Advanced Higher Maths 2015

1
$$\left(\frac{x^2}{3}-\frac{2}{x}\right)^5 = \left(\frac{x^2}{3}+\left(-\frac{2}{x}\right)\right)^5$$

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

$$= \left(\frac{x^2}{3}\right)^5 + 5\left(\frac{x^2}{3}\right)^4\left(-\frac{2}{x}\right) + 10\left(\frac{x^2}{3}\right)^3\left(-\frac{2}{x}\right)^2 + 10\left(\frac{x^2}{3}\right)^2\left(-\frac{2}{x}\right)^3 + 5\left(\frac{x^2}{3}\right)\left(-\frac{2}{x}\right)^4 + \left(-\frac{2}{x}\right)^5$$

$$= \frac{x^{10}}{243} - 5\left(\frac{x^8}{81}\right)\left(\frac{2}{x}\right) + 10\left(\frac{x^6}{27}\right)\left(\frac{4}{x^2}\right) - 10\left(\frac{x^4}{9}\right)\left(\frac{8}{x^3}\right) + 5\left(\frac{x^2}{3}\right)\left(\frac{16}{x^4}\right) - \frac{32}{x^5}$$

$$= \underline{\underline{\frac{x^{10}}{243} - \frac{10}{81}x^7 + \frac{40}{27}x^4 - \frac{80}{9}x + \frac{80}{3x^2} - \frac{32}{x^5}}}$$

2 (a)
$$y = \frac{5x+1}{x^2+2} \Rightarrow \frac{dy}{dx} = \frac{5(x^2+2)-2x(5x+1)}{(x^2+2)^2} = \frac{5x^2+10-10x^2-2x}{(x^2+2)^2}$$

$$= \underline{\underline{\frac{-5x^2-2x+10}{(x^2+2)^2}}} \quad \text{OR} \quad \underline{\underline{\frac{10-2x-5x^2}{(x^2+2)^2}}}$$

(b)
$$f(x) = e^{2x}\sin^2 3x = e^{2x}(\sin 3x)^2$$

$$f'(x) = 2e^{2x}(\sin 3x)^2 + 2e^{2x}(\sin 3x)3\cos 3x$$

$$f'(x) = \underline{\underline{2e^{2x}\sin 3x(\sin 3x + 3\cos 3x)}}$$

OR $2e^{2x}\sin^2 3x + 6e^{2x}\sin 3x\cos 3x$

OR $2e^{2x}\sin^2 3x + 3e^{2x}\sin 6x$

OR $e^{2x}(2\sin^2 3x + 3\sin 6x)$

(3) $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_{20} = 320$

$\Rightarrow \frac{20}{2}\left[2a + 19d\right] = 320$

$10(2a + 19d) = 320$

$2a + 19d = 32 \; \ldots\ldots (1)$

$u_n = a + (n-1)d \qquad u_{21} = 37$

$37 = a + 20d \; \ldots (2)$

$\Rightarrow 74 = 2a + 40d \; \ldots (2)\times 2 \ldots (3)$

$32 = 2a + 19d \; \ldots (1)$

$42 = \qquad 21d \quad \Rightarrow d = 2$

Sub. $d = 2$ into (2):

$a + 20(2) = 37 \quad \Rightarrow a + 40 = 37 \quad \Rightarrow a = -3$

$S_{10} = \frac{10}{2}\left[2(-3) + 9(2)\right] = 5(-6+18) = 5 \times 12 = \underline{\underline{60}}$

(4) $x^4 + y^4 + 9x - 6y = 14 \quad$ i.e. $x^4 + [f(x)]^4 + 9x - 6f(x) = 14 \; \ldots\ldots (1)$

Differentiate both sides of (1) w.r.t. x:

$4x^3 + 4[f(x)]^3 f'(x) + 9 - 6f'(x) = 0$

$4x^3 + 4y^3\frac{dy}{dx} + 9 - 6\frac{dy}{dx} = 0$

$4x^3 + 9 = 6\frac{dy}{dx} - 4y^3\frac{dy}{dx}$

$4x^3 + 9 = \frac{dy}{dx}(6 - 4y^3)$

$\frac{dy}{dx} = \frac{4x^3 + 9}{6 - 4y^3}$

At $A(1,2)$, $x = 1$ and $y = 2$: $\frac{dy}{dx} = \frac{4(1)^3 + 9}{6 - 4(2)^3} = \frac{13}{-26} = -\frac{1}{2} \qquad$ i.e. $m = -\frac{1}{2}$

