right)$

Implicit Functions

An explicit function for two variables, x and y, is one where y is a clearly defined function in x.
That is, the function can be expressed only in terms of x and only one value of y is obtained for any one value of x.

For example, $xy = 4x^3 - x^2 + y$ can be rewritten:

Thus, y is an explicit function of x.

An implicit function for two variables, x and y, is one where y is not an explicit function of x.

Either:

i) the function cannot be expressed only in terms of x

e.g. $y^2 + 2xy - 5x^2 = 0$

or *ii)* there is more than one value of y for any one value of x.

e.g. $x^2 + y^2 = 1$

Implicit Differentiation

To differentiate an implicit function:

- differentiate each term on both sides of the equation.

- differentiate terms in y using the chain rule.

$$\frac{d}{dx}\big(f(y)\big)=\frac{d}{dy}\big(f(y)\big)\times\frac{dy}{dx}$$

- rearrange in terms of $\frac{dy}{dx}$

Example 22: Find $\frac{dy}{dx}$ in terms of x and y when $y^2+2y-3x=x^2$

Where xy terms appear, use the product rule.

Example 23: Find $\frac{dy}{dx}$ in terms of x and y when $y^2 + 2xy - 5x^2 = 0$

Example 24: Find $\frac{dy}{dx}$ in terms of x and y when $\cos(4x-3y)=6x$

Equation of a Tangent to an Implicit Function

As with explicit functions, the tangent to an implicit function at a given point (a,b) will take the form

$$y - b = m(x - a),$$

where m is the derivative of the function at this point.

Example 25:

The point (6,3) lies on the circle $(x-3)^2 + (y-1)^2 = 13$.
Find the equation of the tangent to the circle at this point.

Finding the Second Derivative of an Implicit Function

The second derivative is often easier calculated before the first derivative is fully rearranged.

Example 26: Find $\frac{d^2y}{dx^2}$ when $3x-6y+xy=0$

Logarithmic Differentiation

Expressions which have extended products or quotients, or where the variable appears as a power, can be differentiated easier by introducing natural logarithms, as log rules can simplify the process.

Example 27: Differentiate $y = 2^{\sin x}$

Example 28: Differentiate $y = x^{x^2}$

Example 29: Differentiate $y = \dfrac{(3x+2)^2}{\sqrt{x+4}}$

Parametric Equations

Some curves cannot be described by expressing y directly in terms of x.

Such curves may be represented by 2 functions, describing the behaviour of the x and y coordinates separately, with respect to a third variable (e.g. time or distance).

i.e. $x = f(t) \qquad y = g(t)$

Equations such as these are parametric equations.

The variable (in this case t) is the parameter.

Parametric equations can be sketched by:

i) constructing a table of values.

or ii) obtaining the cartesian equation of the curve by substitution.

Example 30: Describe the position P(x,y) of a particle moving in a curve given the parametric equations $x = 2t + 1$ and $y = 4t^2$.

Standard Forms

If $x = r\cos\theta$ and $y = r\sin\theta$ then $x^2 + y^2 = r^2$

(equation of a CIRCLE)

If $x = a\cos\theta$ and $y = b\sin\theta$ then $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(equation of a ELLIPSE)

Parametric Differentiation

Like other functions, we can differentiate parametric equations to investigate the nature of stationary points and find the gradients of tangents.

Using the chain rule,

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

Example 31: Find $\frac{dy}{dx}$ in terms of t when $x = 3t^2$ and $y = t^4 - 1$

Example 32: Find $\frac{dy}{dx}$ in terms of t when $x = \frac{1}{t+1}$ and $y = \frac{3}{2t}$.

