SCOTTISH QUALIFICATIONS AUTHORITY

2004 Mathematics

Advanced Higher

Finalised Marking Instructions

Solutions to Advanced Higher Mathematics Paper

1. (a)

$$f(x) = \cos^2 x\, e^{\tan x}$$

$$f'(x) = 2(-\sin x)\cos x\, e^{\tan x} + \cos^2 x\ \sec^2 x\, e^{\tan x}$$

1 for Product Rule
2 for accuracy

$$= \left(1 - \sin 2x\right) e^{\tan x}$$

$$f'\left(\frac{\pi}{4}\right) = \left(1 - \sin\frac{\pi}{2}\right) e^{\tan \pi/4} = 0.$$ **1**

(b)

$$g(x) = \frac{\tan^{-1} 2x}{1 + 4x^2}$$

$$g'(x) = \frac{\frac{2}{1+4x^2}\left(1 + 4x^2\right) - \tan^{-1} 2x\,(8x)}{(1 + 4x^2)^2}$$

1 for Product Rule
2 for accuracy

$$= \frac{2 - 8x \tan^{-1} 2x}{(1 + 4x^2)^2}$$

2.

$$\left(a^2 - 3\right)^4 = \left(a^2\right)^4 + 4\left(a^2\right)^3(-3) + 6\left(a^2\right)^2(-3)^2 + 4\left(a^2\right)(-3)^3 + (-3)^4$$

1 for binomial coefficients

$$= a^8 - 12a^6 + 54a^4 - 108a^2 + 81$$ **1 for powers**

1 for coefficients

3.

$$x = 5\cos\theta \Rightarrow \frac{dx}{d\theta} = -5\sin\theta$$

$$y = 5\sin\theta \Rightarrow \frac{dy}{d\theta} = 5\cos\theta$$ **1**

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5\cos\theta}{-5\sin\theta}$$ **1**

When $\theta = \frac{\pi}{4}, \frac{dy}{dx} = -\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1,$ **1**

$x = \frac{5}{\sqrt{2}}, y = \frac{5}{\sqrt{2}}$ **1**

so an equation of the tangent is

$$y - \frac{5}{\sqrt{2}} = -\left(x - \frac{5}{\sqrt{2}}\right)$$ **1**

i.e. $x + y = 5\sqrt{2}.$

4.

$$z^2(z + 3) = (1 + 4i - 4)(1 + 2i + 3)$$ **1 for a method**

$$= (-3 + 4i)(4 + 2i)$$

$$= -20 + 10i$$ **1**

$$z^3 + 3z^2 - 5z + 25 = z^2(z + 3) - 5z + 25$$ **1 for a method**

$$= -20 + 10i - 5 - 10i + 25 = 0$$ **1**

Note: direct substitution of $1 + 2i$ *into* $z^3 + 3z^2 - 5z + 25$ *was acceptable.*

Another root is the conjugate of z, i.e. $1 - 2i$. **1**

The corresponding quadratic factor is $((z - 1)^2 + 4) = z^2 - 2z + 5$.

$$z^3 + 3z^2 - 5z + 25 = (z^2 - 2z + 5)(z + 5)$$

$$z = -5$$ **1**

Note: any valid method was acceptable.

5.

$$\frac{1}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2}$$ **1 for method**

$$= \frac{1}{5(x - 3)} - \frac{1}{5(x + 2)}$$ **1**

$$\int_0^1 \frac{1}{x^2 - x - 6}\,dx = \frac{1}{5}\int_0^1 \frac{1}{|x - 3|} - \frac{1}{|x + 2|}\,dx$$ **1 for method**

1 for accuracy

$$= \frac{1}{5}\left[\ln|x - 3| - \ln|x + 2|\,\right]_0^1$$ **1**

$$= \frac{1}{5}\left[\ln\frac{|x - 3|}{|x + 2|}\right]_0^1$$

$$= \frac{1}{5}\left[\ln\frac{2}{3} - \ln\frac{3}{2}\right]$$ **1**

$$= \frac{1}{5}\ln\frac{4}{9} \approx -0.162$$

6.

$$M_1 = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ **2**

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ **1**

$$M_2M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$ **1**

The transformation represented by M_2M_1 is reflection in $y = -x$. **1**

7.

$$f(x) = e^x \sin x \qquad f(0) = 0$$
$$f'(x) = e^x \sin x + e^x \cos x \qquad f'(0) = 1 \qquad \mathbf{1}$$
$$f''(x) = e^x \sin x + e^x \cos x - e^x \sin x + e^x \cos x \qquad f''(0) = 2 \qquad \mathbf{1}$$
$$= 2e^x \cos x$$
$$f'''(x) = 2e^x \cos x - 2e^x \sin x \qquad f'''(0) = 2 \qquad \mathbf{1}$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots \qquad \mathbf{1}$$

$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 - \ldots \qquad \mathbf{1}$$

OR

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad \mathbf{1}$$

$$\sin x = x - \frac{x^3}{3!} + \ldots \qquad \mathbf{1}$$

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) \qquad \mathbf{1 - method}$$

$$= x - \frac{x^3}{6} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \ldots \qquad \mathbf{1}$$

$$= x + x^2 + \frac{x^3}{3} - \ldots \qquad \mathbf{1}$$

8.

$$231 = 13 \times 17 + 10 \qquad \textbf{1 for method}$$
$$17 = 1 \times 10 + 7$$
$$10 = 1 \times 7 + 3$$
$$7 = 2 \times 3 + 1 \qquad \mathbf{1}$$

Thus the highest common factor is 1.

$$1 = 7 - 2 \times 3$$
$$= 7 - 2 \times (10 - 7) = 3 \times 7 - 2 \times 10 \qquad \textbf{1 for method}$$
$$= 3 \times (17 - 10) - 2 \times 10 = 3 \times 17 - 5 \times 10$$
$$= 3 \times 17 - 5 \times (231 - 13 \times 17) = 68 \times 17 - 5 \times 231. \qquad \mathbf{1}$$

So $x = -5$ and $y = 68$.

9.

$$x = (u - 1)^2 \Rightarrow dx = 2(u - 1)\,du \qquad \mathbf{1}$$

$$\int \frac{1}{(1 + \sqrt{x})^3}\,dx = \int \frac{2(u - 1)}{u^3}\,du \qquad \mathbf{1}$$

$$= 2\int \left(u^{-2} - u^{-3}\right)du \qquad \mathbf{1}$$

$$= 2\left(\frac{-1}{u} + \frac{1}{2u^2}\right) + c \qquad \mathbf{1}$$

$$= \left(\frac{1}{(1 + \sqrt{x})^2} - \frac{2}{(1 + \sqrt{x})}\right) + c \qquad \mathbf{1}$$

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10. $f(x) = x^4 \sin 2x$ so

$$f(-x) = (-x)^4 \sin(-2x) \qquad \mathbf{1}$$
$$= -x^4 \sin 2x \qquad \mathbf{1}$$
$$= -f(x)$$

So $f(x) = x^4 \sin 2x$ is an odd function. **1**

Note: a sketch given with a comment and correct answer, give full marks.
A sketch without a comment, gets a maximum of two marks.

11.

$$V = \int_a^b \pi y^2\, dx \qquad \mathbf{1}$$
$$= \pi \int_0^1 e^{-4x} dx$$

1 for applying formula
1 for accuracy

$$= \pi\left[-\frac{e^{-4x}}{4}\right]_0^1 \qquad \mathbf{1}$$
$$= \pi\left[\frac{-1}{4e^4} + \frac{1}{4}\right] \qquad \mathbf{1}$$
$$= \frac{\pi}{4}\left[1 - \frac{1}{e^4}\right] \approx 0.7706$$

12.

$$\text{LHS} = \frac{d}{dx}(xe^x) = xe^x + 1e^x = (x + 1)e^x$$
$$\text{RHS} = (x + 1)e^x \qquad \mathbf{1}$$

So true when $n = 1$.