Tangent at $A(1,2)$: $y - b = m(x - a)$

$y - 2 = -\frac{1}{2}(x - 1) \quad$ or $\quad 2y - 4 = -x + 1$

$\underline{\underline{2y + x - 5 = 0}} \quad$ (etc)

(5) $\det A = p\begin{vmatrix} p & 1 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 3 & p \\ 0 & -1 \end{vmatrix}$

$$= p(-p+1) - 2(-3-0) + 0$$

$$= -p^2 + p + 6$$

singular $\Leftrightarrow \det A = 0 \quad \Rightarrow \quad 0 = -p^2 + p + 6$

$$p^2 - p - 6 = 0$$

$$(p+2)(p-3) = 0 \quad \Rightarrow \quad \underline{\underline{p=-2}} \text{ or } \underline{\underline{p=3}}$$

(6) $y = 3^{x^2} \quad \Rightarrow \quad \ln y = \ln 3^{x^2} \quad \Rightarrow \quad \ln y = x^2 \ln 3$

i.e. $\ln y = (\ln 3)x^2$

Diff. w.r.t. x: $\quad \dfrac{1}{y}\dfrac{dy}{dx} = (2\ln 3)x$

$$\frac{dy}{dx} = xy\,2\ln 3 \quad = \underline{\underline{(2\ln 3)\,x\,3^{x^2}}}$$

(7)

$3066 = 4 \times 713 + 214$ --- (1)

$713 = 3 \times 214 + 71$ --- (2)

$214 = 3 \times 71 + 1$ --- (3)

$71 = 71 \times 1 + 0$

From (3): $1 = 214 - 3 \times 71$

$1 = 214 - 3 \times (713 - 3 \times 214)$ [using (2)]

$1 = 214 - 3 \times 713 + 9 \times 214$

$1 = 10 \times 214 - 3 \times 713$

$1 = 10 \times (3066 - 4 \times 713) - 3 \times 713$ [using (1)]

$1 = 10 \times 3066 - 40 \times 713 - 3 \times 713$

$1 = 10 \times 3066 - 43 \times 713$

[Compare with $1 = 3066p + 713q$]

$$\underline{\underline{p = 10}}, \quad \underline{\underline{q = -43}}$$

⑧ $x = \sqrt{t+1} = (t+1)^{1/2}$ $\qquad y = \cot t \qquad (0 < t < \pi)$

$$\frac{dx}{dt} = \frac{1}{2}(t+1)^{-1/2} \qquad \frac{dy}{dt} = -\operatorname{cosec}^2 t$$

$$= \frac{1}{2(t+1)^{1/2}}$$

$$= \frac{1}{2\sqrt{t+1}}$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \quad \text{or} \quad \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\operatorname{cosec}^2 t}{\frac{1}{2\sqrt{t+1}}} = \underline{\underline{-2\sqrt{t+1}\operatorname{cosec}^2 t}}$$

⑨ Prove: $\binom{n+2}{3} - \binom{n}{3} = n^2$ $\qquad$ i.e. prove ${}^{n+2}C_3 - {}^{n}C_3 = n^2$

$\underset{\sim}{\text{LHS}}$ $\dfrac{(n+2)!}{3![(n+2)-3]!} - \dfrac{n!}{3!(n-3)!} = \dfrac{(n+2)!}{6(n-1)!} - \dfrac{n!}{6(n-3)!}$

$$= \frac{(n+2)(n+1)n\cancel{(n-1)!}}{6\cancel{(n-1)!}} - \frac{n(n-1)(n-2)\cancel{(n-3)!}}{6\cancel{(n-3)!}}$$

$$= \frac{n(n+2)(n+1)}{6} - \frac{n(n-1)(n-2)}{6}$$

$$= \frac{n}{6}\left[(n+2)(n+1) - (n-1)(n-2)\right]$$

$$= \frac{n}{6}\left[n^2+3n+2 - (n^2-3n+2)\right]$$

$$= \frac{n}{\cancel{6}}\left[\cancel{6}n\right] \quad = nn$$

$$= n^2 = \text{RHS} \quad \text{(as required)}.$$

(10) $\displaystyle\int_0^2 x^2 e^{4x}\,dx = \left[\frac{1}{4}e^{4x}x^2\right]_0^2 - \int_0^2 \frac{1}{4}e^{4x}\,2x\,dx$

