St Andrew's Academy

Department of Mathematics

Advanced Higher

Course Textbook

UNIT 2

Unit 2

N7 Matrices and Systems of Equations GPS 1.1

Matrices

A matrix is a rectangular array of numbers arranged in rows and columns, the array is enclosed by round or square brackets.

e.g. $\begin{pmatrix} x \\ y \end{pmatrix}$ 2 rows, 1 column

$\begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix}$ 2 rows, 2 columns

$\begin{pmatrix} 6 & -8 & 10 \\ 1 & 0 & 6 \end{pmatrix}$ 2 rows, 3 columns

$\begin{pmatrix} 4 & -2 & 3 \end{pmatrix}$ 1 row, 3 columns

Each number in the array is called an entry or an element of the matrix and is identified by first stating the row and then the column in which it appears.

A matrix is usually denoted by a capital letter.

The order of a matrix is given by stating the number of the rows followed by the number of columns.

e.g. $A = \begin{pmatrix} 6 & -8 & 10 \\ 1 & 0 & 6 \end{pmatrix}$

A has 2 rows and 3 columns and is said to be of order 2x3 (read as 2 by 3)

$B = \begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix}$

B has the same number of rows and columns and is called a square matrix of order 2.

In general, a matrix A, with m rows and n columns order (m x n), can be represented as follows, where a_{ij} denotes the row and column of each element.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Matrices are called upper triangular if the values below the diagonal starting in position a_{11} are zero.

e.g. $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$

Systems of Equations

Previously we solved two equations with two unknowns. Let us now consider three equations with three unknowns and two different layouts.

$$x - 2y + 3z = 14$$
$$2x + 3y - 4z = -16$$
$$3x - y - 2z = -1$$

x-2y+3z=14 **1** 2x+3y-4z=-16 **2** 3x-y-2z=-1 **3** Use **1** and **2** to eliminate x. **1**x2 2x-4y+6z=28 2x+3y-4z=-16 Subtract 7y-10z=-44 **2*** Use **1** and **3** to eliminate x. **1**x3 3x-6y+9z=42 3x-y-2z=-1 Subtract 5y-11z=-43 **3*** The system is now x-2y+3z=14 **1** 7y-10z=-44 **2*** 5y-11z=-43 **3*** Use **2*** and **3*** to eliminate y **2***x5 35y-50z=-220 **3***x7 35y-77z=-301 Subtract -27z=-81 z=3 Substitute z=3 into **2*** 7y-10(3)=-44 7y=-14 y=-2 Substitute y=-2 and z=3 into **1** x-2(-2)+3(3)=14 x=1	$\begin{matrix} Row\ 1 \\ Row 2 \\ Row\ 3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 14 \\ 2 & 3 & -4 & -16 \\ 3 & -1 & -2 & -1 \end{array}\right]$ The matrix consisting of the coefficients of x, y and z and the constants on the right hand side is known as the augmented matrix. $\begin{matrix} \\ R2 \to R2 - 2R1 \\ R3 \to R3 - 3R1 \end{matrix}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 14 \\ 0 & 7 & -10 & -44 \\ 0 & 5 & -11 & -43 \end{array}\right]$ $\begin{matrix} \\ \\ R3 \to 7R3 - 5R2 \end{matrix}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 14 \\ 0 & 7 & -10 & -44 \\ 0 & 0 & -27 & -81 \end{array}\right]$ From R3: -27z=-81 z=3 From R2: 7y-10(3)=-44 7y=-14 y=-2 From R1: x-2(-2)+3(3)=14 x=1

Gaussian Elimination

The method shown on the right hand side is known as Gaussian Elimination. There are other methods for solving three equations with three unknowns but for Gaussian Elimination you must reduce the matrix to upper triangular form by producing zeros in the positions shown.
The basic operations which can be used on the augmented matrix are known as elementary row operations (EROs).

EROs: Interchanging rows
Multiplying a row by a constant
Adding two rows

Combinations of EROs create the more complex operations shown in the example above.

Examples: Use Gaussian Elimination to solve the following systems of equations.

a)
$$x+y+2z=3$$
$$4x+2y+z=13$$
$$2x+y-2z=9$$

$$\left[\begin{array}{ccc|c}1&1&2&3\\4&2&1&13\\2&1&-2&9\end{array}\right]$$

$$\begin{array}{l}R2\rightarrow R2-4R1\\R3\rightarrow R3-2R1\end{array}\left[\begin{array}{ccc|c}1&1&2&3\\0&-2&-7&1\\0&-1&-6&3\end{array}\right]$$

$$R3\rightarrow 2R3-R2\left[\begin{array}{ccc|c}1&1&2&3\\0&-2&-7&1\\0&0&-5&5\end{array}\right]$$

From R3: -5z=5
z=-1
From R2: -2y-7(-1)=1
y=3
From R1: x+3+2(-1)=3
x=2

b)
$$x+y+z=2$$
$$4x+2y+z=4$$
$$x-y+z=4$$

$$\left[\begin{array}{ccc|c}1&1&1&2\\4&2&1&4\\1&-1&1&4\end{array}\right]$$

$$\begin{array}{l}R2\rightarrow R2-4R1\\R3\rightarrow R3-R1\end{array}\left[\begin{array}{ccc|c}1&1&1&2\\0&-2&-3&-4\\0&-2&0&2\end{array}\right]*$$

$$R3\rightarrow R3-R2\left[\begin{array}{ccc|c}1&1&1&2\\0&-2&-3&-4\\0&0&3&6\end{array}\right]$$

From R3: 3z=6
z=2
From R2: -2y-3(2)=-4
y=-1
From R1: x+(-1)+2=2
x=1

Note: Examiners will want to see an indication of the EROs.
You must produce the upper triangular matrix
You must show evidence of back substitution

The working above is the expected standard. Nothing can be missed out!

Exercise 1: Use Gaussian Elimination to solve the following systems of equations.

1 a)
$$\begin{aligned} x + y + 2z &= 1 \\ 3x + 3y + z &= 4 \\ 3x + 2y - 2z &= 7 \end{aligned}$$

b)
$$\begin{aligned} x - 2y + z &= 6 \\ 3x + y - 2z &= 4 \\ 7x - 6y - z &= 10 \end{aligned}$$

c)
$$\begin{aligned} 5x - y + 2z &= 25 \\ 3x + 2y - 3z &= 16 \\ 2x - y + z &= 9 \end{aligned}$$

d)
$$\begin{aligned} x + y + z &= 2 \\ 3x - y + 2z &= 4 \\ 2x + 3y + z &= 7 \end{aligned}$$

e)
$$\begin{aligned} 5x - 3y + 6z &= 0 \\ x + 5y + 2z &= 0 \\ -x + 2y + 5z &= 0 \end{aligned}$$

2 A parabola passes through the points (1,2), (2,7) and (3,14).
It has an equation of the form: $y = ax^2 + bx + c$.

a) Use the information to form a 3 x 3 system of equations
b) Solve the system by Gaussian elimination.
c) Write the equation of the parabola.

3 An archaeological dig discovers the remains of a circular Roman amphitheatre. Using a suitable set of axes and convenient units, the archaeologists identify three points on its circumference:
(-2,-1), (-1,2) and (6,3)

a) Assuming the perimeter has an equation of the form:
$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
form a system of equations in g, f, and c.
b) Solve the system and identify the equation of the perimeter.
c) What is the radius of the amphitheatre?

Inconsistency and No Unique Solutions

A set of solutions will not always exist. In some cases, there will not be a set of solutions; here, the equations are **inconsistent**. In some cases, there will be an infinite set of solutions because one of the equations is **redundant** (the third equation will be combination of the other two).

Examples:

1 Show that there will be no solutions to this system of equations

$$x + 2y + z = 6$$
$$2x + y - z = 5$$
$$-2x - y + z = 0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & -1 & 5 \\ -2 & -1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ R2 \rightarrow R2 - 2R1 \\ R3 \rightarrow R3 + 2R1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -3 & -7 \\ 0 & 3 & 3 & 12 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R3 \rightarrow R3 + R2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & -4 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

From R3: 0z=5 i.e. 0=5

So the system of equations is inconsistent and there are no solutions.

2 $2x - y + z = 3$

$$x + 2y - 3z = 1$$
$$4x + 3y - 5z = 5$$

a) Show the system of equations has an infinite set of solutions.

b) By setting λ=1, find the particular solutions.

a)

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 2 & -3 & 1 \\ 4 & 3 & -5 & 5 \end{array}\right]$$

$$\begin{array}{r} \\ R2 \rightarrow 2R2 - R1 \\ R3 \rightarrow R3 - 2R1 \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 5 & -7 & -1 \\ 0 & 5 & -7 & -1 \end{array}\right]$$

$$R3 \to R3 - R2 \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 5 & -7 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The line of zeros tells us that the last equation is redundant and hence there is an infinite set of solutions.

Choose a variable for z, say λ, z= λ.

From R2: 5y-7λ =-1

$$y = \frac{7\lambda - 1}{5}$$

From R1: 2x - $\frac{7\lambda-1}{5}$ + λ =3

$$2x = \frac{15}{5} + \frac{7\lambda - 1}{5} - \frac{5\lambda}{5}$$

$$x = \frac{2\lambda + 14}{10}$$

$$x = \frac{\lambda + 7}{5}$$

This is the General Solution.

b) If $\lambda = 1$: x=$\frac{8}{5}$, y=$\frac{6}{5}$, z=1.

This is the Particular Solution.

3 Find the value of k for which these equations have a set of infinite solutions. State the general solution.

$$x + y - z = k$$
$$3x - 2y + z = 16$$
$$9x - y - z = 5$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & k \\ 3 & -2 & 1 & 16 \\ 9 & -1 & -1 & 5 \end{array}\right]$$

$$\begin{array}{l} R2 \to R2 - 3R1 \\ R3 \to R3 - 9R1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & k \\ 0 & -2 & 1 & 16 - 3k \\ 0 & -10 & 8 & 5 - 9k \end{array}\right]$$

$$R3 \to R3 - 2R1 \left[\begin{array}{ccc|c} 1 & 1 & -1 & k \\ 0 & -5 & 4 & 16 - 3k \\ 0 & 0 & 0 & -27 - 3k \end{array}\right]$$

A set of infinite solutions exist when $-27 - 3k = 0$

$$k = -9$$

Let $z = \lambda$

From R2: $-5y + 4\lambda = 43$

$$y = \frac{1}{5}(4\lambda - 43)$$

From R1: $x + \frac{1}{5}(4\lambda - 43) - \lambda = -9$

$$x = \frac{\lambda - 2}{5}$$

4 For the following equations

$$2x + y + 2z = 1$$
$$3x - z = 3$$
$$4x - y + az = b$$

a) Find the conditions under which the solutions are unique.

b) Consider the case a=-4, b=5

c) Consider the case a=-4, b=2

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 3 & 0 & -1 & 3 \\ 4 & -1 & a & b \end{array}\right]$$

$$\begin{array}{r} R2 \to 2R2 - 3R1 \\ R3 \to R3 - 2R1 \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & -3 & -8 & 3 \\ 0 & -3 & a-8 & b-2 \end{array}\right]$$

$$R3 \to R3 - R1 \left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & -3 & -8 & 3 \\ 0 & 0 & a+4 & b-5 \end{array}\right]$$

a) Solutions are unique when $a + 4 \neq 0$, $a = -4$ and b can take any value.

b) If a=-4, b=5, R3 is a line of zeros so solutions are infinite.

Let $z = \lambda$

From R2: $-3y - 8\lambda = 3$

$$y = -\frac{1}{3}(8\lambda + 3)$$

From R1: $2x - \frac{1}{3}(8\lambda + 3) + 2\lambda = 1$

$$x = \frac{\lambda + 3}{3}$$

c) If a=-4, b=2, R3 gives $0z = -3$

The equations are inconsistent. There are no solutions.

Exercise 2:

1 Show that these systems of equations are inconsistent by reducing the associated matrix to upper triangular form.

a) $2x - y + z = 3$
$x + y - 3z = 1$
$3x + y - 2z = 2$

b) $x + y - z = -1$
$5x + 3y + z = 3$
$2x + y + z = 4$

c) $x + 2y - z = 2$
$3x - y + 2z = -1$
$4x - 6y + 6z = 4$

2 Show that these systems of equations have no unique solution and find the general solution.

a) $-2x + y - 5z = 4$
$3x - y + 2z = -1$
$-4x + y + z = -2$

b) $x + y + 3z = -1$
$2x - y + 4z = 1$
$-x + 5y + z = -5$

c) $x - 2y + z = 3$
$2x + y - 2z = 1$
$3x - y - z = 4$

3 Under what condition does this system of equations have a unique solution? State the solution when this condition holds.

$$x + 2y + 3z = 0$$
$$2x - y + z = 5$$
$$x + 3y + az = -1$$

4 Find the value of a for which these equations have a solution and state the solution for this value if a.

$$x + 3y - z = -3$$
$$x - 2y + z = 4$$
$$x + 4y - 2z = a$$

5
$$2x + y - 3z = 5$$
$$x + 2y + 3z = 1$$
$$2x - y + az = b$$

a) Find the conditions under which the solutions are unique.

b) Find the values of a and b for an infinite number of solutions.

6
$$x + 2y - z = 6$$
$$2x - y - z = 1$$
$$ay + z = b$$

a) Find a and b if this set of equations has more than one solution.

b) Find the general solution by letting z= λ

Matrix Algebra

A matrix (matrices) is a rectangular array of numbers set out in rows and columns, the array is enclosed in round (or square) brackets.

e.g.	$\begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix}$	$\begin{pmatrix} 6 & 8 & 10 \\ -2 & 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 4 & -2 & 7 \end{pmatrix}$
	2 rows 1 column	2 rows 2 columns	2 rows 3 columns	1 row 3 columns

Each number in the array called an **entry** or an **element** of the matrix and is identified by first stating the row and then the column in which it appears.
A matrix is often notated by a capital letter.

The **order** of a matrix is given by stating the number of rows followed by the number of columns.

$$A = \begin{pmatrix} 4 & 7 & 8 \\ -3 & 0 & -4 \end{pmatrix}$$

Matrix A has 2 rows and 3 columns and is said to be of order 2x3 (2 by 3).

$$B = \begin{pmatrix} 3 & -4 \\ 2 & 6 \end{pmatrix}$$

Matrix B has 2 rows and 2 columns and is called a **square matrix** of order 2.

In general, a matrix A, of order $(m \times n)$, has m rows and n columns and can be represented as follows where a_{ij} denotes the element in the ith row and jth column.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Two matrices are **equal** when they are of the same order and their corresponding elements are equal.

The **zero matrix** is a matrix whose elements are all zero.