Assume $\frac{d^k}{dx^k}(xe^x) = (x + k)e^x$ **1**

Consider
$$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(xe^x)\right)$$
$$= \frac{d}{dx}\left((x + k)e^x\right) \qquad \mathbf{1}$$
$$= e^x + (x + k)e^x \qquad \mathbf{1}$$
$$= \left(x + (k + 1)\right)e^x$$

So true for k means it is true for $(k + 1)$, therefore it is true for all integers $n \geqslant 1$. **1**

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13. (a)

$$y = \frac{x - 3}{x + 2} = 1 - \frac{5}{x + 2}$$ **1**

Vertical asymptote is $x = -2$. **1**

Horizontal asymptote is $y = 1$. **1**

(b)

$$\frac{dy}{dx} = \frac{5}{(x + 2)^2}$$ **1**

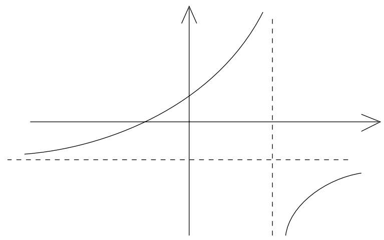
$$\neq 0$$ **1**

(c)

$$\frac{d^2y}{dx^2} = \frac{-10}{(x + 2)^3} \neq 0$$ **1**

So there are no points of inflexion. **1**

(d)



1

The asymptotes are $x = 1$ and $y = -2$. **1**

The domain must exclude $x = 1$. **1**

Note: candidates are not required to obtain a formula for f^{-1}.

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14. (a)

$$\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k},\ \overrightarrow{AC} = 0\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$ **1**

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -4 \\ 0 & 1 & -3 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$$ **1 for method**, **1 for accuracy**

$$-2x - 3y - z = c\ (= -2 + 0 - 3 = -5)$$

i.e. an equation for π_1 is $2x + 3y + z = 5$. **1**

Let an angle be θ, then

$$\cos\theta = \frac{(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}).(\mathbf{i} + \mathbf{j} - \mathbf{k})}{\sqrt{4 + 9 + 1}\sqrt{1 + 1 + 1}}$$ **1**

$$= \frac{2 + 3 - 1}{\sqrt{14 \times 3}}$$ **1**

$$= \frac{4}{\sqrt{42}}$$

$$\theta \approx 51{\cdot}9°$$ **1**

Note: an acute angle is required.

(b) Let $\frac{x - 11}{4} = \frac{y - 15}{5} = \frac{z - 12}{2} = t.$

Then $x = 4t + 11; y = 5t + 15; z = 2t + 12$ **1**

$$(4t + 11) + (5t + 15) - (2t + 12) = 0$$

$$7t = -14 \Rightarrow t = -2$$ **1**

$$x = 3; y = 5 \text{ and } z = 8.$$ **1**

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15. (a)

$$x\frac{dy}{dx} - 3y = x^4$$

$$\frac{dy}{dx} - \frac{3}{x}y = x^3$$ **1**

Integrating factor is $e^{\int -\frac{3}{x}dx}$ **1**

$$= x^{-3}.$$ **1**

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y = 1$$ **1**

$$\frac{d}{dx}\left(\frac{1}{x^3}y\right) = 1$$ **1**

$$\frac{y}{x^3} = x + c$$ **1**

$$y = (x + c)x^3$$

$y = 2$ when $x = 1$, so

$$2 = 1 + c$$ **1**

$$c = 1$$

$$y = (x + 1)x^3$$ **1**

(b)

$$y\frac{dy}{dx} - 3x = x^4$$

$$y\frac{dy}{dx} = x^4 + 3x$$ **1**

$$\int y\,dy = \int (x^4 + 3x)\,dx$$ **1**

$$\frac{y^2}{2} = \frac{x^5}{5} + \frac{3x^2}{2} + c'$$ **1**

When $x = 1, y = 2$ so $c' = 2 - \frac{1}{5} - \frac{3}{2} = \frac{3}{10}$ and so

$$y = \sqrt{2\left(\frac{x^5}{5} + \frac{3x^2}{2} + \frac{3}{10}\right)}.$$ **1**

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16. (a) The series is arithmetic with $a = 8, d = 3$ and $n = 17$. **1**

$$S = \frac{n}{2}\{2a + (n - 1)d\} = \frac{17}{2}\{16 + 16 \times 3\} = 17 \times 32 = 544$$ **1**

(b) $a = 2, S_3 = a + ar + ar^2 = 266$. Since $a = 2$ **1**

$$r^2 + r + 1 = 133$$ **1**

$$r^2 + r - 132 = 0$$

$$(r - 11)(r + 12) = 0$$

$$r = 11 \text{ (since terms are positive)}.$$ **1**

Note: other valid equations could be used.

(c)

$$2(2a + 3 \times 2) = a(1 + 2 + 2^2 + 2^3)$$ **1,1**

$$4a + 12 = 15a$$

$$11a = 12$$

$$a = \frac{12}{11}$$ **1**

The sum $S_B = \frac{12}{11}(2^n - 1)$ and $S_A = \frac{n}{2}\left(\frac{24}{11} + 2(n - 1)\right) = n\left(\frac{1}{11} + n\right)$.

1 for a valid strategy

n	4	5	6	7
S_B	$\frac{180}{11}$	$\frac{372}{11}$	$\frac{756}{11}$	$\frac{1524}{11}$
S_A	$\frac{180}{11}$	$\frac{280}{11}$	$\frac{402}{11}$	$\frac{546}{11}$

The smallest n is 7. **1**

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2004 Applied Mathematics

Advanced Higher – Section A

Finalised Marking Instructions

Advanced Higher Applied Mathematics 2004
Solutions for Section A (Statistics 1 and 2)

A1.	(a)	Stratified	**1**
		and Quota [or Quota (convenience)]	**1**
	(b)	Approach (a) should be best	**1**
		since (b) is not random (other forms e.g. Glasgow not typical, biased)	**1**

A2.	(a)	F~*Bin*(192, 0.002).	**1 for distribution**
			1 for parameters
	(b)	$P(F \geqslant 3) = 1 - P(F \leqslant 2)$	**1**
		$= 1 - (0.6809 + 0.2620 + 0.0501)$	**1**
		$= 0.0070$	**1**
		Notes: applying a Poisson distribution loses (at least) one mark; a Normal distribution loses two marks.	
	(c)	Approximate using the *Poi*(0.384)	**1 for distribution**
			1 for parameters

A3.

Assume that yields are normally distributed . **1**

[Random or independent will not do.]

$\bar{x} = 404.2$; $s = 10.03$ **1**

$t = 2.776$ **1**

A 95% confidence interval for the mean yield, μ, is given by:–

$\bar{x} \pm t\dfrac{s}{\sqrt{n}}$ **1**

$404.2 \pm 2.776\dfrac{10.03}{\sqrt{5}}$

404.2 ± 12.45 **1**

or (391.75, 416.65).

The fact that the confidence interval does not include 382 provides *evidence*, at the 5% level of significance, of a change in the mean yield. (Stating it *is* changed loses one mark.) **1**

Note: the third and fourth marks are lost if a z interval is used.

A4.

TNE = 3% of 500 = 15 **1**

With maximum allowable standard deviation

$P(\text{weight} < 485) = 0.025$ **1**

$\Rightarrow \dfrac{485 - 505}{\sigma} = -1.96$ **1,1**

$\Rightarrow \sigma = \dfrac{20}{1.96} = 10.2$ **1**

There will be a small probability of obtaining a content weight less than 470g with the normal model. **1**

A5. Assume that the distributions of times Before and After have the same shape. **1**

Notes: a Normal distribution with the same shape is a valid comment.
Independent, random, Normal (without shape) are not valid.

Null hypothesis H_0: Median After = Median Before

Alternative hypothesis H_1: Median After < Median Before **1**

Time	19	29	31	35	37	39	39	41	42	43	45	52	59	64
Period	A	A	B	A	A	B	A	A	B	B	A	B	B	B
Rank	1	2	3	4	5	6.5	6.5	8	9	10	11	12	13	14

Rank sum for After times = 37.5 **1**

$W - \frac{1}{2}n(n + 1) = 37.5 - 28 = 9.5$

$P(W - \frac{1}{2}n(n + 1) < 10)$

$= \dfrac{125}{3432}$

$= 0.036$ **1**

Since this value is *less than* 0.05 the null hypothesis
would be rejected in favour of the alternative, **1**
indicating evidence of improved performance. **1**

Notes:
As the computed value, 9.5, is not in the tables, a range of values for the probability was acceptable.
A Normal approximation was accepted.