$$= \frac{1}{4}\left[x^2 e^{4x}\right]_0^2 - \frac{1}{2}\int_0^2 x e^{4x}\,dx$$

$$= \frac{1}{4}\left[4e^8 - 0\right] - \frac{1}{2}\left[\left[\frac{1}{4}e^{4x}x\right]_0^2 - \frac{1}{4}\int_0^2 e^{4x}\,dx\right]$$

$$= e^8 - \frac{1}{8}\left[xe^{4x}\right]_0^2 + \frac{1}{8}\int_0^2 e^{4x}\,dx$$

$$= e^8 - \frac{1}{8}\left[2e^8 - 0\right] + \frac{1}{8}\left[\frac{1}{4}e^{4x}\right]_0^2$$

$$= e^8 - \frac{1}{4}e^8 + \frac{1}{32}\left[e^8 - e^0\right]$$

$$= e^8 - \frac{1}{4}e^8 + \frac{1}{32}e^8 - \frac{1}{32}$$

$$= \frac{32}{32}e^8 - \frac{8}{32}e^8 + \frac{1}{32}e^8 - \frac{1}{32} = \frac{25e^8 - 1}{32}$$

(11) $M_1 = \underline{\underline{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}}$ reflects points about the y-axis.

$M_2 = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}$ is associated with an anti-clockwise rotation of $\frac{\pi}{2}$ about the origin.

$M_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}}$ which is a reflection about the line $\underline{\underline{y = x}}$.

[N.B. Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$. First step was M_2X. Next step is M_1M_2X.

i.e. $M_3 = M_1M_2$ in this question, not M_2M_1.]

(12) Let $2n-1$ and $2n+1$ be two consecutive odd numbers $(n \in \mathbb{N})$

$$(2n+1)^2-(2n-1)^2$$
$$= 4n^2+4n+1-(4n^2-4n+1)$$
$$= 4n^2+4n+1-4n^2+4n-1$$
$= 8n$ which is clearly divisible by 8 (since it is 8 times a natural number).

(13) (a) $z^2 = |z|^2 - 4$ where $z = x+iy$

$$(x+iy)^2 = x^2+y^2-4$$

$x^2+2ixy+i^2y^2 = x^2+y^2-4$ But $i^2=-1$, so

$$\cancel{x^2}-y^2+2ixy = \cancel{x^2}+y^2-4$$

$4 = 2y^2-2ixy$ Comparing real and imaginary parts:

Real: $2y^2=4$, $y^2=2$, $y=\pm\sqrt{2}$

Imag: $-2xy=0$, $xy=0$, $\pm\sqrt{2}x=0$, $x=0$

i.e. solutions are $\pm\sqrt{2}i$

(13) (b) $z^2 = i(|z|^2-4)$ where $z=x+iy$

$$(x+iy)^2 = i(x^2+y^2-4)$$

$x^2+2ixy+i^2y^2 = i(x^2+y^2-4)$ But $i^2=-1$.

$x^2-y^2+2ixy = i(x^2+y^2-4)$ Equate real and imaginary parts:

Real: $x^2-y^2=0$ ......(1)

Imag: $x^2+y^2-4=2xy$ ......(2)

From (1): $x^2=y^2 \Rightarrow y=x$ or $y=-x$

But if $y=x$, (2) becomes $x^2+x^2-4=2x^2$ i.e. $2x^2-4=2x^2$ which has no solutions, so $y \neq x$.

If $y=-x$, (2) becomes $x^2+(-x)^2-4=2x(-x)$
$$\Rightarrow 2x^2-4=-2x^2$$
$$\Rightarrow 4x^2-4=0$$
$$\Rightarrow x^2-1=0$$
$$\Rightarrow (x+1)(x-1)=0$$
$$\Rightarrow x=-1 \text{ or } x=1.$$

If $x=-1$, $y=-x=-(-1)=1$. i.e. $z_1=-1+i$

If $x=1$, $y=-x=-1$. i.e. $z_2=1-i$

(14) $g(x) = f(x) + f(-x)$ ------- (1)

$h(x) = f(x) - f(-x)$ ------- (2)

$$g(-x) = f(-x) + f(-(-x)) = f(-x) + f(x) = g(x)$$

i.e. $g(-x) = g(x) \Rightarrow g(x)$ is an even function.

$$h(-x) = f(-x) - f(-(-x)) = f(-x) - f(x)$$
$$= -f(x) + f(-x)$$
$$= -[f(x) - f(-x)]$$
$$= -h(x)$$

i.e. $h(-x) = -h(x) \Rightarrow h(x)$ is an odd function.