Finding the Transpose

A new matrix can be formed from a matrix A by writing row 1 as column 1, row 2 as column 2, row 3 as column 3 etc. This new matrix is called the **transpose** of A and is denoted by A' (A dashed or A transpose)

If $A = \begin{pmatrix} 3 & 5 & 9 \\ -2 & 7 & -4 \end{pmatrix}$ then $A' = \begin{pmatrix} 3 & -2 \\ 5 & 7 \\ 9 & -4 \end{pmatrix}$

Exercise 3

1 State the order of each of the following matrices:

a) $\begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -1 \\ 4 & 8 \\ 1 & -2 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$

2 State the pairs of equal matrices

$A = (1 \quad 2 \quad 3)$ $B = (3 \quad 2 \quad 1)$ $C = (1 \quad 2 \quad 3)$

$D = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $E = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $F = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $G = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $I = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$ $J = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $K = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

3 Determine the values of x and y in each of the following:

a) $(3x \quad -y) = (12 \quad 3)$ b) $\begin{pmatrix} x+3 \\ 4-y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

c) $\begin{pmatrix} x+2y \\ 2x-y \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$ d) $\begin{pmatrix} x^2 & y^2 \\ y^3 & x^3 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ -27 & 8 \end{pmatrix}$

4 Write down the transpose of each matrix in question 1 and state the order of the transpose.

5 For the matrix $A = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$, show that $(A')' = A$.

6 $P = \begin{pmatrix} x & 9 \\ -3 & y \end{pmatrix}$ and $Q = \begin{pmatrix} 5 & -3 \\ 9 & -4 \end{pmatrix}$

Find x and y given that $P' = Q$.

Addition of Matrices

If two matrices A and B are of the same order, they can be added to make a new matrix $A+B$ formed by adding each element of A to the corresponding element of B.

In general: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \;\Rightarrow\; A+B = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$

Exercise 4

1 Find the sum of the following matrices

a) $\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ b) $\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ c) $\begin{pmatrix} 2a \\ b \end{pmatrix} + \begin{pmatrix} 7a \\ -3b \end{pmatrix}$

d) $\begin{pmatrix} 2u \\ -3v \end{pmatrix} + \begin{pmatrix} -2u \\ 3v \end{pmatrix}$ e) $\begin{pmatrix} 2 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 4 \end{pmatrix}$ f) $\begin{pmatrix} 2 & -3 \end{pmatrix} + \begin{pmatrix} -5 & 8 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ h) $\begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 3 \\ -7 & 1 & -4 \end{pmatrix}$

2 $A = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix}$

Find

a) A+B b) B+C c) (A+B)+C d) A+(B+C)

Is it true that (A+B)+C= A+(B+C)

3 Given that $A = \begin{pmatrix} 3 & -4 \\ -5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 4 \\ 5 & -1 \end{pmatrix}$, find the matrices

a) A+B b) B+A

Comment on your results.

4 For the matrices $A = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 3 & -2 \\ 3 & 8 & 1 & 0 \end{pmatrix}$

show that $(A+B)' = A' + B'$.

Subtraction of Matrices

If two matrices A and B are of the same order, they can be subtracted to make a new matrix $A - B$ formed by subtracting each element of A to the corresponding element of B.

In general: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \Rightarrow A - B = \begin{pmatrix} a-p & b-q \\ c-r & d-s \end{pmatrix}$

Exercise 5

1 Subtract the following matrices

a) $\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ b) $\begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ c) $\begin{pmatrix} 2a \\ b \end{pmatrix} - \begin{pmatrix} 7a \\ -3b \end{pmatrix}$

d) $\begin{pmatrix} 2u \\ -3v \end{pmatrix} - \begin{pmatrix} -2u \\ 3v \end{pmatrix}$ e) $\begin{pmatrix} 2 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 4 \end{pmatrix}$ f) $\begin{pmatrix} 2 & -3 \end{pmatrix} - \begin{pmatrix} -5 & 8 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ h) $\begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 3 \\ -7 & 1 & -4 \end{pmatrix}$

2 $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 3 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 5 & 2 \\ -1 & 0 \end{pmatrix}$

Find:

a) A + B b) A + C c) A + B + C d) A - B

e) C - B f) C - A g) (A+C) + (A+B) h) (A+C) - (A+B)

3 Solve each of the following equations for the 2x2 matrix X:

a) $X + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$ b) $X + \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}$

4 If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -2 \\ 1 & 5 \\ 4 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 2 \\ 1 & 0 \\ -3 & 5 \end{pmatrix}$, evaluate

a) A + B b) B - C c) (A + B) - C d) A + (B - C)

Multiplication of Matrices by a Scalar

If k is a real number and A is a matrix then kA is a new matrix obtained by multiplying each element of A by k.

In general: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow k\,A = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

Examples

If $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$, evaluate

a) $4A$

$$4A = 4\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 4 & 12 \\ -4 & 0 & 16 \end{pmatrix}$$

b) $3A - 2B$

$$\begin{aligned} 3A - 2B &= 3\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \end{pmatrix} - 2\begin{pmatrix} 2 & -1 & 0 \\ 3 & 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 3 & 9 \\ -3 & 0 & 12 \end{pmatrix} - \begin{pmatrix} 4 & -2 & 0 \\ 6 & 8 & 10 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 5 & 9 \\ -9 & -8 & 2 \end{pmatrix} \end{aligned}$$

Exercise 6

1 If $A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$, find a) 2A b) 3A c) -5A d) -A

2 If $A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 5 & 0 \end{pmatrix}$, find their simplest form:

a) A-B b) 2(A+B) c) 2A d) 2B

e) 2A+2B f) 6A g) 3(2A) h) 8B i) 2(4B)

3 If $A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$, simplify:

a) 3A+2B b) 4A-3B c) 5A-4B d) 2(A-5B)

4 Solve each of the following equations for the matrix X:

a) $3X = \begin{pmatrix} 6 & -3 \\ 12 & 9 \end{pmatrix}$ b) $2X + \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ 2 & 8 \end{pmatrix}$

c) $4X - \begin{pmatrix} 3 & 1 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 0 & 13 \end{pmatrix}$ d) $\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} - 3X = \begin{pmatrix} -5 & 10 \\ 8 & 9 \end{pmatrix}$

5 Find the matrix X in each of the following:

a) $2\begin{pmatrix} 1 & -1 & 3 \\ 2 & -7 & 5 \end{pmatrix} + X = 3\begin{pmatrix} 1 & 2 & -4 \\ 3 & -5 & 1 \end{pmatrix}$

b) $5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 3X = 4\begin{pmatrix} -4 & 7 \\ 3 & 8 \end{pmatrix}$

6 Given that $2\begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} 7 & -2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}$, find p, q, r and s.

Multiplication of Matrices

Two matrices can be multiplied together if they are conformable. This means the number of columns in the first matrix is the same as the number of rows in the second matrix.
The product of a $m \times p$ matrix and a $p \times n$ matrix is a $m \times n$ matrix.

To find the elements of the product matrix, we multiply rows and columns together.

Consider $BC = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ for two matrices B and C.

a_{11} is the total of row 1 of B multiplied by column 1 of C.
a_{12} is the total of row 1 of B multiplied by column 2 of C.
a_{1n} is the total of row 1 of B multiplied by column n of C.
a_{21} is the total of row 1 of B multiplied by column n of C.

$\vdots$

a_{mn} is the total of row m of B multiplied by column n of C.

Example: Find AB where $A = \begin{pmatrix} 3 & -2 & 1 \\ 5 & 0 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 5 \\ 8 & -3 \\ 1 & 4 \end{pmatrix}$.

A has order 2x3, B has order 3x2. They are conformable. AB will have order 2x2.

$$AB = \begin{pmatrix} 3 & -2 & 1 \\ 5 & 0 & 6 \end{pmatrix}\begin{pmatrix} -1 & 5 \\ 8 & -3 \\ 1 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 3\times(-1)+(-2)\times 8+1\times 1 & 3\times 5+(-2)\times(-3)+1\times 4 \\ 5\times(-1)+0\times 8+6\times 1 & 5\times 5+0\times(-3)+6\times 4 \end{pmatrix}$$
$$= \begin{pmatrix} -18 & 25 \\ 1 & 49 \end{pmatrix}$$

Exercise 7

1 Find the following matrix products if possible by first considering the order of the matrices.

a) $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ b) $\begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ c) $\begin{pmatrix} 5 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ e) $\begin{pmatrix} 2 & -3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ f) $\begin{pmatrix} 8 & -5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

g) $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ h) $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ i) $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

j) $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ k) $\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ l) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ -4 \end{pmatrix}$

m) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ n) $\begin{pmatrix} 5 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ o) $\begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

p) $\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ q) $\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ r) $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

s) $\begin{pmatrix} cosa & sina \end{pmatrix}\begin{pmatrix} cosa & -sina \\ sina & cosa \end{pmatrix}$

2 By forming a system of simultaneous equations, find x and y

a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$ b) $\begin{pmatrix} x & 0 \\ 1 & y \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

c) $\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \end{pmatrix}$ d) $\begin{pmatrix} x & y \\ y & x \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

3 If $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

a) Find (i) AB (ii) BA and comment on your result.

b) Find (i) $A(BC)$ (ii) $(AB)C$ and comment on your result.

c) Show that $(AB)' = B'A'$

4 Given that $A = \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}$, find A^2 and A^3.

5 If $P = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$, find PQ and QP.

From the above results matrix multiplication is not commutative.

6 Find the following products

a) $\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$ c) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ e) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The 2x2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the **unit** matrix of order 2 and is denoted by I. It behaves like unity in the real number system. If A is a 2×2 matrix, then $IA = AI = A$.

7 If $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find a, b, c and d.

8 If $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find p, q, r and s.

9 If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find p and q such that $A^2 = pA + qI$.

10 If $A = \begin{pmatrix} 3 & -1 \\ 2 & -5 \end{pmatrix}$, find p and q such that $A^2 = pA + qI$.

11 If $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \end{pmatrix}$, find AB and BA.

12 If $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$, find AB and BA.

13 Calculate M^2 and M^3 when $\theta = 60°$, given that $M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

14 A matrix B is such that $B^2 = 6B - 9I$, where I is the 2×2 unit matrix. Find integers p and q such that $B^3 = pB + qI$.

15 A matrix A is such that $A^2 = 3A - 4I$, where I is the 2×2 unit matrix. Find rational numbers p and q such that $A^3 = pA + qI$.

The Determinant of a Matrix

For a 2×2 matrix:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the determinant of the matrix A is denoted by:

$$detA,\ |A| \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

We calculate it like so

$$\boxed{detA = ad - bc}$$

Example: Find the determinant of $A = \begin{pmatrix} 4 & 3 \\ -1 & -2 \end{pmatrix}$

$$\begin{aligned} detA &= \begin{vmatrix} 4 & 3 \\ -1 & -2 \end{vmatrix} \\ &= 4 \times (-2) - 3 \times (-1) \\ &= -5 \end{aligned}$$

If $|A| = 0$, then matrix A is singular.
If $|A| \neq 0$, then matrix A is non-singular.

For a 3×3 matrix:
If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then $detA$ is defined as follows

$$\boxed{detA = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}}$$

In this expression for $detA$, the factor multiplying a given element is the determinant of the matrix from A by omitting the row and column which contains the given element, together with the sign according to this chess board pattern $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$.

Example: Find the determinant of $A = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

$$\begin{aligned} detA &= 3 \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \\ &= 3(-2-1) - 1(-4-0) - 2(2-0) \\ &= 3 \times (-3) - 1 \times (-4) - 2 \times 2 \\ &= -9 + 4 - 4 \\ &= -9 \end{aligned}$$

Exercise 8

1 Evaluate

a) $\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}$ b) $\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}$ c) $\begin{vmatrix} 2 & -1 \\ -4 & -1 \end{vmatrix}$ d) $\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$

2 If $A = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}$, find

a) AB and show that $\det(AB) = detAdetB$

b) BA and show that $\det(AB) = \det(BA)$

3 Evaluate

a) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$ b) $\begin{vmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{vmatrix}$ c) $\begin{vmatrix} ln2 & ln4 \\ ln5 & ln6 \end{vmatrix}$

4 Find the determinant of the following matrices

a) $\begin{pmatrix} -2 & 1 & 4 \\ 3 & -2 & 5 \\ 0 & 1 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 3 \\ 0 & -1 & 4 \\ 2 & 6 & -2 \end{pmatrix}$ c) $\begin{pmatrix} -2 & 0 & 1 \\ 3 & -4 & 5 \\ -7 & -3 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ e) $\begin{pmatrix} -7 & 14 & 7 \\ 2 & -8 & 6 \\ 9 & -3 & 12 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 8 & -10 \\ 2 & 4 & 15 \\ 1 & 12 & 5 \end{pmatrix}$

5 If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix}$, find

a) AB and show that $\det(AB) = detAdetB$

b) BA and show that $\det(AB) = \det(BA)$

6 Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)$.

The Inverse of a 2x2 Matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

and $IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Therefore, $AI = IA = A$.
For this reason the 2×2 unit matrix is called the identity matrix for multiplication of 2×2 matrices.

Consider $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$.

$AB = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $BA = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Therefore, $AB = BA = I$.
For this reason B is called the **multiplicative inverse** of A and is denoted by A^{-1}. In the same way, we can say that A is the multiplicative inverse of B and is denoted by B^{-1}.
The word inverse usually refers to the multiplicative inverse. The inverse for addition is usually referred to as the negative.

If A and B are square matrices of the same order such that $AB = BA = I$, then B is the inverse of A and A is the inverse of B.
It can be shown that if these inverses exist, then they are unique and so we can talk about the inverse of A or the inverse of B.

Not every matrix has an inverse. To find the inverse of a 2×2 matrix, first calculate the determinant. An inverse will exist if the matrix is non-singular i.e. $detA \neq 0$. If $detA = 0$, then no inverse exists.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Example: Find the inverse of $A = \begin{pmatrix} 7 & -2 \\ -2 & 1 \end{pmatrix}$ if it exists.

$detA = 7 \times 1 - (-2) \times (-2) = 3$

Since $detA \neq 0$, then A^{-1} exists.

$A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{7}{3} \end{pmatrix}$

Exercise 9

1 Show that each matrix is the inverse of the other

a) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$

2 Find the inverse of the following 2×2 matrices, if they exist.

a) $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ b) $B = \begin{pmatrix} 7 & 4 \\ 16 & 9 \end{pmatrix}$ c) $C = \begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$

d) $D = \begin{pmatrix} 5 & 7 \\ 6 & 9 \end{pmatrix}$ e) $E = \begin{pmatrix} -2 & 4 \\ 1 & -1 \end{pmatrix}$ f) $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

3 Given that $P = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, find

a) P^{-1} b) Q^{-1} c) $(PQ)^{-1}$

d) $P^{-1}Q^{-1}$ e) $Q^{-1}P^{-1}$ f) $(QP)^{-1}$

$$(AB)^{-1} = B^{-1}A^{-1}$$
$$\det(AB) = detAdetB$$

The above results should be known.

The Inverse of a 3x3 Matrix

To find the inverse of a 3x3 matrix, A, first create an augmented matrix with A and I (the 3x3 unit matrix). Then complete elementary row operations to turn A into I and in this process I becomes A^{-1}

Not every matrix will have an inverse, if $detA = 0$ then no inverse exists.