A6.

Cream	A	B	C
Obs. No. of purchasers	66	99	75
Exp. No. of purchasers	80	80	80

1

$$X^2 = \sum \frac{(O - E)^2}{E}$$

$$= \frac{(66 - 80)^2}{80} + \frac{(99 - 80)^2}{80} + \frac{(75 - 80)^2}{80}$$

$= 2.45 + 4.5125 + 0.3125 = 7.275$ **1**

with 2 d.f. **1**

The critical value of chi-squared at the 5% level is 5.991 **1**
so the null hypothesis would be rejected.
i.e. there is *evidence* of a preference. **1**

The fact that the p-value is less than 0.05 confirms **1**
rejection of the null hypothesis at the 5%level of significance.
Note: using a two-tail test loses a mark.

A7. (a) The fitted value is 13.791 **1**
with residual 10.209. **1**

(b) The wedge-shaped plot casts doubt on the assumption of constant variance of Y_i. (i.e. variance not constant) **1**

(c) Satisfactory now since *variance* seems to be *more constant*. **1**
Note: A phrase such as 'more randomly scattered' is acceptable.

(d) The residuals are normally distributed. **1**

A8. (a)

Pre	36	45	30	63	48	52	44	44	45	51	39	44
Post	39	42	33	70	53	51	48	51	51	51	42	50
Post – Pre	3	–3	3	7	5	–1	4	7	6	0	3	6
Sign	1	–1	1	1	1	–1	1	1	1	0	1	1

Assume that differences are independent. **1**
H_0: Median (Post – Pre) = 0 [or $\eta_d = 0$]
H_1: Median (Post – Pre) > 0 [or $\eta_d > 0$] **1**
Under H_0 the differences Bin (11,0.5) with $b = 2$. **1**
$P(B \leqslant 2) = \left(C_0^{11} + C_1^{11} + C_2^{11}\right)0.5^{11}$
$= (1 + 11 + 55)0.5^{11} = 0.0327.$ **1**

Since 0.0327 < 0.05 the null hypothesis is rejected and there **1**
is *evidence* that the median PCS-12 score has gone up. **1**

Note: applying a two-tailed test loses a mark.

(b) $H_0 : \mu_{\text{Post}} = 50$
$H_1 : \mu_{\text{Post}} \neq 50$ **1**
$\bar{x} = 48.42$
$z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \dfrac{48.42 - 50}{10/\sqrt{12}} = -0.55.$ **1,1**
The critical region is $|z| > 1.96$ at the 5% level of significance. **1**
Since –0.55 is not in the critical region, the null hypothesis is accepted indicating that the Post-operation scores are consistent with a population mean of 50. **1**

Note: a correct use of probability comparisons gets full marks.

A9. (a) P (Alaskan fish classified as Canadian)

$= P(X > 120 \mid X \sim N(100, 20^2))$ **1**

$= P\left(Z > \dfrac{120 - 100}{20}\right)$

$= P(Z > 1)$ **1**

$= 0.1587$ **1**

(b) The probability is the same as in (a) because of symmetry. **1**

(c) P (Canadian origin | Alaskan predicted)

$= \dfrac{\text{P (Alaskan predicted and Canadian origin)}}{\text{P (Alaskan predicted)}}$ **1**

$= \dfrac{\text{P (Alaskan predicted but Canadian origin)}}{\text{P (Ala pred and Alaskan)} + \text{P (Ala pred but Canadian)}}$ **1**

$= \dfrac{0.4 \times 0.1587}{0.6 \times 0.8413 + 0.4 \times 0.1587}$ **1,1**

$= \dfrac{0.06348}{0.50478 + 0.06348}$

$= 0.112.$ **1**

Note: Alternative methods acceptable e.g. Venn or Tree Diagrams

A10. The number, X, of inaccurate invoices in samples of n will have the Bin (n, p) distribution so

$V(X) = npq$ **1**

$\quad = np(1 - p)$

$\Rightarrow$ V (Proportion) $= V\left(\frac{1}{n} X\right) = \dfrac{1}{n^2} V(X)$ **1**

$= \dfrac{p(1 - p)}{n}$ **1**

$\Rightarrow$ Standard deviation of Proportion $= \sqrt{\dfrac{p(1 - p)}{n}}$.

(a) UCL $= p + 3\sqrt{\dfrac{p(1 - p)}{n}}$

$= 0.12 + 3\sqrt{\dfrac{0.12 \times 0.88}{150}}$

$= 0.12 + 0.08 = 0.20.$ **1**

LCL $= 0.12 - 0.08 = 0.04$ **1**

(b) The fact that the point for Week 30 falls below the lower chart limit provides evidence of a drop in the proportion of inaccurate invoices. **1**
or: 8 consecutive points fell below the centre line.

(c) A new chart should be constructed (or set new limits) **1**
using an estimate of p for calculation of limits which is based on data collected since the process change. **1**

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2004 Applied Mathematics

Advanced Higher – Section B

Finalised Marking Instructions

Advanced Higher Applied Mathematics 2004
Solutions for Section B (Numerical Analysis 1 and 2)

B1. $f(x) = \ln(2-x) \qquad f'(x) = \dfrac{-1}{(2-x)} \qquad f''(x) = \dfrac{-1}{(2-x)^2} \qquad f'''(x) = \dfrac{-2}{(2-x)^3}$ **1**

Taylor polynomial is

$$p(1+h) = \ln 1 - h - \frac{h^2}{2} - \frac{2h^3}{6}$$

$$= -h - \frac{h^2}{2} - \frac{h^3}{3} \qquad \textbf{1}$$

For ln 1·1, $h = -0{\cdot}1$ and $p(0{\cdot}9) = 0{\cdot}1 - 0{\cdot}005 + 0{\cdot}00033 = 0{\cdot}0953$. **1,1**

$$p(a+h) = \ln(2-a) - \frac{1}{2-a}h \qquad \textbf{1}$$

Hence expect $f(x)$ to be more sensitive in I_2 since coefficient of h is much larger. **1**

B2. $L(2{\cdot}5)$

$$= \frac{(2{\cdot}5-1{\cdot}5)(2{\cdot}5-3{\cdot}0)(2{\cdot}5-4{\cdot}5)}{(0{\cdot}5-1{\cdot}5)(0{\cdot}5-3{\cdot}0)(0{\cdot}5-4{\cdot}5)}1{\cdot}737 + \frac{(2{\cdot}5-0{\cdot}5)(2{\cdot}5-3{\cdot}0)(2{\cdot}5-4{\cdot}5)}{(1{\cdot}5-0{\cdot}5)(1{\cdot}5-3{\cdot}0)(1{\cdot}5-4{\cdot}5)}2{\cdot}412$$

$$+ \frac{(2{\cdot}5-0{\cdot}5)(2{\cdot}5-1{\cdot}5)(2{\cdot}5-4{\cdot}5)}{(3{\cdot}0-0{\cdot}5)(3{\cdot}0-1{\cdot}5)(3{\cdot}0-4{\cdot}5)}3{\cdot}284 + \frac{(2{\cdot}5-0{\cdot}5)(2{\cdot}5-1{\cdot}5)(2{\cdot}5-3{\cdot}0)}{(4{\cdot}5-0{\cdot}5)(4{\cdot}5-1{\cdot}5)(4{\cdot}5-3{\cdot}0)}2{\cdot}797 \qquad \textbf{2}$$

$$= -\frac{1\times 1{\cdot}737}{10} + \frac{2\times 2{\cdot}412}{4{\cdot}5} + \frac{4\times 3{\cdot}284}{2{\cdot}5\times 2{\cdot}25} - \frac{1\times 2{\cdot}797}{18}$$

$$= -0{\cdot}1737 + 1{\cdot}0720 + 2{\cdot}3353 - 0{\cdot}1554 = 3{\cdot}078 \qquad \textbf{2}$$