(1) + (2): $g(x) + h(x) = 2f(x)$

i.e. $f(x) = \frac{1}{2}[g(x) + h(x)] = \frac{1}{2}g(x) + \frac{1}{2}h(x)$

But $g(x)$ is an even function, so $\frac{1}{2}g(x)$ is also even.
and $h(x)$ is an odd function, so $\frac{1}{2}h(x)$ is also odd.

i.e. $f(x)$ is the sum of an even function and an odd function (as required)

(15) Line L_1 has direction $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and L_2 has direction $\begin{pmatrix}-4\\4\\1\end{pmatrix}$

(a) L_1 passes through $P(2,4,1)$ and therefore has <u>vector</u> equation

$$\underline{r} = 2\underline{i} + 4\underline{j} + \underline{k} + s(\underline{i} + 2\underline{j} - \underline{k}) \quad \text{or} \quad \underline{r} = \begin{pmatrix}2\\4\\1\end{pmatrix} + s\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

and L_2 passes through $Q(-5,2,5)$ and so has <u>vector</u> equation

$$\underline{r} = -5\underline{i} + 2\underline{j} + 5\underline{k} + t(-4\underline{i} + 4\underline{j} + \underline{k}) \quad \text{or} \quad \underline{r} = \begin{pmatrix}-5\\2\\5\end{pmatrix} + t\begin{pmatrix}-4\\4\\1\end{pmatrix}$$

15(b) At the point of intersection (if it exists):

$$\begin{pmatrix}2\\4\\1\end{pmatrix} + s\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}-5\\2\\5\end{pmatrix} + t\begin{pmatrix}-4\\4\\1\end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2+s &= -5-4t \text{ - - - - - (1)}\\ 4+2s &= 2+4t \text{ - - - - - (2)}\\ 1-s &= 5+t \text{ - - - - - (3)} \end{aligned}$$

(1)+(2): $6+3s = -3 \quad \Rightarrow 3s = -9 \quad \Rightarrow s = -3$

Sub. $s=-3$ into (2): $4+2(-3) = 2+4t$

$-2 = 2+4t \quad \Rightarrow 4t = -4 \quad \Rightarrow t = -1$

Sub. $s=-3$ and $t=-1$ into (3):

$1-(-3) = 5+(-1)$

$4 = 4$ i.e. $s=-3$ and $t=-1$ satify (3) too, so

L_1 & L_2 do intersect.

Sub. $s=-3$ into LHS of (1), (2) and (3): $\left.\begin{aligned} x &= 2+(-3) = -1\\ y &= 4+2(-3) = -2\\ z &= 1-(-3) = 4. \end{aligned}\right\}$ $\underline{(-1,-2,4)}$ is the point of intersection.

[or sub. $t=-1$ into RHS of (1), (2) and (3): $\left.\begin{aligned} x &= -5-4(-1) = -1\\ y &= 2+4(-1) = -2\\ z &= 5+(-1) = 4 \end{aligned}\right\}$ $\underline{\underline{(-1,-2,4)}}$]

(c) L_1 and L_2 lie on a plane $\Rightarrow$ normal of plane is a multiple of $\underline{u}_1 \times \underline{u}_2$

$$\begin{array}{cccc} 1 & 2 & -1 & 1\\ -4 & 4 & 1 & -4\\ & 3 & 1 & 2 \end{array} \qquad \begin{pmatrix}2-(-4)\\4-1\\4-(-8)\end{pmatrix} = \begin{pmatrix}6\\3\\12\end{pmatrix} = 3\begin{pmatrix}2\\1\\4\end{pmatrix}$$

Plane is of the form $2x+y+4z=k$ [or $6x+3y+12z=K$]

But $P(2,4,1)$ lies on plane (since L_1 lies on plane) so

$2(2)+4+4(1)=k \quad \Rightarrow k=12$ ie plane is $\underline{\underline{2x+y+4z=12}}$

[or $Q(-5,2,5)$ lies on plane (since L_2 lies on plane) so

$2(-5)+2+4(5)=k \quad \Rightarrow k=12$ i.e. plane is $2x+y+4z=12$

OR $(-1,-2,4)$ lies on plane (since it lies on both L_1 and L_2) so

$2(-1)+(-2)+4(4)=k \quad \Rightarrow k=12$ ie plane is $2x+y+4z=12$]