Example: Find the inverse of

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}$

$detA = 7$ so the inverse exists.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{matrix} R2 = R2 - R1 \\ R3 = R3 - 2R1 \end{matrix} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right)$$

$$R3 = 2R3 - R2 \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -7 & -3 & -1 & 2 \end{array}\right)$$

$$\begin{matrix} R1 = 7R1 + R3 \\ R2 = 7R2 + R3 \end{matrix} \left(\begin{array}{ccc|ccc} 7 & 7 & 0 & 4 & -1 & 2 \\ 0 & -14 & 0 & -10 & 6 & 2 \\ 0 & 0 & -7 & -3 & -1 & 2 \end{array}\right)$$

$$R1 = 2R1 + R2 \left(\begin{array}{ccc|ccc} 14 & 0 & 0 & -2 & 4 & 6 \\ 0 & -14 & 0 & -10 & 6 & 2 \\ 0 & 0 & -7 & -3 & -1 & 2 \end{array}\right)$$

$$R1 = \frac{1}{14}R1 \quad R2 = -\frac{1}{14}R2 \quad R3 = -\frac{1}{7}R3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{7} & \frac{2}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{5}{7} & \frac{-3}{7} & \frac{-1}{7} \\ 0 & 0 & 1 & \frac{3}{7} & \frac{1}{7} & \frac{-2}{7} \end{array}\right)$$

The inverse of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}$ if $A^{-1} = \begin{pmatrix} \frac{-1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{7} & \frac{-3}{7} & \frac{-1}{7} \\ \frac{3}{7} & \frac{1}{7} & \frac{-2}{7} \end{pmatrix}$.

You can check your answer using $AA^{-1} = I$.

b) $A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 0 \end{pmatrix}$

$detA = -6$ so the inverse exists.

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$R2 = R2 - R1 \quad R3 = 2R3 - 3R2 \quad \left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & -8 & -3 & 0 & -3 & 2 \end{array}\right)$$

$$R3 = R3 + 4R2 \quad \left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & -4 & 1 & 2 \end{array}\right)$$

$$R1 = 3R1 + R3 \quad \left(\begin{array}{ccc|ccc} 6 & 6 & 0 & -1 & 1 & 2 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & -4 & 1 & 2 \end{array}\right)$$

$$R1 = R1 - 3R3 \quad \left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & -2 & 2 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & -4 & 1 & 2 \end{array}\right)$$

$$R1 = \frac{1}{6}R1 \quad R2 = \frac{1}{2}R2 \quad R3 = -\frac{1}{3}R3 \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 2 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -3 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \end{array}\right)$$

If $A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 0 \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{4}{3} & \frac{-1}{3} & \frac{-2}{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2 & -2 & 2 \\ -3 & 3 & 0 \\ 8 & -2 & -4 \end{pmatrix}$.

You can check your answer using $AA^{-1} = I$.

Exercise 10: Find the inverses of the following matrices, if they exist

a) $\begin{pmatrix} 3 & 4 & 5 \\ 4 & 3 & 11 \\ 1 & 0 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 4 & 8 & 3 \\ 3 & 5 & 1 \\ 1 & 4 & 3 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 4 & 2 & 1 \\ 3 & 1 & 2 \\ 3 & 5 & 1 \end{pmatrix}$ e) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 8 & 5 \\ 2 & 10 & 7 \\ 9 & 7 & 3 \end{pmatrix}$

Using the Inverse Matrix to Solve a System of Equations

Consider the system of equations: $\begin{aligned} 3x - y &= 5 \\ 2x + y &= 15 \end{aligned}$

Since $\begin{pmatrix} 3x - y \\ 2x + y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$,

the set of equations can be written as $\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix}$.

This is of the form $AX = B$ where $A = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ 15 \end{pmatrix}$.

If A^{-1}exists $\Rightarrow \quad A^{-1}AX = A^{-1}B$

$\Rightarrow \qquad X = A^{-1}B$

In the above example $A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$

$$\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 20 \\ 35 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

Therefore, $x = 4,\ y = 7$.

This extends to systems with more equations and unknowns.

Example: Solve the following system of equations $\begin{aligned} x + y + 2z &= 3 \\ 2x - y - z &= 2 \\ 3x - 2y + 2z &= -2 \end{aligned}$

The equations can be written as $\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 3 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

which is of the form $AX = B$.

Using elementary row operations $A^{-1} = \frac{1}{13}\begin{pmatrix} 4 & 6 & -1 \\ 7 & 4 & -5 \\ 1 & -5 & 3 \end{pmatrix}$

$$\frac{1}{13}\begin{pmatrix} 4 & 6 & -1 \\ 7 & 4 & -5 \\ 1 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 3 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 4 & 6 & -1 \\ 7 & 4 & -5 \\ 1 & -5 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 26 \\ 39 \\ -13 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

Therefore, $x = 2$, $y = 3$, $z = -1$.

Exercise 11:

Solve the following systems of equations using the inverse matrix.

a) $\begin{matrix} x - y = 5 \\ x + y = 11 \end{matrix}$ b) $\begin{matrix} 3x + y = 7 \\ 3x + 2y = 5 \end{matrix}$ c) $\begin{matrix} 2x + y = 5 \\ 2x + 3y = -1 \end{matrix}$

d) $\begin{matrix} 3x - 4y = 18 \\ 5x + y = 7 \end{matrix}$ e) $\begin{matrix} 2x + 3y = 5 \\ 4x - 5y = 21 \end{matrix}$ f) $\begin{matrix} 5x - 3y = 9 \\ 7x - 6y = 9 \end{matrix}$

g) $\begin{matrix} x - 2y + z = 6 \\ 3x + y - 2z = 4 \\ 7x - 6y - z = 10 \end{matrix}$ h) $\begin{matrix} 5x - y + 2z = 25 \\ 3x + 2y - 3z = 16 \\ 2x - y + z = 9 \end{matrix}$

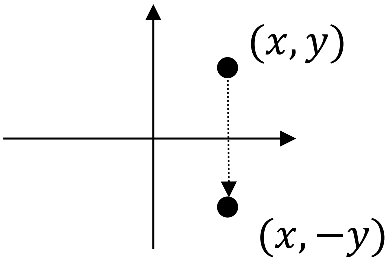
i) $\begin{matrix} x + y + z = 2 \\ 3x - y + 2z = 4 \\ 2x + 3y + z = 7 \end{matrix}$ j) $\begin{matrix} 2x + 4y + 5z = -3 \\ 4x - y - 7z = 6 \\ 6x + 3y - z = 3 \end{matrix}$

Using Matrices to Represent Geometric Transformations

2x2 matrices can be associated with transformations of all points in a Cartesian plane. The transformations we will study are reflections on the x-axis, y-axis, and in the line $y = x$, rotations of 90° and 180°, dilation (enlargement or reduction) or a composition of these.

Examples: Find the matrices associated with

a) reflection of the point (x, y) in the x-axis.



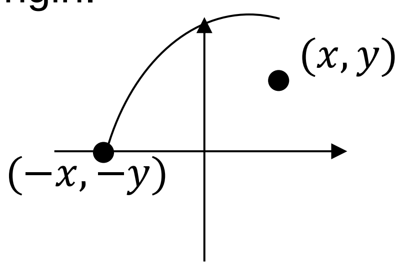
Under this reflection (x, y) maps on to (x', y')

where $(x', y') = (x, -y)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 1x + 0y \\ 0x - 1y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and so $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix associated with the reflection in the x-axis.

b) rotation of the point (x, y) by π radians about the origin.



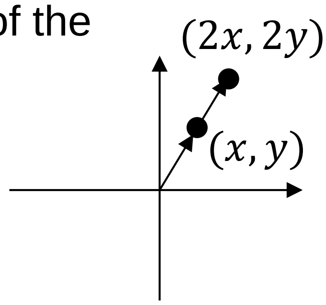
Under this rotation (x, y) maps on to (x', y')

where $(x', y') = (-x, -y)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} -1x + 0y \\ 0x - 1y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and so $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix associated with rotation of π radians about the origin.

c) a dilation where the scale factor is 2 and the centre of the dilation is the origin.



Under this dilation (x, y) maps on to (x', y')

where $(x', y') = (2x, 2y)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 2x + 0y \\ 0x + 2y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and so $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is the matrix associated with dilation, centre the origin and with a scale factor of 2.

Exercise 12

1 Find the matrices associated with the following transformations

a) Reflection in the y-axis.

b) Reflection in the line $y = x$.

c) Reflection in the line $y = -x$.

d) A rotation of $\frac{\pi}{2}$ radians clockwise.

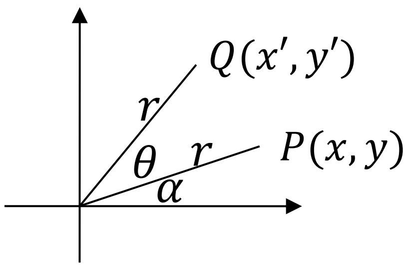
e) A rotation of $\frac{\pi}{2}$ radians anti-clockwise.

f) A dilation, about O, where the scale factor is k.

2 Prove that the matrix associated with a general rotation of θ radians about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

The diagram opposite may be helpful.

OP, which makes an angle of α radians with the x-axis, is rotated through θ radians to OQ.



Hence, show that the matrix associated with a rotation of

a) π radians is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

b) $\frac{\pi}{2}$ radians anti-clockwise is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

c) $\frac{\pi}{2}$ radians clockwise is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Revision of Vectors from Higher

1 Position vector of a point P

Relative to an origin O, P is the point (x, y, z) with position vector

$$\overrightarrow{OP} = \underline{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2 Basic laws

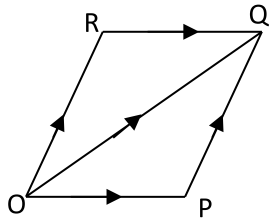
a) The Commutative Law

$$\underline{a} + \underline{b} = \underline{b} + \underline{a}$$

Proof: $\overrightarrow{OP} = \overrightarrow{OP} + \overrightarrow{PQ}$ $\quad$ $\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ}$

$= \underline{a} + \underline{b}$ $\quad$ $= \underline{b} + \underline{a}$

Therefore $\underline{a} + \underline{b} = \underline{b} + \underline{a}$



b) The Associative Law

$$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$$

c) The Zero Vector or Identity Law

$$\underline{a} + \underline{0} = \underline{0} + \underline{a} = \underline{a}$$

d) The Negative of a Vector

$$\underline{a} + (-\underline{a}) = \underline{0}$$

e) Multiplication by a Scalar

If $\underline{a}$ is a non-zero vector and k is a non-zero number, then

(i) $|k\underline{a}|$ is k times $|\underline{a}|$

(ii) if $k > 0$, $k\underline{a}$ is parallel to $\underline{a}$ and in the same direction

(iii) if $k < 0$, $k\underline{a}$ is parallel to $\underline{a}$ and in the opposite direction

f) Unit Vector

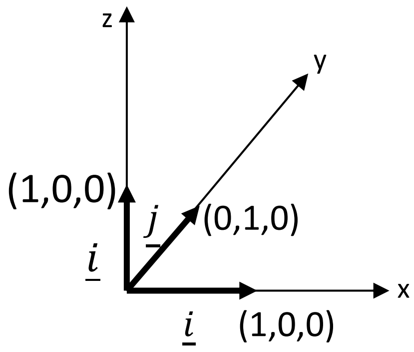
A unit vector is one whose magnitude (length) is one unit.

$\underline{i}$ is a unit vector in the direction of Ox.

$\underline{j}$ is a unit vector in the direction of Oy.

$\underline{k}$ is a unit vector in the direction of Oz.

The position vector of any point can be given in terms of $\underline{i}$, $\underline{j}$ and $\underline{k}$.



e.g. If P is the point (1,2,3), then

$$\underline{p} = \underline{i} + 2\underline{j} + 3\underline{k} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

g) The Magnitude of a Vector

If $\overrightarrow{AB} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $|\overrightarrow{AB}| = \sqrt{x^2 + y^2 + z^2}$.

The distance between $A\ (x_1, y_1, z_1)$ and $B\ (x_2, y_2, z_2)$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

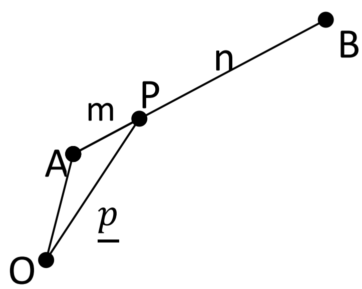
The unit vector in the direction of a vector $\underline{u}$ is given by the formulae $\frac{\underline{u}}{|\underline{u}|}$.

3 Formulae

a) Finding a Point P that Divides a Vector $\overrightarrow{AB}$ in the Ratio $m{:}n$

Find the position vector $\underline{p}$ then P.

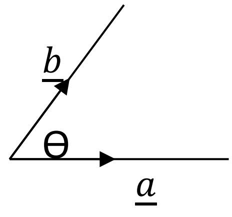
$$\underline{p} = \underline{a} + \frac{m}{m+n}\overrightarrow{AB}$$



b) The Scalar Product (Dot Product)

(i) $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|cos\theta$

where θ is the angle between $\underline{a}$ and $\underline{b}$



(ii) If $\underline{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

then $\underline{a}.\underline{b} = x_1x_2 + y_1y_2 + z_1z_2$

(iii) $cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$

(iv) If $\underline{a}$ and $\underline{b}$ are perpendicular then $\underline{a}.\underline{b} = 0$.

(v) The Distributive Law

$\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$

Examples:

1 If $\underline{a} = 5\underline{i} + 3\underline{j} + 7\underline{k}$ and $\underline{b} = 2\underline{i} - 8\underline{j} + 4\underline{k}$, find the angle between $\underline{a}$ and $\underline{b}$.

$$\underline{a}.\underline{b} = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix}.\begin{pmatrix} 2 \\ -8 \\ 4 \end{pmatrix} = 14$$

$$|\underline{a}| = \sqrt{25 + 9 + 49} = \sqrt{83}$$

$$|\underline{b}| = \sqrt{4 + 64 + 16} = \sqrt{84}$$

$$cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} = \frac{14}{\sqrt{83}\sqrt{84}}$$

$$\theta = 80.3°$$

2 Find the unit vectors which make an angle of $45°$ with the vector $\underline{a} = 2\underline{i} + 2\underline{j} - \underline{k}$ and an angle of $60°$ with the vector $\underline{b} = \underline{j} - \underline{k}$.