Equations of Tangents to Parametric Curves

Example 33:

Find the equation of the tangent to the curve described by the parametric equations

$$x = \frac{1}{t^2 - 1} \quad \text{and} \quad y = 3t \quad \text{at} \quad t = 3$$

Second Derivatives of Parametric Equations

Using the chain rule, $\boxed{\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \times \dfrac{dt}{dx}}$

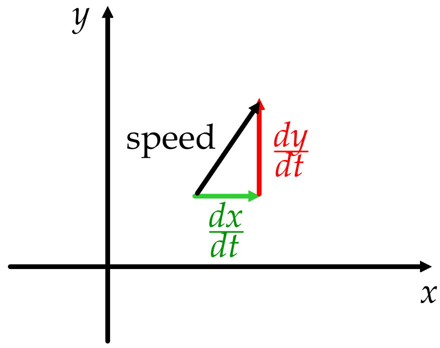
Example 34: Find $\dfrac{d^2y}{dx^2}$ in terms of t given: $x = \dfrac{4}{t^2}$ and $y = t^4 - 3$

Second Derivatives of Parametric Equations

Example 35: Find the turning points of the curve $x = t$, $y = t^3 - 3t$ and determine their nature.

Calculating Speed using Parametric Equations

Given parametric equations for displacement, it is possible to calculate the speed of an object at a given point in time.



Speed is therefore the magnitude of the resultant vector.
(or the length of the hypotenuse of the right angled triangle!)

Therefore: $$speed = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example 36:

The position of a particle with respect to a coordinate axis system at t seconds is given by:

$$x(t) = 5t \quad y(t) = 5t^2 - 8t \qquad 0 \leq t \leq 10$$

Calculate the speed of the particle when $t = 3$.

Past Paper Questions

Differentiation Part 1

2001

Differentiate with respect to x $g(x) = e^{\cot 2x}$, $o < x < \frac{\pi}{2}$. **(2 marks)**

2002

Given that $f(x) = \sqrt{x}e^{-x}$, $x \geq 0$, obtain and simplify $f'(x)$. **(4 marks)**

2003

Given $f(x) = x(1+x)^{10}$, obtain $f'(x)$ and simplify your answer. **(3 marks)**

2004

Given $f(x) = \cos^2 x e^{\tan x}$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$, obtain $f'(x)$ and evaluate $f'\left(\frac{\pi}{4}\right)$. **(3,1 marks)**

2005

(a) Given $f(x) = x^3 \tan 2x$, where $0 < x < \frac{\pi}{4}$, obtain $f'(x)$. **(3 marks)**

(b) For $y = \frac{1+x^2}{1+x}$, where $x \neq -1$, determine $\frac{dy}{dx}$ in simplified form. **(3 marks)**

2006

Differentiate, simplifying your answer: $\frac{1+\ln x}{3x}$, where $x > 0$. **(3 marks)**

2007

Obtain the derivative of the function $f(x) = \exp(\sin 2x)$. **(3 marks)**

2009

Given $f(x) = (x+1)(x-2)^3$, obtain the values of x for which $f'(x) = 0$. **(3 marks)**

2010

Differentiate the following functions

(a) $f(x) = e^x \sin x^2$. **(3 marks)**

(b) $g(x) = \frac{x^3}{1+\tan x}$. **(3 marks)**

2011

Given $f(x) = \sin x \cos^3 x$, obtain $f'(x)$. **(3 marks)**

2012

(a) Given $f(x) = \dfrac{3x+1}{x^2+1}$, obtain $f'(x)$. **(3 marks)**

(b) Let $g(x) = \cos^2 x \exp(\tan x)$. Obtain an expression for $g'(x)$ and simplify your answer. **(4 marks)**

2013

Differentiate $f(x) = e^{\cos x} \sin^2 x$. **(3 marks)**

2014

Given $f(x) = \dfrac{x^2-1}{x^2+1}$, obtain $f'(x)$ and simplify your answer. **(3 marks)**

2015

(a) For $y = \dfrac{5x+1}{x^2+2}$, find $\dfrac{dy}{dx}$. Express your answer as a single, simplified fraction. **(3 marks)**

(b) Given $f(x) = e^{2x} \sin^2 3x$, obtain $f'(x)$. **(3 marks)**

Past Paper Questions

Differentiation Part 2

2001

① Differentiate with respect to x.