(16) $y'' + 2y' + 10y = 3e^{2x}$

A.E. $m^2 + 2m + 10 = 0$

$$m = \frac{-2 \pm \sqrt{4-40}}{2}$$

$$m = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

C.F. $y = e^{-x}(A\cos 3x + B\sin 3x)$

P.I. $Y = ke^{2x}$

$Y' = 2ke^{2x}$

$Y'' = 4ke^{2x}$

$4ke^{2x} + 4ke^{2x} + 10ke^{2x} = 3e^{2x}$

$18ke^{2x} = 3e^{2x} \Rightarrow 18k = 3 \Rightarrow k = \frac{3}{18} = \frac{1}{6}$

ie. $Y = \frac{1}{6}e^{2x}$

G.S. $y = e^{-x}(A\cos 3x + B\sin 3x) + \frac{1}{6}e^{2x}$ .... (1)

But $y = 1$ when $x = 0$:

$1 = e^0(A\underbrace{\cos 0}_{=1} + B\underbrace{\sin 0}_{=0}) + \frac{1}{6}e^0$

$1 = A + \frac{1}{6} \Rightarrow A = \frac{5}{6}$

From (1), $\frac{dy}{dx} = -e^{-x}(A\cos 3x + B\sin 3x) + e^{-x}(-3A\sin 3x + 3B\cos 3x) + \frac{1}{3}e^{2x}$

But $\frac{dy}{dx} = 0$ when $x = 0$:

$0 = -e^0(A\cos 0 + B\underbrace{\sin 0}_{=0}) + e^0(-3A\underbrace{\sin 0}_{=0} + 3B\cos 0) + \frac{1}{3}e^0$

$0 = -A + 3B + \frac{1}{3}$ But $A = \frac{5}{6}$

$0 = -\frac{5}{6} + 3B + \frac{2}{6} \Rightarrow 3B - \frac{1}{2} = 0 \Rightarrow 3B = \frac{1}{2} \Rightarrow B = \frac{1}{6}$.

Particular Solution: $\underline{\underline{y = e^{-x}\left(\frac{5}{6}\cos 3x + \frac{1}{6}\sin 3x\right) + \frac{1}{6}e^{2x}}}$

(17) $$\frac{2x^3-x-1}{(x-3)(x^2+1)} = \frac{2x^3-x-1}{x^3+x-3x^2-3} = \frac{2x^3-x-1}{x^3-3x^2+x-3}$$

$$\begin{array}{rrrr|rrrr} & & & & & & & 2 \\ \hline 1 & -3 & 1 & -3 & 2 & 0 & -1 & -1 \\ & & & & 2 & -6 & 2 & -6 \\ \hline & & & & & 6 & -3 & 5 \end{array}$$

i.e. $$\frac{2x^3-x-1}{(x-3)(x^2+1)} = 2 + \frac{6x^2-3x+5}{(x-3)(x^2+1)}$$

[or $$\frac{2x^3-x-1}{x^3-3x^2+x-3} = \frac{2(x^3-3x^2+x-3)+6x^2-3x+5}{(x^3-3x^2+x-3)} = 2+\frac{6x^2-3x+5}{(x-3)(x^2+1)}$$

Let $$\frac{6x^2-3x+5}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{(Bx+C)}{x^2+1}$$ Mult. by $(x-3)(x^2+1)$

$$6x^2-3x+5 = A(x^2+1) + (Bx+C)(x-3)$$

Let $x=3$: $6(3)^2-3(3)+5 = 10A \Rightarrow 10A=50 \Rightarrow A=5$

Let $x=0$: $5 = A+-3C \Rightarrow 5=5-3C \Rightarrow 3C=0 \Rightarrow C=0$.

Let $x=1$: $6(1)^2-3(1)+5 = 2A-2(B+C)$

$8 = 2(5)-2B$ (since $A=5$ and $C=0$).