Let the unit vector be $\underline{u} = x\underline{i} + y\underline{j} + z\underline{k}$

$$\underline{u}.\underline{a} = |\underline{u}||\underline{a}|cos\theta$$

$$2x + 2y - z = 1 \times 3 \times cos45 \;\Rightarrow\; 2x + 2y - z = \frac{3\sqrt{2}}{2} \quad (1)$$

$$\underline{u}.\underline{b} = |\underline{u}||\underline{b}|cos\theta$$

$$y - z = 1 \times \sqrt{2} \times cos60 \;\Rightarrow\; y - z = \frac{\sqrt{2}}{2} \quad (2)$$

Since $\underline{u}$ is a unit vector $x^2 + y^2 + z^2 = 1$ (3)

(1)-(2) gives $2x + y = \sqrt{2} \;\Rightarrow\; y = \sqrt{2} - 2x$

(1)-2(2) gives $2x + z = \frac{\sqrt{2}}{2} \;\Rightarrow\; z = \frac{\sqrt{2}}{2} - 2x$

Substitute into (3)

$$x^2 + \left(\sqrt{2} - 2x\right)^2 + \left(\frac{\sqrt{2}}{2} - 2x\right)^2 = 1$$

$$9x^2 - 6\sqrt{2}x + \frac{3}{2} = 0$$

$$6x^2 - 4\sqrt{2}x + 1 = 0$$

$x = \frac{1}{3\sqrt{2}}\ or\ \frac{1}{\sqrt{2}}$ by the quadratic formula

$y = \frac{4}{3\sqrt{2}}\ or\ 0$

$z = \frac{1}{3\sqrt{2}}\ or - \frac{1}{\sqrt{2}}$

Hence $\underline{u} = \left(\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)\ or\ \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

Exercise 1

1 Which of the following expressions represent vectors and which represent scalars?

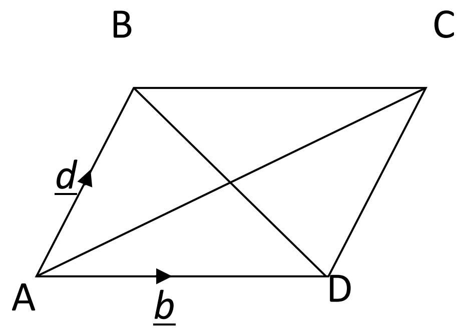
a) $\underline{a}.\underline{c} + \underline{c}.\underline{a} + \underline{a}.\underline{b}$ b) $(\underline{b}.\underline{c})\underline{a} + (\underline{c}.\underline{a})\underline{b} + (\underline{a}.\underline{b})\underline{c}$

c) $[(\underline{b}.\underline{c})(\underline{c}.\underline{a})]\underline{a}$ d) $[(\underline{b}.\underline{c})\underline{c} + (\underline{b}.\underline{a})\underline{a}].(\underline{b} + 2\underline{a})$

Evaluate these when $\underline{a} = \underline{i} + \underline{k},\ \underline{b} = \underline{i} + \underline{j} + 2\underline{k},\ \underline{c} = 2\underline{j} + \underline{k}$

2 A, B and C have coordinates A (1,6,-2), B (2,5,4) and C (4,6,-3). Find the lengths of the sides, cosines of the angles and the area of triangle ABC.

3 Prove that in any parallelogram, the sum of the squares on the diagonals is equal to twice the sum of the squares on the two adjacent sides.



i.e. Prove that $\left|\overrightarrow{BD}\right|^2 + \left|\overrightarrow{AC}\right|^2 = 2\left|\overrightarrow{AB}\right|^2 + 2\left|\overrightarrow{AD}\right|^2$ by expressing $\overrightarrow{BD}$ as $\underline{b} - \underline{d}$ and $\overrightarrow{AC}$ as $\underline{b} + \underline{d}$.

4 Find the two unit vectors which make an angle of 45^0 with the vectors $\underline{a} = \underline{i}$ and $\underline{b} = \underline{k}$.

5a) Find the unit vectors which make an angle of 45^0 with the vector $\underline{a} = -\underline{i} + \underline{k}$ and an angle of 60^0 with the vector $\underline{b} = -2\underline{i} + 2\underline{j} + \underline{k}$.

b) Show that these two unit vectors are perpendicular.

The Vector Product

Definition

The vector product is denoted by $\underline{a} \times \underline{b}$ which reads as $\underline{a}$ cross $\underline{b}$.

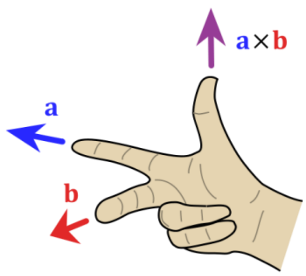
(i) $\underline{a} \times \underline{b}$ has magnitude $|\underline{a}||\underline{b}| \sin\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$. $0 \le \theta \le 180°$.

(ii) $\underline{a} \times \underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$.

(iii) The direction of $\underline{a} \times \underline{b}$ is determined by the "right hand rule".

Right Hand Rule

If the thumb of the right hand is held perpendicular to the middle finger and the forefinger then the forefinger represents $\underline{a}$, the middle finger represents $\underline{b}$ and the thumb represents $\underline{a} \times \underline{b}$.



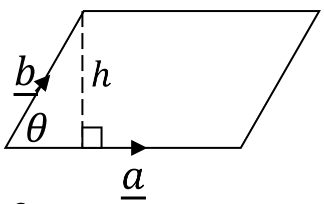
Properties

(i) $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

This follows directly from the right hand rule. If you rotate your wrist in order to represent $-\underline{b}$ ($\underline{b}$ in the other direction) then your thumb will point down.

(ii) $|\underline{a} \times \underline{b}|$ is the area of a parallelogram with sides determined by $\underline{a}$ and $\underline{b}$.

$Area = base \times height = |\underline{a}| \times h = |\underline{a}||\underline{b}| sin\theta$



(iii) $\underline{a} \times \underline{b} = 0 \iff \underline{a}$ is parallel to $\underline{b}$ or one of $\underline{a}$ or $\underline{b}$ is $\underline{0}$.

(iv) If $\underline{a}$ and $\underline{b}$ are non-zero vectors, the following statements are equivalent

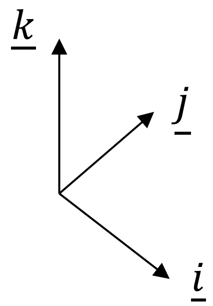
$$\underline{a} \times \underline{b} = 0 \iff \underline{a} \text{ is parallel to } \underline{b} \iff \underline{a} = k\underline{b}$$

(v) $k\underline{a} \times \underline{b} = k(\underline{a} \times \underline{b})$, $\quad k\underline{a} \times n\underline{b} = kn(\underline{a} \times \underline{b})$

(vi) $\underline{a} \times \left(\underline{b} + \underline{c}\right) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$, $\quad \left(\underline{b} + \underline{c}\right) \times \underline{a} = \underline{b} \times \underline{a} + \underline{c} \times \underline{a}$

The Vector Product in Component Form

Let $\underline{i}$, $\underline{j}$ and $\underline{k}$ be unit vectors, mutually perpendicular to form a right hand system.



Therefore $\underline{i} \times \underline{j} = \underline{k}$ $\qquad$ Also $\underline{i} \times \underline{i} = \underline{0}$

$\underline{i} \times \underline{k} = -\underline{j}$ $\qquad$ $\underline{j} \times \underline{j} = \underline{0}$

$\underline{j} \times \underline{i} = -\underline{k}$ $\qquad$ $\underline{k} \times \underline{k} = \underline{0}$

$\underline{j} \times \underline{k} = \underline{i}$

$\underline{k} \times \underline{i} = \underline{j}$

$\underline{k} \times \underline{j} = -\underline{i}$

Let $\underline{a} = x_1\underline{i} + y_1\underline{j} + z_1\underline{k}$ and $\underline{b} = x_2\underline{i} + y_2\underline{j} + z_2\underline{k}$

Then $\underline{a} \times \underline{b} = \left(x_1\underline{i} + y_1\underline{j} + z_1\underline{k}\right) \times \left(x_2\underline{i} + y_2\underline{j} + z_2\underline{k}\right)$

$= x_1\underline{i} \times \left(x_2\underline{i} + y_2\underline{j} + z_2\underline{k}\right) + y_1\underline{j} \times \left(x_2\underline{i} + y_2\underline{j} + z_2\underline{k}\right) + z_1\underline{k} \times \left(x_2\underline{i} + y_2\underline{j} + z_2\underline{k}\right)$

$= x_1\underline{i} \times x_2\underline{i} + x_1\underline{i} \times y_2\underline{j} + x_1\underline{i} \times z_2\underline{k} + y_1\underline{j} \times x_2\underline{i} + y_1\underline{j} \times y_2\underline{j} + y_1\underline{j} \times z_2\underline{k}$

$+ z_1\underline{k} \times x_2\underline{i} + z_1\underline{k} \times y_2\underline{j} + z_1\underline{k} \times z_2\underline{k}$

$= x_1x_2\underline{0} + x_1y_2\underline{k} - x_1z_2\underline{j} - y_1x_2\underline{k} + y_1y_2\underline{0} + y_1z_2\underline{i} + z_1x_2\underline{j} - z_1y_2\underline{i} + z_1z_2\underline{0}$

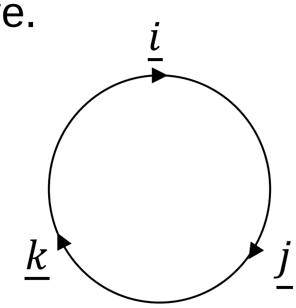
$= x_1y_2\underline{k} - x_1z_2\underline{j} - y_1x_2\underline{k} + y_1z_2\underline{i} + z_1x_2\underline{j} - z_1y_2\underline{i}$

These results can be summarised in a table

X	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{0}$	$\underline{k}$	$-\underline{j}$
$\underline{j}$	$-\underline{k}$	$\underline{0}$	$\underline{i}$
$\underline{k}$	$\underline{j}$	$-\underline{i}$	$\underline{0}$

or

If moving in a clockwise direction, the vector product is positive.
If moving in an anti-clockwise direction, the vector product is negative.



$= (y_1z_2 - y_2z_1)\underline{i} + (x_2z_1 - x_1z_2)\underline{j} + (x_1y_2 - x_2y_1)\underline{k}$

These components of $\underline{a} \times \underline{b}$ can be found more easily by rearranging the components of $\underline{a}$ and $\underline{b}$ under the unit vectors.

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

The component in the direction $\underline{i}$ is found by evaluating the determinant formed by the components of $\underline{j}$ and $\underline{k}$.

The component in the direction $\underline{j}$ is found by evaluating the determinant formed by the components of $\underline{i}$ and $\underline{k}$.

The component in the direction $\underline{k}$ is found by evaluating the determinant formed by the components of $\underline{i}$ and $\underline{j}$.

Examples:

1 If $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$, find a) $\underline{a} \times \underline{b}$ b) $\underline{b} \times \underline{a}$

a) $\underline{a} \times \underline{b}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$= \big(2 \times 1 - 3 \times (-1)\big)\underline{i} - (1 \times 1 - 2 \times 3)\underline{j} + (1 \times (-1) - 2 \times 2)\underline{k}$

$= 5\underline{i} + 5\underline{j} - 5\underline{k}$

b) $\underline{b} \times \underline{a}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$= \big((-1) \times 3 - 2 \times 1\big)\underline{i} - (2 \times 3 - 1 \times 1)\underline{j} + (2 \times 2 - 1 \times (-1))\underline{k}$

$= -5\underline{i} - 5\underline{j} + 5\underline{k}$

2 If $\underline{a} = 3\underline{i} + 5\underline{j} + 7\underline{k}$, $\underline{b} = \underline{i} + \underline{k}$ and $\underline{c} = 2\underline{i} - \underline{j} + 3\underline{k}$ verify that

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$$

(This is known as the vector triple product)

$\underline{b} \times \underline{c}$ $\underline{a} \times (\underline{b} \times \underline{c})$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$
$$= \underline{i} - \underline{j} - \underline{k}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 5 & 7 \\ 1 & -1 & -1 \end{vmatrix}$$
$$= 2\underline{i} + 10\underline{j} - 8\underline{k}$$

$$(\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$$
$$= (6 - 5 + 21)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - (3 + 0 + 7)\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 10 \\ -8 \end{pmatrix}$$

Therefore $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$

3 Find the area of the triangle with vertices A (1,3,2), B (4,3,0) and C (2,1,1)

Area of a triangle = $\frac{1}{2}$ x Area of a parallelogram = $\frac{1}{2}\ |\overrightarrow{CA} \times \overrightarrow{CB}|$

$\overrightarrow{CA} = \underline{a} - \underline{c} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ $\quad \overrightarrow{CB} = \underline{b} - \underline{c} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ $\quad \overrightarrow{CA} \times \overrightarrow{CB} = \begin{pmatrix} 4 \\ -7 \\ -6 \end{pmatrix}$

Area of a triangle= $\frac{1}{2} \times \sqrt{16 + 49 + 36} = \frac{1}{2}\sqrt{101}$

4 Find a unit vector perpendicular to both $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + 2\underline{k}$.

The vector $\underline{n} = \underline{a} \times \underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$.

Obtain the unit vector that has the same direction as $\underline{n}$

$$\underline{u} = \frac{\underline{n}}{|\underline{n}|} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$
$$\underline{a} \times \underline{b} = \underline{i} - 5\underline{j} - 3\underline{k}$$
$$|\underline{a} \times \underline{b}| = \sqrt{1 + 25 + 9} = \sqrt{35}$$
$$\underline{u} = \frac{1}{\sqrt{35}}(\underline{i} - 5\underline{j} - 3\underline{k})$$

Exercise 2

1 If $\underline{a} = 3\underline{i} + 2\underline{j} - \underline{k}$, $\underline{b} = \underline{i} - \underline{j} - 2\underline{k}$ and $\underline{c} = 4\underline{i} - 3\underline{j} + 4\underline{k}$, evaluate

a) $\underline{a} \times (\underline{b} \times \underline{c})$ b) $(\underline{a} \times \underline{b}) \times \underline{c}$ c) $(\underline{a} \times \underline{b}).(\underline{a} \times \underline{c})$

d) $(\underline{a} \times \underline{b}).(\underline{b} \times \underline{c})$ e) $[\underline{a} \times (\underline{b} \times \underline{c})].\underline{c}$

2 If $\underline{a} = 3\underline{i} + 2\underline{j} + 5\underline{k}$, $\underline{b} = 4\underline{i} + 3\underline{j} + 2\underline{k}$ and $\underline{c} = 2\underline{i} + \underline{j} + 10\underline{k}$, find

a) $\underline{a} \times \underline{b}$ b) $(\underline{a} \times \underline{b}).\underline{c}$ c) $\underline{b}.(\underline{a} \times \underline{c})$

3 If $\underline{a} = 3\underline{i} + \underline{j} + 2\underline{k}$, $\underline{b} = 2\underline{j} - \underline{k}$ and $\underline{c} = \underline{i} + \underline{j} + \underline{k}$, and $\underline{d} = \underline{b} \times (\underline{c} \times \underline{a}) + (\underline{a}.\underline{c})\underline{a}$, show that $\underline{b}$ is perpendicular to $\underline{d}$.

4 Find a vector perpendicular to each of the vectors $\underline{a} = 4\underline{i} - 2\underline{j} + 3\underline{k}$ and $\underline{b} = 5\underline{i} + \underline{j} - 4\underline{k}$.

5 If $\underline{a} = \underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$,

a) Find (i) $\underline{a} \times \underline{b}$ (ii) $\underline{a} \times (\underline{a} + \underline{b})$

b) Show that $\underline{a}.(\underline{a} + \underline{b}) = 0$

6 Find the unit vectors perpendicular to both $\underline{a} = 4\underline{i} - \underline{k}$ and $\underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$.

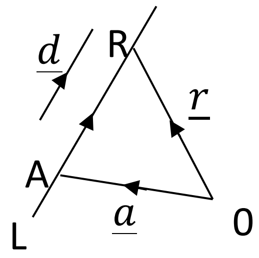
7 Find the area of the triangle ABC where A (4, -8, -13), B (5, -2, -3) and C (5, 4, 10).