B3. $\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0$ **1**

Maximum rounding error = $\varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon$. **1**

$\Delta^2 f_0 = 2{\cdot}618 - 2\times 2{\cdot}369 + 2{\cdot}124 = 0{\cdot}004$ **1**
and $4\varepsilon = 4 \times 0{\cdot}0005 = 0{\cdot}002$. **1**

$\Delta^2 f_0$ appears to be significantly different from 0. **1**

B4. (a) Difference table is:

i	x	$f(x)$	diff1	diff2	diff3
0	0	1·023	352	–95	3
1	0·5	1·375	257	–92	–4
2	1	1·632	165	–96	
3	1·5	1·797	69		
4	2	1·866			

2

(b) $p = 0{\cdot}3$

$$f(0{\cdot}65) = 1{\cdot}375 + 0{\cdot}3\,(0{\cdot}257) + \frac{(0{\cdot}3)\,(-0{\cdot}7)}{2}\,(-0{\cdot}092)$$ **2**

$$= 1{\cdot}375 + 0{\cdot}077 + 0{\cdot}010 = 1{\cdot}462$$ **1**

(or, with $p = 1{\cdot}3$, $1{\cdot}023 + 0{\cdot}458 - 0{\cdot}019$).

B5. $f(x) = (((x - 1{\cdot}1)x + 1{\cdot}7)x)x - 3{\cdot}2$ and $f(1{\cdot}3) = 0{\cdot}1124$. **1,1**

Since f is positive and increasing at $x = 1{\cdot}3$, root appears to occur for $x < 1{\cdot}3$. **1**

$f(x)_{\text{min}} = (((x - 1{\cdot}15)x + 1{\cdot}65)x)x - 3{\cdot}25$

$f(1{\cdot}3)_{\text{min}} = -0{\cdot}132$ (opposite sign), so root may occur for $x > 1{\cdot}3$. **1,1**

B6. In diagonally dominant form,

$$\begin{aligned} 4x_1 - 0.3x_2 + 0.5x_3 &= 6.1 \\ 0.5x_1 - 7x_2 + 0.7x_3 &= 3.7 \\ 0.3x_1 \qquad\quad + 2x_3 &= 8.6. \end{aligned}$$ **1**

The diagonal coefficients of x are large relative to the others, so system is likely to be stable. (Or, this implies equations are highly linearly independent, or, determinant of system is large.) **1**

Rewritten equations are:

$$x_1 = (6.1 + 0.3x_2 - 0.5x_3)/4$$
$$x_2 = (-3.7 + 0.5x_2 + 0.7x_3)/7$$
$$x_3 = (8.6 - 0.3x_1)/2$$

Gauss Seidel table is:

x_1	x_2	x_3
0	0	0
1·525	–0·420	4·071
0·985	–0·051	4·152
1·002	–0·042	4·150
1·003	–0·042	

Hence (2 decimal places) $x_1 = 1{\cdot}00$; $x_2 = -0{\cdot}04$; $x_3 = 4{\cdot}15$. **4**

B7. Tableau is: $\begin{pmatrix} 2{\cdot}6 & 0 & 1{\cdot}622 & 0{\cdot}742 & 0{\cdot}479 & 0 \\ 0 & 6{\cdot}469 & 1{\cdot}923 & -0{\cdot}538 & 1 & 0 \\ 0 & 0 & 3{\cdot}604 & -0{\cdot}415 & 0{\cdot}128 & 1 \end{pmatrix}$

$$\sim \begin{pmatrix} 2{\cdot}6 & 0 & 0 & 0{\cdot}929 & 0{\cdot}421 & -0{\cdot}450 \\ 0 & 6{\cdot}469 & 0 & -0{\cdot}317 & 0{\cdot}932 & -0{\cdot}534 \\ 0 & 0 & 3{\cdot}604 & -0{\cdot}415 & 0{\cdot}128 & 1 \end{pmatrix} \quad \begin{matrix} (R_1 - 1{\cdot}622R_3/3{\cdot}604) \\ (R_2 - 1{\cdot}923R_3/3{\cdot}604) \end{matrix}$$ **3**

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0{\cdot}357 & 0{\cdot}162 & -0{\cdot}173 \\ 0 & 1 & 0 & -0{\cdot}049 & 0{\cdot}144 & -0{\cdot}083 \\ 0 & 0 & 1 & -0{\cdot}115 & 0{\cdot}036 & 0{\cdot}277 \end{pmatrix}$$ (dividing by diagonal elements) **1**

Hence $\mathbf{A}^{-1} = \begin{pmatrix} 0{\cdot}36 & 0{\cdot}16 & -0{\cdot}17 \\ -0{\cdot}05 & 0{\cdot}14 & -0{\cdot}08 \\ -0{\cdot}12 & 0{\cdot}04 & 0{\cdot}28 \end{pmatrix}$. **1**

accuracy **1**

B8. (a)

x	y	$f(x, y)$	$hf(x, y)$	
1	1	0·414	0·041	
1·1	1·041	0·514	0·051	**2**
1·2	1·092	0·620	0·062	
1·3	1·154			**1**

Global truncation error is first order. **1**

(b) Predictor-corrector calculation (with one corrector application) is:

x	y	$y' = \sqrt{x^2 + 2y - 1} - 1$	y_P	y_P'	$\frac{1}{2}h(y' + y_P')$	
1	1	0·4142	1·0414	0·5142	0·0464	
1·1	1·0464					**4**

The difference in the (rounded) second decimal place between the values of $x(1{\cdot}1)$ in the two calculations suggests that the second decimal place cannot be relied upon in the first calculation. **1**

B9. Trapezium rule calculation is:

x	$f(x)$	m	$mf_1(x)$	$mf_2(x)$
1	1·2690	1	1·2690	1·2690
1·25	1·1803	2		2·3606
1·5	0·9867	2	1·9734	1·9734
1·75	0·6839	2		1·3678
2	0·2749	1	0·2749	0·2749
			3·5173	7·2457

2

Hence $I_1 = 3{\cdot}5173 \times 0{\cdot}5/2 = 0{\cdot}8793$ and $I_2 = 7{\cdot}2457 \times 0{\cdot}25/2 = 0{\cdot}9057$. **1**

Difference table is:

−887	−1049
−1936	−1092
−3028	−1062
−4090	

2

$|\text{max truncation error}| = 1 \times 0{\cdot}1092/12 \approx 0{\cdot}009$ **1**

Hence $I_2 = 0{\cdot}91$ or $0{\cdot}9$. **1**

Expect to reduce error by factor 4. **1**

With n strips and step size $2h$, Taylor series for expansion of an integral I approximated by the trapezium rule is:

$I = I_n + C(2h)^2 + D(2h)^4 + \ldots = I_n + 4Ch^2 + 16Dh^4 + \ldots$ (a)

With $2n$ strips and step size h, we have: $I = I_{2n} + Ch^2 + Dh^4 + \ldots$ (b) **2**

4(b) − (a) gives $3I = 4I_{2n} - I_n - 12Dh^4 + \ldots$

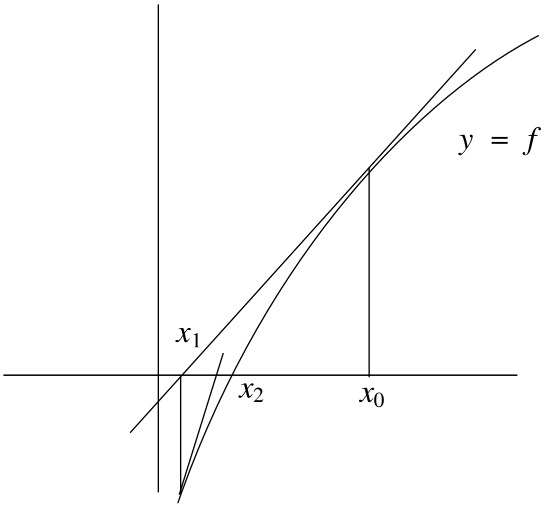
i.e. $I \approx (4I_{2n} - I_n)/3 = I_{2n} + (I_{2n} - I_n)/3$ **1**