(a) $f(x) = (2+x)\tan^{-1}\sqrt{x-1}, \quad x > 1$

(b) $g(x) = e^{\cot 2x}, \quad 0 < x < \frac{\pi}{2}$ **(4, 2 marks)**

② A curve has equation $xy + y^2 = 2$.

(a) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

(b) Hence find an equation of the tangent to the curve at the point $(1,1)$.

(3, 2 marks)

2002

① A curve is defined by the parametric equations

$$x = t^2 + t - 1, \quad y = 2t^2 - t + 2$$

for all t. show that the point $A(-1,5)$ lies on the curve and obtain an equation of the tangent to the curve at the point A.

(6 marks)

② Given $y = (x+1)^2(x+2)^{-4}$ and $x > 0$, use logarithmic differentiation to show that $\frac{dy}{dx}$ can be expressed in the form $\left(\frac{a}{x+1} + \frac{b}{x+2}\right)y$, stating the values of the constants a and b. **(3 marks)**

2003

① (a) Given $f(x) = x(1+x)^{10}$, obtain $f'(x)$ and simplify your answer.

(b) Given $y = 3^x$, use logarithmic differentiation to obtain $\frac{dy}{dx}$ in terms of x.

(3, 3 marks)

② The equation $y^3 + 3xy = 3x^2 - 5$ defines a curve passing through the point $A(2,1)$. Obtain an equation for the tangent to the curve at A.

2004

① (a) Given $f(x)=\cos^2 x\, e^{\tan x}$, $\frac{-\pi}{2}<x<\frac{\pi}{2}$, obtain $f'(x)$ and evaluate $f'\left(\frac{\pi}{4}\right)$.

(b) Differentiate $g(x)=\frac{\tan^{-1}2x}{1+4x^2}$. **(4, 3 marks)**

② A curve is defined by the equations $x=5\cos\theta$, $y=5\sin\theta$, $(0\le\theta\le 2\pi)$.

Use parametric differentiation to find $\frac{dy}{dx}$ in terms of θ.

Find the equation of the tangent to the curve at the point where $\theta=\frac{\pi}{4}$.

(2, 3 marks)

2005

Given the equation $2y^2-2xy-4y+x^2=0$ of a curve, obtain the x coordinate of each point at which the curve has a horizontal tangent.

(4 marks)

2006

① Differentiate, simplifying your answers:

(a) $2\tan^{-1}\sqrt{1+x}$, where $x>-1$;

(b) $\frac{1+\ln x}{3x}$, where $x>0$. **(3, 3 marks)**

② Given $xy-x=4$, use implicit differentiation to obtain $\frac{dy}{dx}$ in terms of x and y.

Hence obtain $\frac{d^2y}{dx^2}$ in terms of x and y. **(2, 3 marks)**

2007

A curve is defined by the parametric equations $x=\cos 2t$, $y=\sin 2t$, $0<t<\frac{\pi}{2}$.

(a) Use parametric differentiation to find $\frac{dy}{dx}$.

Hence find the equation of the tangent when $t=\frac{\pi}{8}$.

(b) Obtain an expression for $\frac{d^2y}{dx^2}$ and hence show that $\sin 2t\,\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=k$,

where k is an integer. State the value of k.

(5, 5 marks)

2008

① (a) Differentiate $f(x)=\cos^{-1}(3x)$ where $\frac{-1}{3}<x<\frac{1}{3}$.

(b) Given $x=2\sec\theta$, $y=3\sin\theta$, use parametric differentiation to find $\frac{dy}{dx}$ in terms of θ. **(2, 3 marks)**

② A curve is defined by the equation $xy^2+3x^2y=4$ for $x>0$ and $y>0$.

Use implicit differentiation to find $\frac{dy}{dx}$.

Hence find an equation of the tangent to the curve where $x=1$.

(3, 3 marks)

2009

① (a) Given $f(x)=(x+1)(x-2)^3$, obtain the values of x for which $f'(x)=0$.