8 Prove algebraically the vector triple product $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$

The Equations of a Straight Line

In Vector Form

Let $\underline{d} \neq \underline{0}$ be a fixed vector. Let A, with position vector $\underline{a}$, be a fixed point and R, with position vector $\underline{r}$ be a variable point.
Let R lie on a straight line L which passes through A and is parallel to $\underline{d}$.
Since R lies on L, $\overrightarrow{AR}$ is parallel to $\underline{d}$ (or is zero).



$\overrightarrow{AR} = t\underline{d}$ (where t is some parameter)

$\underline{r} - \underline{a} = t\underline{d}$

$\underline{r} = \underline{a} + t\underline{d}$

This is the vector equation of a line L through A parallel to $\underline{d}$.
The scalar t is a parameter and may take any real value including zero.
The vector $\underline{d}$ is called the direction vector of the line.
The cosines of the angles between the vector $\underline{d}$ and the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ are called the Direction Cosines of $\underline{d}$.

In Parametric Form

If $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underline{d} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ then the vector equation becomes

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t\begin{pmatrix} l \\ m \\ n \end{pmatrix}$, equating components gives $\begin{array}{c} x = a + tl \\ y = b + tm \\ z = c + tn \end{array}$

These are the parametric equations of the line.

In Symmetric or Cartesian Form

We can eliminate the parameter, t, to obtain the following

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

This is called the symmetric form of the equation of a line through (a,b,c) in the direction $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$.

NOTE: If any of the direction vector is zero, the parametric equations should be used.

Finding the equation of a line given two points on the line

A particular straight line in space can be precisely specified in a variety of ways:

a) by means of two points
b) by means of one point on the line and the direction vector of the line
c) as the intersection of two planes (see later)

Examples

1 Find the equation of the line joining the points A (1,0,2) and B (2,1,0)

$\overrightarrow{AB}=\underline{b}-\underline{a}=\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ This is the direction vector of the line.

A (1,0,2) lies on the line.

Substitute into $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$

$\frac{x-1}{1}=\frac{y-0}{1}=\frac{z-2}{-2}$ in symmetric form

$x=1+t$
$y=t$ in parametric form
$z=2-2t$

NOTE: B (2,1,0) also lies on the line, giving

$\frac{x-2}{1}=\frac{y-1}{1}=\frac{z}{-2}$ and $x=2+t$, $y=1+$, $z=-2t$

This illustrates that the equation of a line is not unique.
However, subtracting 1 from each part of the first equation does create the second equation.

An alternative check, to show that the lines are the same, is a point on one line lies on the other and their direction vectors are parallel.

2 Find the symmetrical form of the equation of the line through the point (6,3,-5).

a) in direction $\begin{pmatrix} 4 \\ -8 \\ 7 \end{pmatrix}$

b) parallel to the line $\frac{x}{3} = \frac{y-10}{-2} = \frac{z+8}{13}$

a) $\frac{x-6}{4} = \frac{y-3}{-8} = \frac{z+5}{7}$

b) direction $= \begin{pmatrix} 3 \\ -2 \\ 13 \end{pmatrix}$, $\frac{x-6}{3} = \frac{y-3}{-2} = \frac{z+5}{13}$

Exercise 3

1 Find the symmetrical and vector equation of the line through the point $(5, -2, 6)$ in the direction $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$.

2 Find the symmetrical and vector equation of the line through the point $(2, -3, -7)$ in the direction $\frac{x+5}{7} = \frac{y-13}{3} = \frac{z-4}{-2}$.

3 Find the symmetrical equations of the lines joining the following pairs of points

a) (3,2,-7), (5,-13,-4)

b) (-8,-13,-9), (12,7,1)

c) (3,0,0), (0,0,5)

d) (0,0,0), (-10,4,-6)

The Equation of a Plane

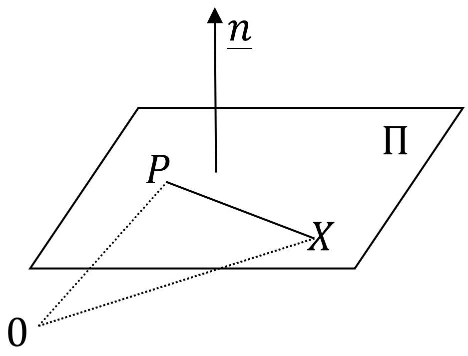
The normal vector, $\underline{n}$, of a plane is perpendicular to the plane if it is perpendicular to every vector which lies on the plane, Π.

The Equation of a Plane in Scalar Product Form of the Vector Equation

Let $\underline{n} \neq \underline{0}$ be a fixed vector and let P with position vector $\underline{p}$ be a fixed point.

Let Π denote the plane through P normal to $\underline{n}$.

Let X with position vector $\underline{x}$ be a variable point.



Then the following statements are equivalent:

(i) X lies on Π

(ii) $\underline{n}$ is perpendicular to PX

(iii) $\underline{n}.\left(\underline{x} - \underline{p}\right) = 0$

Therefore $\underline{n}.\underline{x} - \underline{n}.\underline{p} = 0$

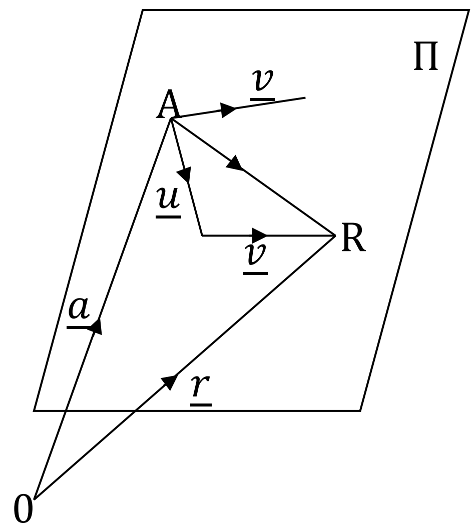
$\Rightarrow \underline{n}.\underline{x} = \underline{n}.\underline{p}$

This equation is true if and only if X lies on Π.

Since $\underline{n}$ and $\underline{p}$ are fixed, $\underline{n}.\underline{p} = k$ where k is a constant.

Therefore the equation of a plane can be written in the form $\underline{n}.\underline{x} = k$ where k is a constant, $\underline{n}$ is the normal vector to the plane and $\underline{x}$ is the position vector of any point on the plane.

The Equation of a Plane in Parametric Form for the Vector Equation



Consider the plane Π which is parallel to vectors $\underline{u}$ and $\underline{v}$ (where $\underline{u}$is not parallel to $\underline{v}$) and which also contains the point A whose position vector is $\underline{a}$.
Let R be any point on Π with position vector $\underline{r}$ and so $\overrightarrow{AR}$ lies on Π.
If R is any point on this plane, $\overrightarrow{AR} = \lambda\underline{u} + \mu\underline{v}$ where λ and μ are parameters.
If $\underline{r}$ is the position vector of R, $\underline{r} = \underline{a} + \overrightarrow{AR}$
i.e. $\underline{r} = \underline{a} + \lambda\underline{u} + \mu\underline{v}$.
Thus any equation of the form $\underline{r} = \underline{a} + \lambda\underline{u} + \mu\underline{v}$, where λ and μ are parameters, represents the plane parallel to the vectors $\underline{u}$ and $\underline{v}$ and containing point $\underline{a}$.

The Equation of a Plane in Symmetrical or Cartesian Form
In coordinate terms, if $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$ and $\underline{x} = x\underline{i} + y\underline{j} + z\underline{k}$ then the equation of the plane, with the aid of the scalar product is written

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = k \text{ or } ax + by + cz = k$$

NOTE $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector. Its components are called the direction ratios. If this is equal to the unit vector then the components are direction cosines.

Examples

1 Find a parametric equation of the plane containing the three points A, B and C whose coordinates are (2,1,3), (7,2,3) and (5,3,5) respectively.

The position vectors of A, B and C are

$\underline{a} = 2\underline{i} + \underline{j} + 3\underline{k}$, $\underline{b} = 7\underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{c} = 5\underline{i} + 3\underline{j} + 5\underline{k}$

$\overrightarrow{AB} = \left(7\underline{i} + 2\underline{j} + 3\underline{k}\right) - \left(2\underline{i} + \underline{j} + 3\underline{k}\right) = 5\underline{i} + \underline{j}$

$\overrightarrow{AC} = \left(5\underline{i} + 3\underline{j} + 5\underline{k}\right) - \left(2\underline{i} + \underline{j} + 3\underline{k}\right) = 3\underline{i} + 2\underline{j} + 2\underline{k}$

The parametric equation is therefore

$\underline{r} = \underline{a} + \lambda\underline{u} + \mu\underline{v}$

$\underline{r} = 2\underline{i} + \underline{j} + 3\underline{k} + \lambda\left(5\underline{i} + \underline{j}\right) + \mu\left(3\underline{i} + 2\underline{j} + 2\underline{k}\right)$

2 Find the Cartesian equation of the plane containing the points A (0,1,-1), B (1,1,0) and C (1,2,0).

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\1 & 1 & 0\\0 & 1 & -1\end{vmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$ This is the normal vector.

The equation of the plane is of the form $-x + z = k$.

Since A (0,1,-1) lies on the plane, substitute in the above equation to find k.

$-0 + (-1) = k \quad \Rightarrow k = -1$

The equation of the plane is $-x + z = -1$ or $x - z = 1$.

3 Find the Cartesian equation of the plane through (-1,2,3) containing the direction vectors $8\underline{i} + 5\underline{j} + \underline{k}$ and $-4\underline{i} + 5\underline{j} + 7\underline{k}$.

The normal vector $\underline{n}$ is perpendicular to both the given vectors

so $\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 8 & 5 & 1 \\ -4 & 5 & 7 \end{vmatrix} = \begin{pmatrix} 30 \\ -60 \\ 60 \end{pmatrix}$.

Any vector parallel to $\underline{n}$ will be perpendicular to the plane, so take $\underline{n} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

The equation of the plane is therefore in the form

$$x - 2y + 2z = k$$

Since A (-1, 2,3) lies on the plane, substitute in the above equation to find k

$$(-1) - 2(2) + 2(3) = k \;\Rightarrow\; k = 1$$

The equation of the plane is $x - 2y + 2z = 1$.

4 Find the Cartesian equation of the plane containing the point P (3,-2,-7) and the line $\frac{x-5}{3} = \frac{y}{1} = \frac{z+6}{4}$.

The points P (3,-2,-7) and Q (5,0,-6) lie on the line.

$$\overrightarrow{PQ} = \begin{pmatrix} 5 \\ 0 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

The direction of the line is $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = \begin{pmatrix} 7 \\ -5 \\ -4 \end{pmatrix}$$

The equation is $7x - 5y - 4z = k$

Substitute P (3,-2,-7) into equation gives

$$7(3) - 5(-2) - 4(-7) = k \;\Rightarrow\; k = 59$$

$$7x - 5y - 4z = 59$$

It will not always be possible to find an equation of a plane for three points.

Consider the points A (1,2,3), B (0,3,2) and C (3,0,5).

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The normal vector is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ which is impossible.

Therefore the points A, B and C must be collinear and an infinity of planes must pass through A, B and C.

Notice $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

Exercise 4

1 Find the Cartesian equation of the plane

a) with normal $\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$ and through the point (3,-1,1)

b) with normal $2\underline{i} - \underline{j} - \underline{k}$ and through (4,1,-2)

2 Find the Cartesian equation of the plane through the given point, containing the stated directions

a) $(1,2,-3)$, $\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

b) $(1,-1,1)$, $3\underline{i} - 5\underline{j} - 7\underline{k}, -4\underline{i} - \underline{j} + 6\underline{k}$

3 Find the Cartesian equation of the plane through the following sets of three points

a) $(2,1,3)$, $(4,1,4)$, $(2,3,6)$

b) $(3,0,0)$, $(0,5,0)$, $(0,0,7)$

4 Find the Cartesian equation of the plane containing both the point and the line given

a) $(5,8,-4)$, $\begin{matrix} x = -4t \\ y = 5 + t \\ z = -1 \end{matrix}$

b) $(2,-5,3)$, $\frac{x-1}{-3} = \frac{y+7}{5} = \frac{z-3}{2}$

5 A plane is parallel to $\frac{x-1}{2} = \frac{y}{3} = \frac{z-1}{4}$ and $\frac{x+1}{-1} = \frac{y}{2} = \frac{z}{1}$, it also passes through the point (1,0,-1). Find its equation.

6 A plane passes through the points (0,1,2) and (1,-1,0) and is parallel to the direction $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Find its equation.

The Angle Between Two Lines

The angle between two lines is the angle between their direction vectors and can be found using the scalar product.

Example

Find the size of the angle between the lines $x-1=y=z-1$ and $x=1+t,\ y=5t,\ z=-t$.

The lines can be expressed in the symmetrical form

$\frac{x-1}{1}=\frac{y}{1}=\frac{z-1}{1}$ and $\frac{x-1}{1}=\frac{y}{5}=\frac{z}{-1}$

The directions are $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\5\\-1\end{pmatrix}$

$$cos\theta=\frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$$

$$cos\theta=\frac{5}{\sqrt{3}\times 3\sqrt{3}}=\frac{5}{9}$$

$$\theta=56.3°$$

The Angle Between Two Planes

The angle between two planes is equal to the angle between the two normal vectors.

Example

Find the angle between the planes $x+2y+z=0$ and $x+y=0$.

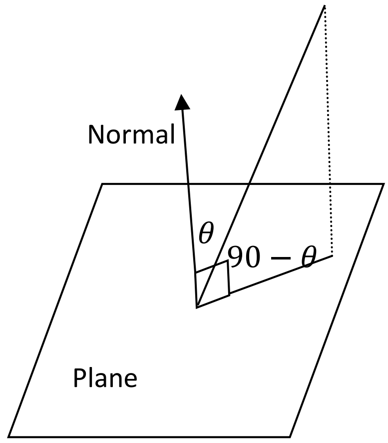
The normal vectors are $\underline{a}=\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\underline{b}=\begin{pmatrix}1\\1\\0\end{pmatrix}$

$$cos\theta=\frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$$

$$cos\theta=\frac{3}{\sqrt{6}\times\sqrt{2}}=\frac{\sqrt{3}}{2}$$

$$\theta=\frac{\pi}{6}$$

The Angle Between a Line and a Plane



If θ is the angle between a line and the normal vector to the plane, then $90-\theta$ is the angle between the line and the plane.

NOTE

(i) $90-\theta$ is the angle between the line and its projection on the plane.

(ii) $90-\theta$ is the smallest angle between the line and the plane.

Example

Find the angle between the line $x=t, y=t, z=0$ and the plane $x+z=0$

Direction Vector= $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and Normal Vector = $\begin{pmatrix}1\\0\\1\end{pmatrix}$

$$cos\theta = \frac{1}{\sqrt{2}\times\sqrt{2}} = \frac{1}{2}$$

$$\theta = 60$$

$$90-\theta = 30°$$

The Intersection of Two Lines

Two distinct lines in a plane are either parallel or intersecting.
In three dimensions there are three possibilities: they may be parallel; intersecting; or skew (neither parallel or intersecting).

The following example demonstrates how to find whether two lines intersect and if they do, how to find the point of intersection.