$I_R = (4 \times 0{\cdot}9057 - 0{\cdot}8793)/3 = 0{\cdot}914$ **1**

B10. Gradient of $y = f(x)$ at x_0 is $f'(x_0) = \dfrac{f(x_0)}{x_0 - x_1}$. **1**

Hence $x_1 - x_0 = -\dfrac{f(x_0)}{f'(x_0)}$, i.e. $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

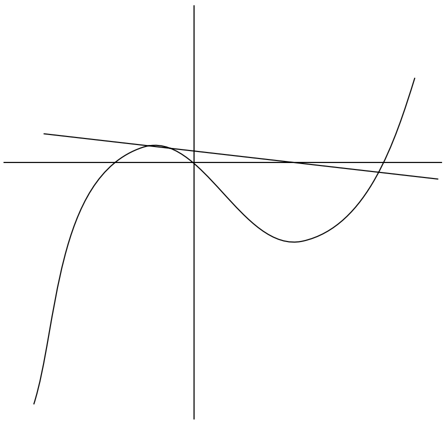
Likewise $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ and in general $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. **1**



1

$f(x) = e^{-x} + x^4 - 2x^3 - 5x^2 - 1$ and $f'(x) = -e^{-x} + 4x^3 - 6x^2 - 10x$; $x_0 = 3{\cdot}5$ **1**

Root is 3·47 (2 decimal places). **1**



In a situation such as diagrammed, the Newton-Raphson method depends for convergence on the point of intersection of tangent with x-axis being closer to the root than the initial point. In the interval $[-0{\cdot}3,\ 0]$ there must be a TV of $f(x)$ so that $f'(x) = 0$ and the point of intersection may be far from initial point; so iteration may lead to a different root. **2**

For bisection, $f(-1{\cdot}1) = 0{\cdot}080$; $f(-1) = -0{\cdot}281$

$f(-1{\cdot}05) = -0{\cdot}124$

$f(-1{\cdot}075) = -0{\cdot}208$; **1**

$f(-1{\cdot}0875) = 0{\cdot}024$ **1**

Hence root lies in $[-1{\cdot}0875.\ -\ 1{\cdot}075]$. **1**

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Advanced Higher – Section C

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Solutions for Section C (Mechanics 1 and 2)

C1.

$$\mathbf{r}(t) = (2t^2 - t)\mathbf{i} - (3t + 1)\mathbf{j}$$

$$\Rightarrow \mathbf{v}(t) = (4t - 1)\mathbf{i} - 3\mathbf{j} \qquad \mathbf{1}$$

$$\Rightarrow |\mathbf{v}(t)| = \sqrt{(4t - 1)^2 + 9} \qquad \mathbf{1}$$

When the speed is 5,

$$(4t - 1)^2 + 9 = 25 \qquad \mathbf{1}$$

$$(4t - 1)^2 = 16$$

$$4t - 1 = \pm 4$$

$$t = \frac{5}{4} \text{ seconds (as } t > 0). \qquad \mathbf{1}$$

C2. (a)

$$\mathbf{v}_F = 25\sqrt{2}(\cos 45°\mathbf{i} + \sin 45°\mathbf{j}) \qquad \mathbf{1}$$

$$= 25(\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}_F = 25t(\mathbf{i} + \mathbf{j}) \quad \text{as } \mathbf{r}_F(0) = \mathbf{0} \qquad \mathbf{1}$$

$$\mathbf{v}_L = 20\mathbf{j}$$

$$\mathbf{r}_L = 20t\mathbf{j} + \mathbf{c}$$

But $\mathbf{r}_L(0) = 10\mathbf{i}$ so $\mathbf{r}_L = 10\mathbf{i} + 20t\mathbf{j}$ **1**

The position of the ferry relative to the freighter is

$$\mathbf{r}_F - \mathbf{r}_L = (25t - 10)\mathbf{i} + 5t\mathbf{j} \qquad \mathbf{1}$$

(b) When $t = 1$

$$|\mathbf{r}_F - \mathbf{r}_L| = \sqrt{15^2 + 5^2} \qquad \mathbf{1}$$

$$= \sqrt{250} = 5\sqrt{10} \text{ km} \qquad \mathbf{1}$$

C3. (a) Using $T = \frac{2\pi}{\omega} \Rightarrow 8\pi = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{1}{4}$. **1**

Maximum acceleration = $\omega^2 a$ **1**

$$\tfrac{1}{4} = \tfrac{1}{16}a \Rightarrow a = 4 \qquad \mathbf{1}$$

(b) Maximum speed = $\omega a = \frac{1}{4} \times 4 = 1$. **1**

Using

$$v^2 = \omega^2(a^2 - x^2) \qquad \mathbf{1}$$

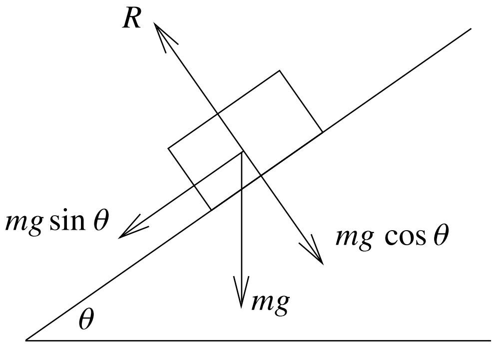
$$\left(\frac{1}{2}\right)^2 = \frac{1}{16}(16 - x^2) \qquad \mathbf{1}$$

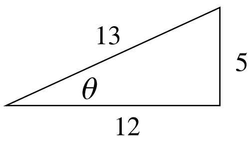
$$4 = 16 - x^2$$

$$x^2 = 12$$

$$x = \pm 2\sqrt{3} \text{ m} \qquad \mathbf{1}$$

C4.





Resolving perp. to plane: $R = mg\cos\theta$
Parallel to the plane (by Newton II)

$$ma = -\mu R - mg\sin\theta$$

$$= -\mu mg\cos\theta - mg\sin\theta \qquad \textbf{2E1}$$

$$a = -g(\mu\cos\theta + \sin\theta)$$

$$= \frac{-(5 + 12\mu)g}{13} \qquad \textbf{2E1}$$

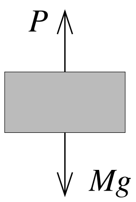
Using $v^2 = u^2 + 2as$

$$0 = gL - \frac{2(5 + 12\mu)gL}{13} \qquad \textbf{1}$$

$$gL = \frac{2(5 + 12\mu)gL}{13}$$

$$10 + 24\mu = 13 \Rightarrow \mu = \frac{1}{8} \qquad \textbf{2E1}$$

C5.



$$P = Mg \qquad \textbf{1}$$

Combined mass = $M + 0{\cdot}01M = 1.01M$.
By Newton II

$$1{\cdot}01Ma = (P + 0{\cdot}05P) - 1{\cdot}01Mg \qquad \textbf{1M,1}$$

$$1.01Ma = 1.05Mg - 1.01Mg$$

$$1.01a = 0.04g \qquad \textbf{1}$$

$$a = \frac{4}{101}g\,(\approx\ 0.3)\ \text{m s}^{-2} \qquad \textbf{1}$$

C6. (i) By conservation of energy, the speed of block A (v_A) immediately before the collision is given by

$$v_A = \sqrt{2gh}. \qquad \textbf{1}$$

By conservation of momentum, the speed of the composite block (v_C) after the collision is given by

$$2mv_C = mv_A$$
$$v_C = \tfrac{1}{2}\sqrt{2gh} \qquad \textbf{1M,1}$$

(ii) By the work/energy principle

Work done against friction = Loss of KE + Change in PE **1**

$$F \times h = \tfrac{1}{2}(2m).\tfrac{1}{4}.2gh + 2mg \times \tfrac{1}{2}h \qquad \textbf{1,1}$$
$$F = \frac{mg}{2} + mg$$
$$F = \frac{3}{2}W \text{ since } W = mg. \qquad \textbf{1}$$