$2B = 2$

$B=1$

i.e. $$\frac{6x^2-3x+5}{(x-3)(x^2+1)} = \frac{5}{x-3} + \frac{x}{x^2+1}$$

$$\int \frac{2x^3-x-1}{(x-3)(x^2+1)}\,dx = \int\left(2+\frac{5}{x-3}+\frac{x}{x^2+1}\right)dx = \int 2\,dx + 5\int\frac{1}{x-3}\,dx + \frac{1}{2}\int\frac{2x}{x^2+1}\,dx$$

$$= \underline{\underline{2x + 5\ln(x-3) + \tfrac{1}{2}\ln(x^2+1) + C}} \qquad (x>3)$$

or $$\underline{\underline{2x + \ln(x-3)^5 + \ln(x^2+1)^{1/2} + C}}$$

or $$\underline{\underline{2x + \ln\left((x-3)^5\sqrt{x^2+1}\right) + C}}$$

vol. of prism

$\downarrow$

(18) (a) $\frac{dV}{dt} = -k\sqrt{h}$ .....(1), $k > 0$ $\qquad V = Ah. \Rightarrow \frac{dV}{dh} = A$ ....(2)

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$$

$$-k\sqrt{h} = A\frac{dh}{dt} \quad \text{[using (1) and (2)]}$$

$$-\frac{k}{A}\sqrt{h} = \frac{dh}{dt} \quad \text{......(3)}$$

(b) (3): $\frac{dh}{dt} = -\frac{k}{A}h^{1/2}$

$$\int \frac{dh}{h^{1/2}} = -\frac{k}{A}\int dt \quad \Rightarrow \int h^{-1/2}dh = \frac{-k}{A}\int 1.dt$$

$$\Rightarrow \frac{h^{1/2}}{\frac{1}{2}} = -\frac{k}{A}t + C \quad \Rightarrow 2h^{1/2} = -\frac{k}{A}t + C \quad \text{....(4)}$$

But when $t = 0$, $h = 144$ and $\frac{dh}{dt} = -0.3$

(4) becomes: $2\sqrt{144} = -\frac{k}{A}(0) + C \Rightarrow C = 2\times 12 = 24$

Sub. $C = 24$ into (4): $2\sqrt{h} = -\frac{k}{A}t + 24.$

$$2Ah^{1/2} = -kt + 24A \quad \text{....(5)}$$

Diff. (5) w.r.t. t:

$$Ah^{-1/2}\frac{dh}{dt} = -k \qquad \text{But } \frac{dh}{dt} = -0.3 \text{ when } h = 144$$

$$\frac{A}{\sqrt{144}} \times -0.3 = -k$$

$$k = \frac{0.3A}{12} = \frac{3A}{120} = \frac{A}{40}$$

Sub. $k = \frac{A}{40}$ into (5): $2A\sqrt{h} = -\frac{A}{40}t + 24A$ $\qquad$ Divide by A:

$$2\sqrt{h} = -\frac{t}{40} + 24 \Rightarrow \sqrt{h} = 12 - \frac{t}{80} \Rightarrow h = \left(12 - \frac{t}{80}\right)^2 \text{.....(6)}$$

(c) Empty $\Rightarrow h = 0$

$$\left(12 - \frac{1}{80}t\right)^2 = 0$$

$$\Rightarrow 12 = \frac{1}{80}t$$

$$\Rightarrow 960 = t \quad \text{ie } t = 960 \text{ hours} = \frac{960}{24} \text{ days} = \underline{\underline{40 \text{ days}}}.$$

(d) $$\frac{dV}{dt} = -k\sqrt{h} \quad [\text{using (i)}]$$

$$\Rightarrow \frac{dV}{dt} = -k\sqrt{\left(12 - \frac{1}{80}t\right)^2} \quad [\text{using (6)}]$$

$$\Rightarrow \frac{dV}{dt} = -k\left(12 - \frac{1}{80}t\right) \ldots (7) \text{ where } k = \frac{A}{40}.$$

But for a cylinder, the base is a circle, so $A = \pi r^2$

$$A = \pi \times 20^2 = 400\pi$$

$$\Rightarrow k = \frac{A}{40} = \frac{400\pi}{40} = 10\pi$$

so (7) becomes: $$\frac{dV}{dt} = -10\pi\left(12 - \frac{1}{80}t\right)$$

At end of 4^{th} day, $t = 4 \times 24 = 96$ (hours)

$$\frac{dV}{dt} = -10\pi\left(12 - \frac{1}{80} \times 96\right) = -\cancel{10}^{2}\pi\left(\frac{54}{\cancel{5}_1}\right) = \underline{\underline{-108\pi}} \text{ cm}^3/\text{hr}$$