Example

Line 1 $\frac{x+9}{4}=\frac{y+5}{1}=\frac{z+1}{-2}$ Line 2 $\frac{x-8}{-5}=\frac{y-2}{-4}=\frac{z-5}{8}$ Line 3 $\frac{x-8}{-5}=\frac{y-2}{-4}=\frac{z+15}{8}$

a) Show that lines 1 and 2 do not intersect.

b) Find where lines 1 and 3 intersect.

a) If there is a point (p,q,r) lying on both lines then

$\frac{p+9}{4}=\frac{q+5}{1}=\frac{r+1}{-2}=\lambda$ and $\frac{p-8}{-5}=\frac{q-2}{-4}=\frac{r-5}{8}=\mu$

Therefore $p=4\lambda-9=-5\mu+8$ (4)

$q=\lambda-5=-4\mu+2$ (5)

$r=-2\lambda-1=8\mu+5$ (6)

(4) $4\lambda+5\mu=17$

(5) $\lambda+4\mu=7$

Solving together gives $\lambda=3$ and $\mu=1$.

Now substitute in to (6): LHS=-7 RHS=13

Since LHS≠RHS, equation (6) is not satisfied so lines 1 and 2 do not intersect.

b) If there is a point (p,q,r) lying on both lines then

$\frac{p+9}{4}=\frac{q+5}{1}=\frac{r+1}{-2}=\lambda$ and $\frac{p-8}{-5}=\frac{q-2}{-4}=\frac{r+15}{8}=\mu$

Therefore $p=4\lambda-9=-5\mu+8$ (7)

$q=\lambda-5=-4\mu+2$ (8)

$r=-2\lambda-1=8\mu-15$ (9)

(7) $4\lambda+5\mu=17$

(8) $\lambda+4\mu=7$

Solving together gives $\lambda=3$ and $\mu=1$.

Now substitute in to (9): LHS=-7 RHS=-7

Since LHS=RHS, equation (9) is satisfied so lines 1 and 3 intersect.

Using $\lambda=3$ (or $\mu=1$) in (7), (8) and (9) gives

$p=3,\ q=-2,\ r=-7$

Lines 1 and 3 intersect at $(3,-2,-7)$.

The Intersection of Two Planes

Two non-parallel planes will always meet in a straight line. Given the equation of two planes, we can proceed as follows:

Example

Find the equation of the line of intersection of the planes
$3x - 5y + z = 8$ and $2x - 3y + z = 3$

Method 1

For any point (x, y, z) which lies on both plane, the values of x, y and z fit both equations simultaneously. Hence eliminating z from both equations (by subtraction in this case) gives $x - 2y = 5$.

There are infinitely many pairs of values of x and y which satisfy this equation, but if we choose a value of x then the value of y is fixed and vice versa.

Let $y = t$, then $x = 5 + 2t$ and substituting these expressions for x and y into $3x - 5y + z = 8$ gives

$3(5 + 2t\ ) - 5t + z = 8 \quad \Rightarrow \quad z = -7 - t$

The parametric equations of the line are
$$x = 5 + 2t$$
$$y = t$$
$$z = -7 - t$$

The equation in Cartesian form is $\frac{x-5}{2} = \frac{y}{1} = \frac{z+7}{-1}$

Method 2

To find the equation of a line, we require its direction vector and a point on the line. The direction vector of the line is perpendicular to both normal vectors of the planes.

Normal vectors of the planes are $\underline{a} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

The direction vector of the line of intersection must be parallel to $\underline{a} \times \underline{b}$

$$\underline{a} \times \underline{b} == \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -5 & 1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

Let $z = 0$, hence $3x - 5y = 8$ and $2x - 3y = 3$

Solving these equations simultaneously gives $x = -9, y = -7$

Therefore $(-9, -7, 0)$ lies on the line

The equation of the line in Cartesian form is

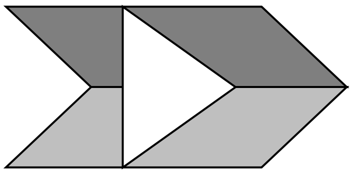
$\frac{x+9}{-2} = \frac{y+7}{-1} = \frac{z}{1}$ or $\frac{x-5}{-2} = \frac{y}{-1} = \frac{z+7}{1}$

Subtracting 7 from each part gives $\frac{x-5}{2} = \frac{y}{1} = \frac{z+7}{-1}$

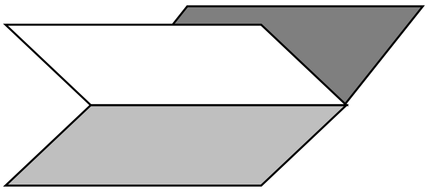
The Intersection of Three Planes

If we consider the equations of three planes together then various situations can occur:

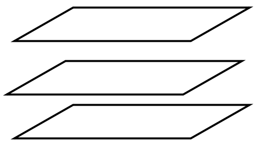
a) There is a unique solution
There is a single point of intersection.



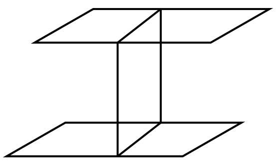
b) There is a linear solution. In this case there are infinitely many points which are common to all three planes.



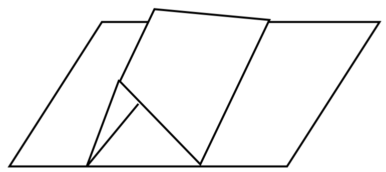
c) There are no solutions
(i) The three planes are parallel



(ii) Two are parallel



(iii) One plane is parallel to the intersection of the other two.



Example

1 Show that the planes

$$A:\ 2x - y + 5z = -4$$
$$B:\ 3x - y + 2z = -1$$
$$C:\ 4x - y - z = 2$$

intersect on a line and find the equation of the line.

Method 1

Plan

- find the line of the intersection of two planes (say A and B)
- use one of the methods on page 25-26
- show that this line is parallel to C
- show that a point on the line lies on plane C
- the last two points show that A, B and C lie on a line

The direction vector of the line is $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ 1 \end{pmatrix}$

Let $z = 0$: $2x - y = -4$
$3x - y = -1$

Subtract $x = 3$

$y = 10$

So the line of intersection of A and B is $\frac{x-3}{3} = \frac{y-10}{11} = \frac{z}{1}$

The normal to C is $\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 11 \\ 1 \end{pmatrix} . \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} = 12 - 11 - 1 = 0$$

Hence, plane C is parallel to the line.

$(3,10,0)$ lies on the line and $4 \times 3 - 10 - 0 = 2$

So $(3,10,0)$ lies on C. Hence plane C intersects A and B on the line given above.

Method 2

Plan - use Guassian Elimination

$$\left[\begin{array}{ccc|c} 2 & -1 & 5 & -4 \\ 3 & -1 & 2 & -1 \\ 4 & -1 & -1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 5 & -4 \\ 0 & 1 & -11 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The line of zeros indicates a redundant equation so the planes intersect on a line.

From Row 3: $z = t$

From Row 2: $y - 11z = 10 \;\;\Rightarrow\;\; y = 11t + 10$

From Row 3: $2x - y + 5 = -4 \;\;\Rightarrow\;\; x = 3t + 3$

Therefore the equation of the plane is $\frac{x-3}{3} = \frac{y-10}{11} = \frac{z}{1}$

2 Show that these planes do not intersect at a common point or line

$$x - y + z = 10$$
$$2x - y + 3z = 5$$
$$4x - 2y + 6z = 7$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 2 & -1 & 3 & 5 \\ 4 & -2 & 6 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & 1 & -15 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

The last line gives $0z = -3$ which is impossible.

Hence the planes do not intersect at a common point or a line.

Exercise 5

1 Calculate the acute angle between the following pairs of lines:

a) $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z}{2}$, $\;\; x = 3t + 3,\; y = 1,\; z = -4t - 2$

b) $x - 1 = y = z - 1,\; x = 1 + t,\; y = 5t,\; z = -t$

c) $\frac{x+4}{3} = \frac{y-1}{5} = \frac{z+3}{1}$, $\;\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$

d) $\underline{r} = 4\underline{i} - \underline{j} + \lambda(\underline{i} + 2\underline{j} - 2\underline{k})$, $\qquad \underline{r} = \underline{i} - \underline{j} + 2\underline{k} - \mu(2\underline{i} + 4\underline{j} - 4\underline{k})$

2 Find the acute angle between the following planes:

a) $2x + 2y - 3z = 3,\ \ x + 3y - 4z = 6$

b) $5x - 14y + 2z = 13,\ \ 6x + 7y + 6z = -23$

c) $\underline{r}.\left(\underline{i} - \underline{j}\right) = 4,\ \ \underline{r}.\left(\underline{j} + \underline{k}\right) = 1$

d) $\underline{r}.\left(\underline{i} + \underline{j} + \underline{k}\right) = 1,\ \ \underline{r}.\left(\underline{i} - \underline{j} + \underline{k}\right) = 0$

3 Find the acute angle between the following lines and planes:

a) $\frac{x}{4} = \frac{y-1}{-1} = \frac{z+3}{-5},\quad x - 2y + 4z = -3$

b) $\frac{x-2}{2} = \frac{y+1}{6} = \frac{z+3}{3},\quad 2x - y - 2z = 4$

c) $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6},\quad 10x + 2y - 11z = 3$

d) $\underline{r} = \underline{i} - \underline{j} + \lambda\left(\underline{i} + \underline{j} + \underline{k}\right),\ \underline{r}.\left(\underline{i} - 2\underline{j} + 2\underline{k}\right) = 4$

4 Find the coordinates of the points of intersection of these lines:

a) $\frac{x-4}{1} = \frac{y-8}{2} = \frac{z-3}{1},\quad \frac{x-7}{6} = \frac{y-6}{4} = \frac{z-5}{5}$

a) $\frac{x-2}{1} = \frac{y-9}{2} = \frac{z-13}{3},\quad \frac{x+3}{-1} = \frac{y-7}{2} = \frac{z+2}{-3}$

5 Find the equation of the line of intersection of the following two planes:

a) $x + y + 2z = 2,\quad x - y - z = 5$

b) $2x - y = 3,\quad x + y + 4z = 1$

c) $2x + 3y + z = 8,\quad x + y + z = 10$

6 Find the co-ordinates of the point of intersection of the following line and plane:

a) $x = 4 + t,\ y = 1 - t,\ z = 3t,\qquad 2x + 4y + z = 9$

b) $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-1}{4},\qquad x - 2y + 3z = 26$

7 A plane contains the line $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-1}{2}$ and the plane is parallel to the line $\frac{x}{1}=\frac{y+7}{2}=\frac{z}{3}$.
Find the equation of the plane.

8a) Find the co-ordinates of the point A in which the line L with equation $\frac{x+1}{2}=\frac{y-2}{-1}=\frac{z+3}{2}$ meets the plane Π with equation $3x-y+z=10$.

b) Hence find the equation in cartesian form for the line through A lying wholly in the plane and perpendicular to the line.

9a) Find the parametric equations for the line L joining the points A (2,-4,3) and B (4,0,-5).

b) Verify that L is perpendicular to the line M joining the point A to C (-2,-6,1).

10a) Show that the line L with parametric equations
$x=2-t,\ y=-3+2t,\ z=-1-4t$
lies on the plane Π with equation $2x+3y+z=-6$.

b) Find the parametric equations for the line M through the point (3,2,-4) perpendicular to Π.

c) Prove that M meets Π at a point lying on L.

11a) Show that the line L joining the points A (2,1,-1) and B (3,-2,1) is perpendicular to the line M with parametric equations
$x=11+4t,\ y=3+2t,\ z=1+t$.

b) Find the equation of the plane Π through L perpendicular to M and prove that Π meets M at a point C equidistant from A and B.

12a) Find the parametric equations for the line joining A (1,-1,2) and B (4,5,-7).

b) Prove that AB intersects the line with parametric equations
$x=6+4t,\ y=2+t,\ z=1+2t$ at right angles.

c) Find the coordinates of the point of intersection of the lines.

Integration

Standard Indefinite Integrals	
f(x)	$\int f(x)\,dx$
x^n	$\frac{x^{n+1}}{n+1}+C$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)}+C$
sinx	-cosx+C
cosx	sinx+c
sin(ax+b)	$-\frac{1}{a}$cos(ax+b)+C
cos(ax+b)	$\frac{1}{a}$sin(ax+b) +C

In addition, the following trigonometric identities were useful.

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Exercise 1: Integrate the following functions

a) $f(x)=8x^3$

b) $f(x)=\frac{6}{x^3}$

c) $f(x)=\sqrt{x}$

d) $f(x)=\frac{1}{\sqrt{x}}$

e) $f(x)=\frac{3x^4+6}{x^2}$

f) $f(x)=\frac{1-3x}{\sqrt{x}}$

g) $f(x)=(3x+4)^5$

h) $f(x)=(1-2x)^4$

i) $f(x)=\frac{1}{(2x+3)^2}$

j) $f(x)=\frac{1}{\sqrt{4x+1}}$

k) $f(x)=\frac{3}{(3x+1)^{\frac{3}{2}}}$

l) f(x)=sin3x

m) $f(x)=\cos\left(\frac{1}{3}x\right)$

n) $f(x)=\sin^2 x$

o) $f(x)=\cos^2\left(\frac{1}{2}x\right)$

New Integrals	
f(x)	$\int f(x)\,dx$
e^x	e^x+C
e^{ax+b}	$\frac{1}{a}e^{ax+b}+C$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b)+C$
$\sec^2 x$	$\tan x+C$
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b)+C$
$\operatorname{cosec}^2 x$	$\cot x+C$
$\operatorname{cosec}^2(ax+b)$	$\frac{1}{a}\cot(ax+b)+C$