(b) Calculate the gradient of the curve defined by $\frac{x^2}{y}+x=y-5$ at the point $(3,-1)$.

(3, 4 marks)

2010

Given $y=t^3-\frac{5}{2}t^2$ and $x=\sqrt{t}$ for $t>0$, use parametric differentiation to express $\frac{dy}{dx}$ in terms of t in simplified form.

Show that $\frac{d^2y}{dx^2}=at^2+bt$, determining the values of the constants a and b.

Obtain an equation for the tangent to the curve which passes through the point of inflexion.

(4, 3, 3 marks)

2011

Obtain $\frac{dy}{dx}$ when y is defined as a function of x by the equation $y+e^y=x^2$.

(3 marks)

2012

① The radius of a cylindrical column of liquid is decreasing at the rate of $0\cdot02ms^{-1}$, while the height is increasing at the rate of $0\cdot01ms^{-1}$.
Find the rate of change of the volume when the radius is $0\cdot6$ metres and the height is 2 metres.
(Recall that the volume of a cylinder is given by $V=\pi r^2h$.) **(5 marks)**

(2) A curve is defined parametrically, for all t, by the equations

$$x = 2t + \frac{1}{2}t^2, \qquad y = \frac{1}{3}t^3 - 3t.$$

Obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of t.

Find the values of t at which the curve has stationary points and determine their nature.

Show that the curve has exactly two points of inflexion.

(5, 3, 2 marks)

2013

A curve has equation $x^2 + 4xy + y^2 + 11 = 0$.

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-2, 3)$. **(6 marks)**

Past Paper Answers
Differentiation 1

2001 (Q.A2)
$-2e^{cot2x}cosec^2 2x$

2002 (Q.A4a)
$f'(x) = \frac{1}{2\sqrt{x}}e^{-x}(1-2x)$

2003 (Q.A1)
$f'(x) = (1+x)^9(1+11x)$

2004 (Q.1)
$f'(x) = e^{tanx}(1 - sin2x);\ f'\left(\frac{\pi}{4}\right) = 0$

2005 (Q.1)
a) $f'(x) = x^2(3tan2x + 2xsec^2 2x)$ b) $\frac{dy}{dx} = \frac{x^2+2x-1}{(1+x)^2}$

2006 (Q.2b)
$-\frac{lnx}{3x^2}$

2007 (Q.2a)
$f'(x) = 2e^{\sin 2x}\cos 2x$

2009 (Q.1a)
$x = -\frac{1}{4},\ x = 2$

2010 (Q.1)
a) $e^x(\sin x^2 + 2x\cos x^2)$ b) $g'(x) = \frac{x^2[3(1+\tan x - x\sec^2 x}{(1+\tan x)^2}$

2011 (Q.3b)
$f'(x) = \cos^2 x\,(\cos^2 x - 3\sin^2 x)$

2012 (Q.1)
a) $f'(x) = \frac{-3x^2-2x+3}{(x^2+1)^2}$ b) $g'(x) = e^{\tan x}(1-\sin 2x)$

2013 (Q.2)
$f'(x) = e^{\cos x}(\sin 2x - \sin^3 x)$

2014 (Q.1a)
$f'(x) = \frac{4x}{(x^2+1)^2}$

2015 (Q.1)
a) $f'(x) = \frac{-5x^2-2x+10}{(x^2+2)^2}$ b) $f'(x) = 2e^{2x}\sin 3x\,(\sin 3x + 3\cos 3x)$