C7. (a) The equations of motion give

$$\ddot{y} = -g \qquad \mathbf{v}(0) = V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{j}$$
$$\dot{y} = -gt + V\sin\alpha$$
$$y = V\sin\alpha t - \tfrac{1}{2}gt^2 \qquad \textbf{1}$$

Maximum height when $\dot{y} = 0 \Rightarrow t = \dfrac{V}{g}\sin\alpha$, and so **1**

$$H = V\sin\alpha \times \frac{V}{g}\sin\alpha - \frac{1}{2}g\frac{V^2}{g^2}\sin^2\alpha \qquad \textbf{1}$$
$$= \frac{V^2}{2g}\sin^2\alpha$$

(b) (i)

$$h = \frac{V^2}{2g}\sin^2 2\alpha$$
$$= \frac{V^2}{2g}4\sin^2\alpha\cos^2\alpha \qquad \textbf{1}$$
$$= \frac{2V^2}{g}\sin^2\alpha\left(1 - \sin^2\alpha\right) \qquad \textbf{1}$$
$$= 4H\left(1 - \frac{2gH}{V^2}\right) \quad \text{since} \quad \sin^2\alpha = \frac{2gH}{V^2} \qquad \textbf{1}$$

(ii) Since $h = 3H$

$$3H = 4H\left(1 - \sin^2\alpha\right) \qquad \textbf{1}$$
$$\frac{3}{4} = 1 - \sin^2\alpha$$
$$\sin^2\alpha = \frac{1}{4} \qquad \textbf{1}$$
$$\sin\alpha = \pm\frac{1}{2} \qquad \textbf{1}$$
$$\Rightarrow \alpha = \frac{\pi}{6} \text{ and so } 2\alpha = \frac{\pi}{3} \qquad \textbf{1}$$

C8. (a) Radius of horizontal circle $r = L\sin 60° = \frac{\sqrt{3}}{2}L$. **1**

$$AB = \frac{r}{\sin 30°} = 2 \times \frac{\sqrt{3}}{2}L = \sqrt{3}L$$

Extension of AB, $x = (\sqrt{3} - 1)L$ **1**

Tension in AB, $T_1 = \frac{\lambda x}{L}$ **1**

$= 2(\sqrt{3} - 1)mg$. **1**

(b) Resolving vertically (where T_2 is the tension in BC)

$$T_1 \cos 30° = mg + T_2 \cos 60°$$ **1**

$$\frac{\sqrt{3}}{2} \times 2(\sqrt{3} - 1)mg = mg + \frac{1}{2}T_2$$ **1**

$$T_2 = (6 - 2\sqrt{3} - 2)mg$$

$$= 2(2 - \sqrt{3})mg$$ **1**

(c) Resolving horizontally (using $L = 1$)

$$T_1 \sin 30° + T_2 \sin 60° = m\left(\frac{\sqrt{3}}{2}\right)\omega^2$$ **1**

$$\frac{1}{2} \times 2(\sqrt{3} - 1)mg + \frac{\sqrt{3}}{2} \times 2(2 - \sqrt{3})mg = m\left(\frac{\sqrt{3}}{2}\right)\omega^2$$ **1**

$$(2\sqrt{3} - 2 + 4\sqrt{3} - 6)g = \sqrt{3}\omega^2$$

$$(6\sqrt{3} - 8)g = \sqrt{3}\,\omega^2$$ **1**

$$\omega^2 = \frac{2(3\sqrt{3} - 4)g}{\sqrt{3}}$$

$$\omega = \sqrt{\frac{2(3\sqrt{3} - 4)g}{\sqrt{3}}}$$ **1**

C9. (i)

$$m\frac{dv}{dt} = -mkv^3 \qquad \mathbf{1}$$

$$v\frac{dv}{dx} = -kv^3 \qquad \mathbf{1}$$

$$\frac{dv}{dx} = -kv^2$$

Separating the variables and integrating gives

$$\int v^{-2}dv = \int -k\,dx \qquad \mathbf{1}$$

$$\Rightarrow -v^{-1} = -kx + c \qquad \mathbf{1}$$

At $x = 0, v = U$

$$-U^{-1} = c$$

so

$$v^{-1} = kx + U^{-1} \qquad \mathbf{1}$$

$$v = \frac{U}{1 + kUx}.$$

(ii) Now $v = \frac{dx}{dt}$, so

$$\frac{dx}{dt} = \frac{U}{1 + kUx} \qquad \mathbf{1}$$

$$\int (1 + kUx)\,dx = \int U\,dt \qquad \mathbf{1}$$

$$x + \tfrac{1}{2}kUx^2 = Ut + c_1$$

Since $x = 0$ when $t = 0$, then $c_1 = 0$ **1**

$$kUx^2 + 2x = 2Ut$$

(iii)

$$V = \tfrac{1}{2}U \Rightarrow \tfrac{1}{2}U(1 + kUx) = U$$

$$\Rightarrow 1 + kUx = 2 \Rightarrow x = \frac{1}{kU} \qquad \mathbf{1M,1}$$

The time taken

$$2Ut = kU\frac{1}{k^2U^2} + \frac{2}{kU} = \frac{3}{kU}$$

$$\Rightarrow t = \frac{3}{2kU^2} \qquad \mathbf{1}$$

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Advanced Higher – Section D

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Solutions for Section D (Mathematics 1)

D1.

$$(4x-5y)^4 = (4x)^4 - 4\times(4x)^3(5y) + 6\times(4x)^2(5y)^2 - 4\times(4x)(5y)^3 + (5y)^4 \qquad \textbf{3E1}$$

$$= 256x^4 - 1280x^3y + 2400x^2y^2 - 2000xy^3 + 625y^4. \qquad \textbf{1}$$

When $y = \frac{1}{x}$, the term independent of x is 2400. **1**

D2.

$$y = x^2 \ln x$$

$$\frac{dy}{dx} = 2x\ln x + x^2.\frac{1}{x} = 2x\ln x + x \qquad \textbf{2E1}$$

$$\frac{d^2y}{dx^2} = 2\ln x + 2x.\frac{1}{x} + 1 = 2\ln x + 3 \qquad \textbf{2E1}$$

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2x\ln x + 3x - 2x\ln x - x = 2x \qquad \textbf{1}$$

Thus $k = 2$. **1**

D3. (a)

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & -6 \\ 3 & -1 & 1 & 7 \\ 2 & 1 & -\lambda & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & -6 \\ 0 & -4 & 7 & 25 \\ 0 & -1 & 4-\lambda & 10 \end{array}\right) \quad \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \qquad \textbf{1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & -6 \\ 0 & -4 & 7 & 25 \\ 0 & 0 & 9-4\lambda & 15 \end{array}\right) \quad \begin{array}{l} \\ \\ 4R_3 - R_2 \end{array} \qquad \textbf{1}$$

There is no solution when $\lambda = \frac{9}{4}$. **1**

(b) When $\lambda = 1$,

$$5c = 15 \Rightarrow c = 3$$

$$-4b + 21 = 25 \Rightarrow b = -1$$

$$a - 1 - 6 = -6 \Rightarrow a = 1 \qquad \textbf{2E1}$$

i.e. $a = 1, b = -1, c = 3$

D4. $x + 1 = u \Rightarrow dx = du$ **1 for differentials**

$x = u - 1 \Rightarrow x^2 + 2 = u^2 - 2u + 3.$

$$\int \frac{x^2+2}{(x+1)^2}\,dx = \int \frac{u^2-2u+3}{u^2}\,du$$ **1 for substitution**

$$= \int 1 - \frac{2}{u} + 3u^{-2}\,du$$ **1 for simplifying**

$$= u - 2\ln|u| - 3u^{-1} + c$$ **1**

$$= x - 2\ln|x+1| - \frac{3}{x+1} + c$$ **1**

D5. (a)

$$\frac{(x-1)(x-4)}{x^2+4} = A + \frac{Bx+C}{x^2+4}$$

$$x^2 - 5x + 4 = Ax^2 + 4A + Bx + C$$ **M1**

$$A = 1,\ B = -5,\ C = 0$$ **2E1**

i.e. $$f(x) = 1 - \frac{5x}{x^2+4}.$$

(b) As $x \to \pm\infty$, $y \to 1$. **1**

[No vertical asymptotes since $x^2 + 4 \neq 0$.]