Examples:

a) $\int e^{2x}dx = \frac{1}{2}e^{2x} + C$

b) $\int e^{1-5x}dx = -\frac{1}{5}e^{1-5x} + C$

c) $\int \frac{1}{3x+2}dx = \frac{1}{3}\ln(3x+2) + C$

d) $\int sec^2(4x-3) = \frac{1}{4}\tan(4x-3)$

Exercise 2: Integrate the following functions

a) $f(x)=e^{5x}$

b) $f(x)=e^{-7x}$

c) $f(x)=3e^{2x}$

d) $f(x)=\frac{1}{3x}$

e) $f(x)=\frac{1}{x+5}$

f) $f(x)=\frac{1}{2x-3}$

g) $f(x)=(e^x + e^{-x})^2$

h) $f(x)=\frac{1+e^x}{e^x}$

i) $f(x)=e^{2x} + \frac{1}{e^{3x}}$

j) $f(x)=\frac{6}{3x+2}$

k) $f(x)=\frac{3}{1-2x}$

l) $f(x)=\frac{9}{5-7x}$

m) $f(x)=\sec^2 4x$

n) $f(x)=\sec^2(\pi+2x)$

o) $f(x)=7\operatorname{cosec}^2 2x$

Definite Integrals

Reminder: $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

Examples: Evaluate

a) $\int_{-3}^{-2} \frac{1}{x^2} dx$
$= \int_{-3}^{-2} x^{-2}\, dx$
$= [-x^{-1}]_{-3}^{-2}$
$= \left[-\frac{1}{x}\right]_{-3}^{-2}$
$= -\frac{1}{-2} - \left(-\frac{1}{-3}\right)$
$= \frac{1}{2} - \frac{1}{3}$
$= \frac{1}{6}$

b) $\int_1^5 \frac{1}{\sqrt{2x-1}} dx$
$= \int_1^5 (2x-1)^{-\frac{1}{2}}\, dx$
$= \left[\frac{(2x-1)^{\frac{1}{2}}}{2\times\frac{1}{2}}\right]_1^5$
$= \left[(2x-1)^{\frac{1}{2}}\right]_1^5$
$= \sqrt{9} - \sqrt{1}$
$= 2$

c) $\int_0^{\frac{\pi}{2}} sin2x\, dx$
$= \left[-\frac{1}{2}cos2x\right]_0^{\frac{\pi}{2}}$
$= -\frac{1}{2}cos\pi - \left(-\frac{1}{2}cos0\right)$
$= \frac{1}{2} + \frac{1}{2}$
$= 1$

d) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sec^2\frac{1}{2}x\, dx$
$= \left[2tan\frac{1}{2}x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$
$= 2tan\frac{\pi}{4} - \left(2tan\left(-\frac{\pi}{4}\right)\right)$
$= 2\times 1 - 2\times(-1)$
$= 4$

e) $\int_0^2 e^{-3x}\, dx$
$= \left[-\frac{1}{3}e^{-3x}\right]_0^2$
$= -\frac{1}{3}e^{-6} - \left(-\frac{1}{3}e^0\right)$
$= -\frac{1}{3}e^{-6} + \frac{1}{3}$
$= \frac{1}{3}(1 - e^{-6})$

f) $\int_4^5 \frac{2}{x-3} dx$
$= [2\ln(x-3)]_4^5$
$= 2ln2 - 2ln1$
$= 2ln2$
$= ln4$

Exercise 3: Evaluate

a) $\int_1^2 \left(\frac{2}{x^2} - \frac{5}{x^4}\right) dx$ b) $\int_{\frac{1}{4}}^1 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ c) $\int_1^4 \frac{2x^2+3}{\sqrt{x}} dx$

d) $\int_1^6 \sqrt{x+3}dx$ e) $\int_0^1 \frac{1}{(4x+5)^2} dx$ f) $\int_3^{12} (x-4)^{\frac{1}{3}} dx$

g) $\int_0^{\frac{\pi}{4}} cos3xdx$ h) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cosec^2xdx$ i) $\int_0^{2\pi} sin^2xdx$

j) $\int_0^2 e^{-3x}dx$ k) $\int_0^1 e^{1-x}dx$ l) $\int_0^1 e^{\frac{x}{2}}dx$

m) $\int_5^9 \frac{1}{x-3} dx$ n) $\int_0^1 \frac{1}{3x+2} dx$ o) $\int_{-4}^0 \frac{1}{1-2x} dx$

Integration by Substitution: Substitution not given

It is sometimes difficult to reduce an integral to one of the standard integrals. Such integrals may be made simpler by changing the variable by means of a substitution of a new variable.

For this method to work, one part of the function must be a derivative of another part. Every part of the function will need to be written in the new variable, however the answer should be given in the original variable.

Examples: Complete the following integrations

a) $\int x^4(1+2x^5)^3 dx$

Let $u=1+2x^5$

$\frac{du}{dx}=10x^4$

$\frac{1}{10}du=x^4dx$

The integral becomes

$\int \frac{1}{10} u^3 du$

$= \frac{1}{10} \times \frac{u^4}{4} + C$

$= \frac{1}{40}(1+2x^5)^4 + C$

b) $\int 2xe^{x^2}dx$

Let u=x^2

$\frac{du}{dx}$=2x

du=2xdx

The integral becomes

$\int e^u du$

$= e^u + C$

$= e^{x^2} + C$

c) $\int sin^2 x cosx dx$

Let u=sinx

$\frac{du}{dx}$=cosx

du=cosxdx

The integral becomes

$\int u^2 du$

$= \frac{u^3}{3} + C$

$= \frac{1}{3} sin^3 x + C$

d) $\int \frac{lnx}{x}$

Let u=lnx

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx$

The integral becomes

$\int udu$

$= \frac{u^2}{2} + C$

$= \frac{1}{2}(lnx)^2 + C$

e) $\int \frac{x^4}{(x^5+1)^3}dx$

Let u=x^5+1

$\frac{du}{dx}$=$5x^4$

$\frac{1}{5}$du=x^4dx

The integral becomes

$\int \frac{1}{5u^3}dx$

$= \int \frac{1}{5}u^{-3}du$

$= \frac{1}{5} \times \frac{u^{-2}}{-2} + C$

$= -\frac{1}{10(x^5+1)^2} + C$

f) $\int \frac{2x}{x^2+4}dx$

Let u=x^2+4

$\frac{du}{dx}$=2x

du=2xdx

The integral becomes

$\int \frac{1}{u}du$

$= lnu + C$

$= \ln(x^2+4) + C$

g) $\int tanxdx$

Let u=cosx

$\frac{du}{dx}$=-sinx

-du=sinxdx

$= \int \frac{sinx}{cosx}dx$

The integral becomes

$\int -\frac{1}{u}du$

$= -lnu + C$

$= -ln(cosx) + C$

Examples f and g demonstrated the special integral of the form

$$\int \frac{f'(x)}{f(x)}dx = ln[f(x)] + C$$

Exercise 4: Complete the following integrations

a) $\int x(x^2-3)^5dx$ b) $\int x^2(x^3-1)^2dx$ c) $\int x\sqrt{1-x^2}dx$

d) $\int cosxsin^4xdx$ e) $\int \frac{x}{(1-x^2)^3}dx$ f) $\int \frac{x^3}{1+x^4}dx$

g) $\int \frac{sinx}{cos^3x}dx$ h) $\int \frac{e^x}{3e^x-1}dx$ i) $\int \frac{sec^2x}{tanx}dx$

Integration by Substitution: Substitution will given

In this type, one part of the function will not be a derivative of another part but the appropriate substitution will be given. As with the previous examples, every part of the function will need to be written in the new variable and the answer should be given in the original variable.

Examples: Complete the following integrations

a) $\int x\sqrt{3x-2}dx$ where u=3x-2

The integral becomes

$$\int \frac{1}{3}(u+2)u^{\frac{1}{2}}\frac{1}{3}du$$

$$= \int \left(\frac{1}{9}u^{\frac{3}{2}} + \frac{2}{9}u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{9}\times\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{9}\times\frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{45}(3x-2)^{\frac{5}{2}} + \frac{4}{27}(3x-2)^{\frac{3}{2}} + C$$

Let u=3x-2

$\frac{du}{dx}$=3

$\frac{1}{3}$du=dx

3x-2=u

3x=u+2

x=$\frac{1}{3}$(u+2)

b) $\int x(x+2)^5dx$ where u=x+2

The integral becomes

$$\int (u-2)u^5du$$

$$= \int (u^6 - 2u^5)\,du$$

$$= \frac{u^7}{7} - \frac{2u^6}{6} + C$$

$$= \frac{1}{7}(x+2)^7 - \frac{1}{3}(x+2)^6 + C$$

Let u=x+2

$\frac{du}{dx}$=1

du=dx

x+2=u

x=u-2

c) $\int \frac{x}{\sqrt{x+4}}dx$ where u=x+4

The integral becomes

$$\int \frac{u-4}{u^{\frac{1}{2}}}du$$

$$= \int \left(u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\right)du$$

Let u=x+4

$\frac{du}{dx}$=1

du=dx

x+4=u

x=u-4

$$= \frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}} + C$$
$$= \frac{2}{3}(x+4)^{\frac{3}{2}} - 8(x+4)^{\frac{1}{2}} + C$$

Exercise 5: Integrate the following functions

a) $\int 9x(3x+2)^3 dx \;\; where\; u = 3x+2$

b) $\int 7x(2x+3)^5 dx \; where\; u = 2x+3$

c) $\int 3x\sqrt{1+x}dx \;\; where\; u = 1+x$

d) $\int \frac{3x}{\sqrt{2x+3}}dx \;\; where\; u = 2x+3$

A Special Substitution

In integrals which contain $\sqrt{a^2 - x^2}$, the substitution $x = sin\theta$ can be used.

Examples: Complete the following integrals.

a) $$\int \sqrt{1-x^2}dx \;\; where\; x = sin\theta$$

The integral becomes

$$\int cos^2\theta d\theta$$
$$= \int \frac{1}{2}(1 + co\,s2\theta)d\theta$$
$$= \int \left(\frac{1}{2} + \frac{1}{2}cos2\theta\right)d\theta$$
$$= \frac{1}{2}\theta + \frac{1}{4}sin2\theta + C$$
$$= \frac{1}{2}sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C$$

Let $x = sin\theta$

$$\frac{dx}{d\theta} = cos\theta$$
$$dx = cos\theta d\theta$$
$$\sqrt{1-x^2} = \sqrt{1-sin^2\theta}$$
$$= \sqrt{cos^2\theta}$$
$$= cos\theta$$
$$sin\theta = x$$
$$\theta = sin^{-1}x$$
$$sin2\theta = 2sin\theta cos\theta$$
$$= 2x\sqrt{1-x^2}$$

b) $\int \frac{x^2}{\sqrt{9-x^2}}dx \;\; where\; x = 3sin\theta$

The integral becomes

$$\int \frac{9sin^2\theta 3cos\theta}{3cos\theta}d\theta$$

$$= \int 9sin^2\theta d\theta$$

$$= \int \frac{9}{2}(1 - cos2\theta)d\theta$$

$$= \int \left(\frac{9}{2} - \frac{9}{2}cos2\theta\right)d\theta$$

$$= \frac{9}{2}\theta - \frac{9}{4}sin2\theta + C$$

$$= \frac{9}{2}sin^{-1}\frac{x}{3} - \frac{9}{4} \times \frac{2}{9}x\sqrt{9-x^2} \; + C$$

$$= \frac{9}{2}sin^{-1}\frac{x}{3} - \frac{1}{2}x\sqrt{9-x^2} \; + C$$

Let $x = 3sin\theta$

$\frac{dx}{d\theta} = 3cos\theta$

$dx = 3cos\theta d\theta$

$\sqrt{9-x^2} = \sqrt{9 - 9sin^2\theta}$

$= \sqrt{9cos^2\theta}$

$= 3cos\theta$

$3sin\theta = x$

$sin\theta = \frac{x}{3}$

$\theta = sin^{-1}\frac{x}{3}$

$3cos\theta = \sqrt{9-x^2}$

$cos\theta = \frac{1}{3}\sqrt{9-x^2}$

$sin2\theta = 2sin\theta cos\theta$

$= 2\frac{x}{3} \times \frac{1}{3}\sqrt{9-x^2}$

$= \frac{2}{9}x\sqrt{9-x^2}$

Exercise 6: Integrate the following functions

a) $\int \frac{x}{\sqrt{1-x^2}}dx \; where\; x = sin\theta$

b) $\int \sqrt{4-x^2}dx \;\; where\; x = 2sin\theta$

c) $\int \frac{x}{\sqrt{9-x^2}}dx \;\; where\; x = 3sin\theta$

d) $\int \frac{x^2}{\sqrt{4-x^2}}dx \; wher\; x = 2sin\theta$

Substitution and Definite Integrals

We can avoid going back to the original variable in the following questions by also changing the limits on the original function. The substitution may or may not be given.

Examples: Evaluate

a) $\int_1^2 (2x+4)(x^2+4x)^3 dx$

Let $u = x^2 + 4x$

$\frac{du}{dx} = 2x + 4$

$du = (2x+4)dx$

$x = 1 \implies u = 1^2 + 4(1) = 5$

$x = 2 \implies u = 2^2 + 4(2) = 12$

The integral becomes

$\int_5^{12} u^3 du$

$= \left[\frac{u^4}{4}\right]_5^{12}$

$= \frac{12^4}{4} = \frac{5^4}{4} = 5027.75$

b) $\int_0^1 \frac{dx}{(1+x^2)^2}$ $where\ x = tan\theta$

The integral becomes

$$\int_0^{\frac{\pi}{4}} \frac{sec^2\theta}{sec^4\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} cos^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}cos2\theta\right) d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{4}sin2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \times \frac{\pi}{4} + \frac{1}{4}sin2 \times \frac{\pi}{4} - \left(\frac{1}{2} \times 0 + \frac{1}{4}sin2 \times 0\right)$$

$$= \frac{\pi}{8} + \frac{1}{4}sin\frac{\pi}{2} - 0$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

Let $x = tan\theta$

$$\frac{dx}{d\theta} = sec^2\theta$$

$$dx = sec^2\theta d\theta$$

$$1 + x^2 = 1 + tan^2\theta$$

$$= 1 + \frac{sin^2\theta}{cos^2\theta}$$

$$= \frac{cos^2\theta}{cos^2\theta} + \frac{sin^2\theta}{cos^2\theta}$$

$$= \frac{1}{cos^2\theta}$$

$$(1 + x^2)^2 = \frac{1}{cos^4\theta} = sec^4\theta$$

$$tan\theta = x$$

$$\theta = tan^{-1}x$$

$$x = 0 \implies \theta = tan^{-1}0 = 0$$

$$x = 1 \implies \theta = tan^{-1}1 = \frac{\pi}{4}$$

Exercise 7: Evaluate

a) $\int_e^{e^3} \frac{dx}{xlnx}$ $where\ u = lnx$

b) $\int_2^4 \frac{x(x^2+4)}{x^2-2} dx$ $where\ u = x^2 - 2$

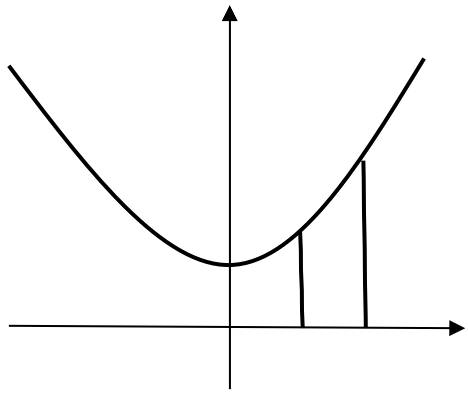
c) $\int_0^{\frac{\pi}{3}} sinxcos^4xdx$ $where\ u = cosx$

The Area Enclosed by the Curve and the x-axis

This has been covered at higher. The area between y=f(x), x=a, x=b and the x-axis is $\int_a^b y dx = \int_a^b f(x)dx = F(b) - F(a)$ where F(x) is the integral of y=f(x).

Examples: Find the areas enclosed by

a) f(x)=x^2+2 between x=1, x=3 and the x-axis.



The area is above the x-axis.

$\int_1^3 (x^2 + 2)dx$

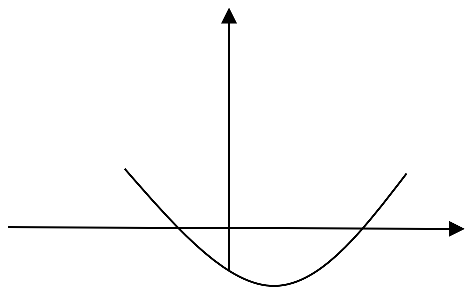
$= \left[\frac{x^3}{3} + 2x\right]_1^2$

$= \frac{3^3}{3} + 2 \times 3 - (\frac{1^3}{3} + 2 \times 1)$

$= 9 + 6 - (\frac{1}{3} + 2)$

$= 12\frac{2}{3} units^2$

b) f(x)=x^2-4x-5 between x=-1, x=5



The area is below the x-axis.