Past Paper Answers
Differentiation 2

2001

qA2 1) a) $tan^{-1}\sqrt{x-1}+\frac{2+x}{2x\sqrt{x-1}}$ b) $-2e^{cot2x}cosec^2 2x$

qA7 2) a) $\frac{dy}{dx}=-\frac{y}{x+2y}$ b) $3y+x-4=0$

2002

qA3 proof; $y=5x+10$

qA4b proof; a = 2, b = -4

2003

qA1 a) $f'(x)=(1+x)^9(1+11x)$ b) $\frac{dy}{dx}=3^x ln3$

qA3 $x-y-1=0$

2004

q1 a) $f'(x)=e^{tanx}(1-sin2x);\ f'\left(\frac{\pi}{4}\right)=0$ b) $g'(x)=\frac{2-8xtan^{-1}2x}{(1+4x^2)^2}$

q3 $\frac{dy}{dx}=-\cot\theta;\ x+y-5\sqrt{2}=0$

2005 (Q.2)

$x=0, x=4$

2006

q2 a) $\frac{1}{(2+x)\sqrt{(1+x)}}$ b) $-\frac{lnx}{3x^2}$

q4 $\frac{dy}{dx}=\frac{1-y}{x};\ \frac{d^2y}{dx^2}=\frac{2(y-1)}{x^2}$

2007 (Q.13)

a) $\frac{dy}{dx}=-cot2$t; $x+y-\sqrt{2}=0$ b) $\frac{d^2y}{dx^2}=-cosec^3 2t;$ k = -1

2008

q2 a) $f'(x)=-\frac{3}{\sqrt{1-9x^2}}$ b) $\frac{dy}{dx}=\frac{3}{2}cot\theta cos^2\theta$

q5 $\frac{dy}{dx}=-\frac{y(y+6x)}{x(2y+3x)};$ $7x+5y-12=0$

2009 (Q.1)

a) $x=-\frac{1}{4},\ x=2$ b) $m=-\frac{1}{2}$

2010 (Q.13)

$\frac{dy}{dx}=2\sqrt{t^3}(3t-5)$; proof a = 30, b = -30; $8x+2y-5=0$

2011 (Q.3)

a) $\frac{dy}{dx} = \frac{2x}{1+e^y}$ b) $f'(x) = \cos^2 x\,(\cos^2 x - 3\sin^2 x)$

2012

q12 $\frac{dV}{dt} = -0.044\ \text{m}^3/\text{s}$

q13 $\frac{dy}{dx} = \frac{t^2-3}{2+t}$; $\frac{d^2y}{dx^2} = \frac{(t+1)(t+3)}{(t+2)^2}$; min TP at $t = \sqrt{3}$, max TP at $t = -\sqrt{3}$;

points of inflexion when t = -3 and t = -1

2013 (Q.11)

$\frac{dy}{dx} = 4$; $\frac{d^2y}{dx^2} = 33$

Advanced Higher Mathematics
Formal Homework Assignment

Differentiation

1. Find the derivative of $f(x) = (2x - 1)^2$ from first principles.

2. Differentiate, with respect to x:

a) $f(x) = x^2\cos 4x$ b) $f(x) = \dfrac{e^{3x}}{e^{3x+1}}$

3. If $f(x) = \sin x \sec x$, show that $f'\left(\frac{\pi}{3}\right) = 4$

4. A curve is defined by the equations: $x = t^2 + \dfrac{2}{t}$ and $y = t^2 - \dfrac{2}{t}$.
Find the coordinates of the turning point on the curve and determine its nature.

5. Given $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{\pi}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

6. Differentiate: $y = \dfrac{\cos^{-1} 2x}{x\sqrt{x}}$

7. Given $y = \dfrac{2^x}{x+2}$, find the rate of change of y with respect to x when x = 2.

8. A Serpentine curve has equation $x^2 y = ax - a^2 y$.

a) Show that $\dfrac{dy}{dx} = \dfrac{a(a^2 - x^2)}{(a^2 + x^2)^2}$.

b) Let m_1, m_2 and m_3 be the gradients of the tangents at $x = \frac{1}{2}a$, $x = a$ and $x = 2a$ respectively. Show that $m_1 + m_2 + 4m_3 = 0$.

9. Reflect on your understanding of the new techniques in differentiation, particularly in relation to:
(a) connections to your existing knowledge and
(b) any elements of these new areas which interest you.