(c)

$$f(x) = 1 - \frac{5x}{x^2+4}$$

$$f'(x) = -\frac{5(x^2+4) - 10x^2}{(x^2+4)^2} = 0 \text{ at S.V.}$$ **1**

$$\Rightarrow 20 - 5x^2 = 0 \Rightarrow x = \pm 2$$ **1**

$$\Rightarrow (2, -\tfrac{1}{4}) \text{ and } (-2,\ 2\tfrac{1}{4})$$ **1**

(d) $y = 0 \Rightarrow x = 1$ or $x = 4$. **1**

Area $= -\int_1^4 \left(1 - \frac{5x}{x^2+4}\right) dx$ **1**

$$= -\left[x - \tfrac{5}{2}\ln(x^2+4)\right]_1^4$$ **1**

$$= -\left[4 - \tfrac{5}{2}\ln 20\right] + \left[1 - \tfrac{5}{2}\ln 5\right]$$ **1**

$$= \frac{5}{2}\ln 4 - 3 = 5\ln 2 - 3 \text{ (acceptable but not required)}$$

$$\approx 0{\cdot}47 \text{ (acceptable but not required)}$$

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Advanced Higher – Section E

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Solutions for Section E (Statistics 1)

E1. (a) Stratified **1**
and Quota [or Quota (convenience)] **1**
(b) Approach (a) should be best **1**
since (b) is not random (other forms e.g. Glasgow not typical, biased) **1**

E2. (a) $F \sim Bin(192, 0.002)$. **1 for distribution**
1 for parameters
(b) $P(F \geqslant 3) = 1 - P(F \leqslant 2)$ **1**
$= 1 - (0.6809 + 0.2620 + 0.0501)$ **1**
$= 0.0070$ **1**
Notes: applying a Poisson distribution loses (at least) one mark; a Normal distribution loses two marks.
(c) Approximate using the $Poi(0.384)$ **1 for distribution**
1 for parameters

E3. Assume that yields are normally distributed. **1**
Assume that the standard deviation is unchanged. **1**
$\bar{x} = 404.2$.
A 95% confidence interval for the mean yield, μ, is given by:–
$\bar{x} \pm 1.96 \dfrac{\sigma}{\sqrt{n}}$ **1**
$404.2 \pm 1.96 \dfrac{10.}{\sqrt{5}}$
404.2 ± 8.8 **1**
or (395.4, 413.0).
The fact that the confidence interval does not include 382 **1**
provides evidence, at the 5% level of significance, of a **1**
change in the mean yield.

E4. TNE = 3% of 500 = 15 **1**
With maximum allowable standard deviation
$P(\text{weight} < 485) = 0.025$ **1**
$\Rightarrow \dfrac{485 - 505}{\sigma} = -1.96$ **1,1**
$\Rightarrow \sigma = \dfrac{20}{1.96} = 10.2$ **1**

There will be a small probability of obtaining a content weight less than 470g with the normal model. **1**

E5. (a) P (Alaskan fish classified as Canadian)

$= \mathrm{P}(X > 120 \mid X \sim \mathrm{N}(100, 20^2))$ **1**

$= \mathrm{P}\left(Z > \dfrac{120 - 100}{20}\right)$

$= \mathrm{P}(Z > 1)$ **1**

$= 0.1587$ **1**

(b) The probability is the same as in (a) because of symmetry. **1**

(c) P (Canadian origin | Alaskan predicted)

$= \dfrac{\text{P (Alaskan predicted and Canadian origin)}}{\text{P (Alaskan predicted)}}$ **1**

$= \dfrac{\text{P (Alaskan predicted but Canadian origin)}}{\text{P (Ala pred and Alaskan)} + \text{P (Ala pred but Canadian)}}$ **1**

$= \dfrac{0.4 \times 0.1587}{0.6 \times 0.8413 + 0.4 \times 0.1587}$ **1,1**

$= \dfrac{0.06348}{0.50478 + 0.06348}$

$= 0.112.$ **1**

Note: Alternative methods acceptable e.g. Venn or Tree Diagrams

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Advanced Higher – Section F

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Solutions for Section F (Numerical Analysis 1)

F1. $f(x) = \ln(2-x) \qquad f'(x) = \dfrac{-1}{(2-x)} \qquad f''(x) = \dfrac{-1}{(2-x)^2} \qquad f'''(x) = \dfrac{-2}{(2-x)^3}$ **1**

Taylor polynomial is

$$p(1+h) = \ln 1 - h - \frac{h^2}{2} - \frac{2h^3}{6}$$

$$= -h - \frac{h^2}{2} - \frac{h^3}{3} \qquad \mathbf{1}$$

For ln 1·1, $h = -0{\cdot}1$ and $p(0{\cdot}9) = 0{\cdot}1 - 0{\cdot}005 + 0{\cdot}00033 = 0{\cdot}0953.$ **1,1**

$$p(a+h) = \ln(2-a) - \frac{1}{2-a}h \qquad \mathbf{1}$$

Hence expect $f(x)$ to be more sensitive in I_2 since coefficient of h is much larger. **1**

F2. $L(2{\cdot}5)$

$$= \frac{(2{\cdot}5-1{\cdot}5)(2{\cdot}5-3{\cdot}0)(2{\cdot}5-4{\cdot}5)}{(0{\cdot}5-1{\cdot}5)(0{\cdot}5-3{\cdot}0)(0{\cdot}5-4{\cdot}5)}1{\cdot}737 + \frac{(2{\cdot}5-0{\cdot}5)(2{\cdot}5-3{\cdot}0)(2{\cdot}5-4{\cdot}5)}{(1{\cdot}5-0{\cdot}5)(1{\cdot}5-3{\cdot}0)(1{\cdot}5-4{\cdot}5)}2{\cdot}412$$

$$+ \frac{(2{\cdot}5-0{\cdot}5)(2{\cdot}5-1{\cdot}5)(2{\cdot}5-4{\cdot}5)}{(3{\cdot}0-0{\cdot}5)(3{\cdot}0-1{\cdot}5)(3{\cdot}0-4{\cdot}5)}3{\cdot}284 + \frac{(2{\cdot}5-0{\cdot}5)(2{\cdot}5-1{\cdot}5)(2{\cdot}5-3{\cdot}0)}{(4{\cdot}5-0{\cdot}5)(4{\cdot}5-1{\cdot}5)(4{\cdot}5-3{\cdot}0)}2{\cdot}797 \qquad \mathbf{2}$$

$$= -\frac{1\times 1{\cdot}737}{10} + \frac{2\times 2{\cdot}412}{4{\cdot}5} + \frac{4\times 3{\cdot}284}{2{\cdot}5\times 2{\cdot}25} - \frac{1\times 2{\cdot}797}{18}$$

$$= -0{\cdot}1737 + 1{\cdot}0720 + 2{\cdot}3353 - 0{\cdot}1554 = 3{\cdot}078 \qquad \mathbf{2}$$

F3. $$\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0 \qquad \mathbf{1}$$

Maximum rounding error = $\varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon.$ **1**

$\Delta^2 f_0 = 2{\cdot}618 - 2\times 2{\cdot}369 + 2{\cdot}124 = 0{\cdot}004$ **1**
and $4\varepsilon = 4 \times 0{\cdot}0005 = 0{\cdot}002.$ **1**

$\Delta^2 f_0$ appears to be significantly different from 0. **1**

F4. (a) Difference table is:

i	x	$f(x)$	diff1	diff2	diff3
0	0	1·023	352	−95	3
1	0·5	1·375	257	−92	−4
2	1	1·632	165	−96	
3	1·5	1·797	69		
4	2	1·866			

2

(b) $p = 0{\cdot}3$

$$f(0{\cdot}65) = 1{\cdot}375 + 0{\cdot}3(0{\cdot}257) + \frac{(0{\cdot}3)(-0{\cdot}7)}{2}(-0{\cdot}092)$$ **2**

$$= 1{\cdot}375 + 0{\cdot}077 + 0{\cdot}010 = 1{\cdot}462$$ **1**

(or, with $p = 1{\cdot}3$, $1{\cdot}023 + 0{\cdot}458 - 0{\cdot}019$).

F5. Trapezium rule calculation is:

x	$f(x)$	m	$mf_1(x)$	$mf_2(x)$
1	1·2690	1	1·2690	1·2690
1·25	1·1803	2		2·3606
1·5	0·9867	2	1·9734	1·9734
1·75	0·6839	2		1·3678
2	0·2749	1	0·2749	0·2749
			3·5173	7·2457

2

Hence $I_1 = 3{\cdot}5173 \times 0{\cdot}5/2 = 0{\cdot}8793$ and $I_2 = 7{\cdot}2457 \times 0{\cdot}25/2 = 0{\cdot}9057$. **1**

Difference table is:

−887	−1049
−1936	−1092
−3028	−1062
−4090	

2

$|\text{max truncation error}| = 1 \times 0{\cdot}1092/12 \approx 0{\cdot}009$ **1**

Hence $I_2 = 0{\cdot}91$ or $0{\cdot}9$. **1**

Expect to reduce error by factor 4. **1**

With n strips and step size $2h$, Taylor series for expansion of an integral I approximated by the trapezium rule is:

$I = I_n + C(2h)^2 + D(2h)^4 + \ldots = I_n + 4Ch^2 + 16Dh^4 + \ldots$ (a)

With $2n$ strips and step size h, we have: $I = I_{2n} + Ch^2 + Dh^4 + \ldots$ (b) **2**

4(b) – (a) gives $3I = 4I_{2n} - I_n - 12Dh^4 + \ldots$

i.e. $I \approx (4I_{2n} - I_n)/3 = I_{2n} + (I_{2n} - I_n)/3$ **1**

$I_R = (4 \times 0{\cdot}9057 - 0{\cdot}8793)/3 = 0{\cdot}914$ **1**

SCOTTISH
QUALIFICATIONS
AUTHORITY

2004 Applied Mathematics

Advanced Higher – Section G

Finalised Marking Instructions

Advanced Higher Applied Mathematics 2004
Solutions for Section G (Mechanics 1)

G1.

$$\mathbf{r}(t) = \left(2t^2 - t\right)\mathbf{i} - (3t + 1)\mathbf{j}$$

$$\Rightarrow \mathbf{v}(t) = (4t - 1)\mathbf{i} - 3\mathbf{j} \qquad \mathbf{1}$$

$$\Rightarrow |\mathbf{v}(t)| = \sqrt{(4t - 1)^2 + 9} \qquad \mathbf{1}$$

When the speed is 5,

$$(4t - 1)^2 + 9 = 25 \qquad \mathbf{1}$$

$$(4t - 1)^2 = 16$$

$$4t - 1 = \pm 4$$

$$t = \frac{5}{4} \text{ seconds } (\text{as } t > 0). \qquad \mathbf{1}$$

G2. (a)

$$\mathbf{v}_F = 25\sqrt{2}\left(\cos 45°\mathbf{i} + \sin 45°\mathbf{j}\right) \qquad \mathbf{1}$$

$$= 25(\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}_F = 25t(\mathbf{i} + \mathbf{j}) \qquad \text{as } \mathbf{r}_F(0) = \mathbf{0} \qquad \mathbf{1}$$

$$\mathbf{v}_L = 20\mathbf{j}$$

$$\mathbf{r}_L = 20t\mathbf{j} + \mathbf{c}$$

$$\text{But } \mathbf{r}_L(0) = 10\mathbf{i} \text{ so } \mathbf{r}_L = 10\mathbf{i} + 20t\mathbf{j} \qquad \mathbf{1}$$

The position of the ferry relative to the freighter is

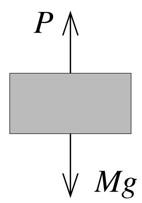
$$\mathbf{r}_F - \mathbf{r}_L = (25t - 10)\mathbf{i} + 5t\mathbf{j} \qquad \mathbf{1}$$

(b) When $t = 1$

$$|\mathbf{r}_F - \mathbf{r}_L| = \sqrt{15^2 + 5^2} \qquad \mathbf{1}$$

$$= \sqrt{250} = 5\sqrt{10} \text{ km} \qquad \mathbf{1}$$

G3.



$$P = Mg \qquad \mathbf{1}$$

Combined mass = $M + 0{\cdot}01M = 1.01M$.
By Newton II

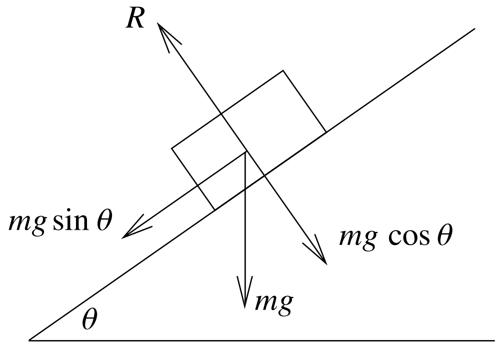
$$1{\cdot}01Ma = (P + 0{\cdot}05P) - 1{\cdot}01Mg \qquad \mathbf{1M,1}$$

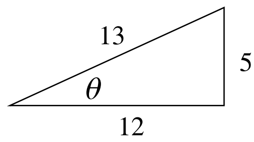
$$1.01Ma = 1.05Mg - 1.01Mg$$

$$1.01a = 0.04g \qquad \mathbf{1}$$

$$a = \frac{4}{101}g\,(\approx 0.3) \text{ m s}^{-2} \qquad \mathbf{1}$$

G4.





Resolving perp. to plane: $R = mg\cos\theta$
Parallel to the plane (by Newton II)

$$ma = -\mu R - mg\sin\theta$$

$$= -\mu mg\cos\theta - mg\sin\theta \qquad \textbf{2E1}$$

$$a = -g(\mu\cos\theta + \sin\theta)$$

$$= \frac{-(5 + 12\mu)g}{13} \qquad \textbf{2E1}$$

Using $v^2 = u^2 + 2as$

$$0 = gL - \frac{2(5 + 12\mu)gL}{13} \qquad \textbf{1}$$

$$gL = \frac{2(5 + 12\mu)gL}{13}$$

$$10 + 24\mu = 13 \Rightarrow \mu = \frac{1}{8} \qquad \textbf{2E1}$$

G5. (a) The equations of motion give

$$\ddot{y} = -g \qquad \mathbf{v}(0) = V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{j}$$
$$\dot{y} = -gt + V\sin\alpha$$
$$y = V\sin\alpha t - \frac{1}{2}gt^2 \qquad \mathbf{1}$$

Maximum height when $\dot{y} = 0 \Rightarrow t = \frac{V}{g}\sin\alpha$, and so **1**

$$H = V\sin\alpha \times \frac{V}{g}\sin\alpha - \frac{1}{2}g\frac{V^2}{g^2}\sin^2\alpha \qquad \mathbf{1}$$
$$= \frac{V^2}{2g}\sin^2\alpha$$

(b) (i)

$$h = \frac{V^2}{2g}\sin^2 2\alpha$$
$$= \frac{V^2}{2g}4\sin^2\alpha\cos^2\alpha \qquad \mathbf{1}$$
$$= \frac{2V^2}{g}\sin^2\alpha\left(1 - \sin^2\alpha\right) \qquad \mathbf{1}$$
$$= 4H\left(1 - \frac{2gH}{V^2}\right) \qquad \text{since} \quad \sin^2\alpha = \frac{2gH}{V^2} \qquad \mathbf{1}$$

(ii) Since $h = 3H$

$$3H = 4H\left(1 - \sin^2\alpha\right) \qquad \mathbf{1}$$
$$\frac{3}{4} = 1 - \sin^2\alpha$$
$$\sin^2\alpha = \frac{1}{4} \qquad \mathbf{1}$$
$$\sin\alpha = \pm\frac{1}{2} \qquad \mathbf{1}$$
$$\Rightarrow \alpha = \frac{\pi}{6} \text{ and so } 2\alpha = \frac{\pi}{3} \qquad \mathbf{1}$$