$\int_{-1}^5 (x^2 - 4x - 5)dx$

$= \left[\frac{x^3}{3} - 2x^2 - 5x\right]_{-1}^5$

$= \frac{5^3}{3} - 2(5)^2 - 5(5) - \left(\frac{(-1)^3}{3} - 2(-1)^2 - 5(-1)\right)$

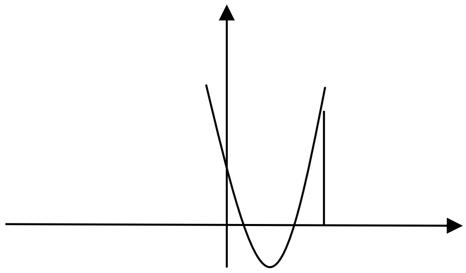
$= \frac{125}{3} - 50 - 25 - (\frac{-1}{3} - 2 + 5)$

$= -\frac{78}{3}$

The negative value indicates the area is below the x-axis as shown.

$Area = \frac{78}{3} units^2$

c) $f(x)=x^2-3x+2$ between x=0 and x=3.



Some of the area is below the x-axis.

$$Area\ 1 = \int_0^1 (x^2 - 3x + 2)dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^1$$

$$= \frac{1^3}{3} - \frac{3\times1^2}{2} + 2\times1 - \left(\frac{0^3}{3} - \frac{3\times0^2}{2} + 2\times0\right)$$

$$= \frac{5}{6}$$

$$Area\ 2 = \int_1^2 (x^2 - 3x + 2)dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2$$

$$= \frac{2^3}{3} - \frac{3\times2^2}{2} + 2\times2 - \left(\frac{1^3}{3} - \frac{3\times1^2}{2} + 2\times1\right)$$

$$= -\frac{1}{6}$$

$$Area\ 3 = \int_2^3 (x^2 - 3x + 2)dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_2^3$$

$$= \frac{3}{3} - \frac{3\times3^2}{2} + 2\times3 - \left(\frac{2^3}{3} - \frac{3\times2^2}{2} + 2\times2\right)$$

$$= \frac{5}{6}$$

$$Total\ Area = \frac{5}{6} + \frac{1}{6} + \frac{5}{6}$$

$$= 1\frac{5}{6}\,units^2$$

Exercise 8

1 Find the areas enclosed by the following curves and the x-axis between the lines given:

a) $y=3x^2+2$, x=0 and x=2

b) $y=x^3-x$, x=1 and x=2

2 Find the area enclosed by the following curves and the x-axis:

a) $y=6+x-x^2$

b) $y=x(x-2)(x-3)$

The Area Enclosed by the Two Curves

The area enclosed between two curves has already been met at Higher.

$$Area = \int_a^b (f(x) - g(x))dx$$

where the curves $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$ provided the graph of $f(x)$ is above the graph of $g(x)$, their positions relative to the x-axis.

Examples: Find the areas between

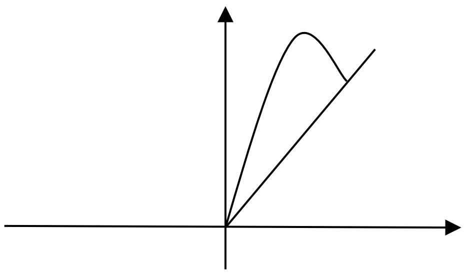
a) y=x(4-x) and y=x

The curves meet where x(4-x)=x

x^2-3x=0

x(x-3)=0

x=0 and x=3



$$Area = \int_0^3 (x(4-x) - x)dx$$

$$= \int_0^3 (3x - x^2)dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{3\times3^2}{2} - \frac{3^3}{3} - 0$$

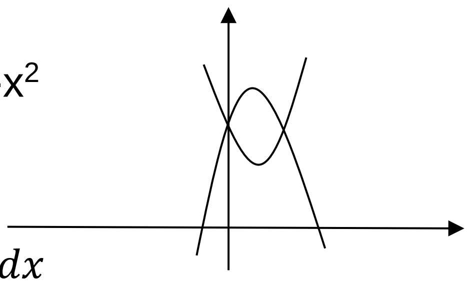
$$= \frac{9}{2} units^2$$

b) y=$(x-1)^2$+4 and y=5+4x-x^2

The curves meet where $(x-1)^2$+4=5+4x-x^2

2x^2-6x=0

x=0 and x=3



$$Area = \int_0^3 ((x-1)^2 + 4 - (5 + 4x - x^2))dx$$

$$= \int_0^3 (6x - 2x^2)dx$$

$$= \left[3x^2 - \frac{2x^3}{3}\right]_0^3$$

$$= 3 \times 3^2 - \frac{2\times3^3}{3} - 0$$

$$= 9units^2$$

Exercise 9: Find the areas enclosed by the following curves

a) *y=x(10-x) and y=4x*

b) *y=4x-x^2 and y=x^2-4x+6*

c) $y = 2\sqrt{x}$ and $y = \frac{x^2}{4}$

d) *y=x^3+x^2-5x and y=x^2-x*

e) *y=sinx and y=cosx between x=0 and x=2π*

e) *y^2=4ax and x^2=4ay*

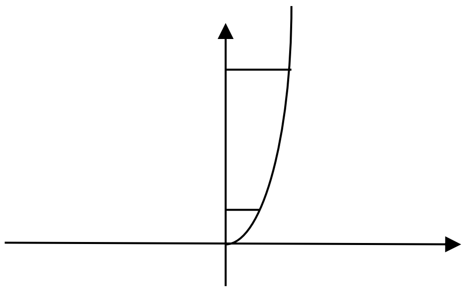
The Area Enclosed by a Curve and the y-axis

The area between x=f(y), y=a, y=b and the x-axis is $\int_a^b xdy = \int_a^b f(y)dy = F(b) - F(a)$ where F(y) is the integral of x=f(y) and F(b) and F(a) are the values of F(y) at y=a and y=b.

Examples:

Find the areas enclosed by the curves, the lines and the y-axis.

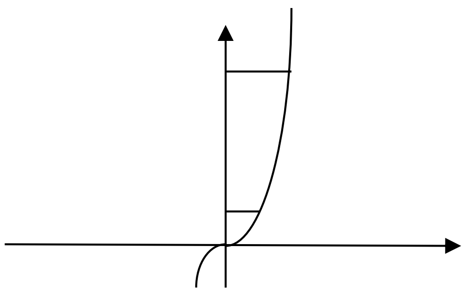
a) y=x^2, y=1 and y=4



From $y = x^2$ we get $x = \sqrt{y}$.

$$Area = \int_1^4 y^{\frac{1}{2}}dy$$
$$= \left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^4$$
$$= \frac{2}{3} \times 4^{\frac{3}{2}} - \frac{2}{3} \times 1^{\frac{3}{2}}$$
$$= \frac{14}{3}units^2$$

b) y=x^3, y=1 and y=8



From $y = x^2$ we get $x = \sqrt[3]{y}$.

$$Area = \int_1^8 y^{\frac{1}{3}}dy$$
$$= \left[\frac{3}{4}y^{\frac{4}{3}}\right]_1^8$$
$$= \frac{3}{4} \times 8^{\frac{4}{3}} - \frac{3}{4} \times 1^{\frac{4}{3}}$$
$$= 11\frac{1}{4}units^2$$

Exercise 10

Find the areas enclosed by the curves, the y-axis and the lines given:

a) $x = y^2, y = 3$

b) $y = x^3, y = 1, y = 8$

c) $x = \frac{1}{\sqrt{y}}, y = 2, y = 3$

d) $y^2 = 1 - x, y = 0, y = 1$

e) $y = \frac{1}{x^3}, y = 8, y = 27$

f) $y = lnx, y = 2, y = 5$

New Special Integrals

Integrals of the form $\int \frac{1}{\sqrt{1-x^2}} dx$ and $\int \frac{1}{\sqrt{a^2-x^2}} dx$

If $f(x) = sin^{-1}x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$ then $\boxed{\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}x + C}$

If $f(x) = sin^{-1}\left(\frac{x}{2}\right)$

$\Rightarrow f'(x) = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{1}{2} = \frac{1}{\sqrt{1-\frac{x^2}{4}}} \times \frac{1}{2} = \frac{1}{\sqrt{\frac{4-x^2}{4}}} \times \frac{1}{2} = \frac{1}{\sqrt{4-x^2}}$

then $\int \frac{1}{\sqrt{4-x^2}} dx = sin^{-1}\left(\frac{x}{2}\right) + C$

In general $\boxed{\int \frac{1}{\sqrt{a^2-x^2}} dx = sin^{-1}\left(\frac{x}{a}\right) + C}$

Integrals of the form $\int \frac{1}{1+x^2} dx$ and $\int \frac{1}{a^2+x^2} dx$

If $f(x) = tan^{-1}x \Rightarrow f'(x) = \frac{1}{1+x^2}$ then $\boxed{\int \frac{1}{1+x^2} dx = tan^{-1}x + C}$

If $f(x) = tan^{-1}\left(\frac{x}{3}\right)$

$\Rightarrow f'(x) = \frac{1}{1+\left(\frac{x}{3}\right)^2} \times \frac{1}{3} = \frac{1}{1+\frac{x^2}{9}} \times \frac{1}{3} = \frac{1}{\frac{9+x^2}{9}} \times \frac{1}{3} = \frac{1}{\frac{9+x^2}{3}} = \frac{3}{9+x^2}$

then $\int \frac{3}{9+x^2} dx = tan^{-1}\left(\frac{x}{3}\right) + C$ so $\int \frac{1}{9+x^2} dx = \frac{1}{3} tan^{-1}\left(\frac{x}{3}\right) + C$

In general $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right) + C$

Examples: Integrate the following

a) $\int \frac{1}{\sqrt{16-x^2}}dx = sin^{-1}\left(\frac{x}{4}\right) + C$

b) $\int \frac{1}{36+x^2}dx = \frac{1}{6}tan^{-1}\left(\frac{x}{6}\right) + C$

c)
$$\int \frac{1}{\sqrt{25-9x^2}}dx = \int \frac{1}{\sqrt{9\left(\frac{25}{9}-x^2\right)}}dx$$
$$= \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{25}{9}-x^2\right)}}dx$$
$$= \frac{1}{3}sin^{-1}\left(\frac{x}{\frac{5}{3}}\right) + C$$
$$= \frac{1}{3}sin^{-1}\left(\frac{3x}{5}\right) + C$$

d)
$$\int \frac{1}{9+4x^2}dx = \frac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}dx$$
$$= \frac{1}{4} \times \frac{1}{\frac{3}{2}}tan^{-1}\left(\frac{x}{\frac{3}{2}}\right) + C$$
$$= \frac{1}{6}tan^{-1}\left(\frac{2x}{3}\right) + C$$

Exercise 11

1 Integrate

a) $\int \frac{1}{\sqrt{49-x^2}}dx$ b) $\int \frac{1}{49+x^2}dx$ c) $\int \frac{1}{\sqrt{9-x^2}}dx$ d) $\int \frac{1}{100+x^2}dx$

e) $\int \frac{1}{\sqrt{36-25x^2}}dx$ f) $\int \frac{1}{36+25x^2}dx$ g) $\int \frac{3}{\sqrt{36-9x^2}}dx$ h) $\int \frac{2}{25+4x^2}dx$

2 Evaluate

a) $\int_1^{\sqrt{3}} \frac{2}{1+x^2}dx$ b) $\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}dx$ c) $\int_{\frac{1}{2}}^{1} \frac{3}{\sqrt{1-x^2}}dx$ d) $\int_0^3 \frac{1}{9+x^2}dx$

A Variety of Integrals

All of the integrals below have been met before:

$\int \frac{1}{ax+b}dx$	$\int \frac{1}{(ax+b)^n}dx$	$\int \frac{1}{a^2+x^2}dx$	$\int \frac{1}{a^2-x^2}dx$	$\int \frac{bx}{a^2+x^2}dx$	$\int \frac{bx+c}{a^2+x^2}dx$

Examples: Integrate

a) $\int \frac{1}{2x+1}dx = \frac{1}{2}ln\|2x+1\| + C$ b) $\int \frac{1}{1-Q}dQ = -ln\|1-Q\| + C$	These are of the form $\int \frac{1}{ax+b}dx$. This is the standard integral with solution $\frac{1}{a}ln\|ax+b\| + C$
c) $\int \frac{1}{(2x+1)^4}dx = \int (2x+1)^{-4}dx$ $= \frac{(2x+1)^{-3}}{-3\times 2} + C$ $= \frac{-1}{6(2x+1)^3} + C$	This is of the form: $\int (ax+b)^{-n}dx = \frac{(ax+b)^{-n+1}}{a\times(-n+1)} + C$
d) $\int \frac{1}{4+x^2}dx = \frac{1}{2}tan^{-1}\left(\frac{x}{2}\right) + C$	This is the standard integral: $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right) + C$
e) $\int \frac{1}{4-x^2}dx = \frac{1}{4}\int \left(\frac{1}{2+x} + \frac{1}{2-x}\right)dx$ $= \frac{1}{4}\left(ln\|2+x\| - ln\|2-x\|\right) + C$ $= \frac{1}{4}(ln\|2+x\| - ln\|2-x\| + lnA)$ $= \frac{1}{4}ln\left\|\frac{A(2+x)}{2-x}\right\|$	The denominator can be factorised and therefore partial fractions are used to create an integrand we can deal with. Finding partial fractions was covered in Unit 1, the detail has not been shown here.
f) $\int \frac{x}{4+x^2}dx = \frac{1}{2}\int \frac{2x}{4+x^2}dx$ $= \frac{1}{2}\ln(4+x^2) + C$	This is very similar to d) but the solution is very different. The integral is rearranged so that the numerator is the derivative of the denominator. Then: $\int \frac{f'(x)}{f(x)}dx = ln\|f(x)\| + C$

g) $\int \frac{x+2}{4+x^2}dx = \int \frac{x}{4+x^2}dx + \int \frac{2}{4+x^2}dx$ $= \frac{1}{2}\ln(4+x^2) + \ tan^{-1}\left(\frac{x}{2}\right) + C$	This is a combination of d) and f).

The above list of integrals is not exhaustive. The following exercise contains the six types and also questions that require the substitution method from Unit 1 and integrals giving rise to $\sin^{-1}x$.

Exercise 12: Integrate

a) $\int \frac{1}{(1-4x)^2}dx$ b) $\int \frac{2}{1-9x^2}dx$ c) $\int \frac{dx}{9+x^2}$ d) $\int \frac{5}{1-t}dt$

e) $\int \frac{6}{\sqrt{36-Q^2}}dQ$ f) $\int \frac{2x}{9+x^2}dx$ g) $\int \frac{x^4}{(x^5+1)^3}dx$ h) $\int \frac{x+3}{x^2+9}dx$

i) $\int \frac{4}{t(t+3)}dt$ j) $\int \frac{dx}{(4+x)^2}$ k) $\int \frac{5}{2x+3}dx$ l) $\int \frac{x}{\sqrt{1-x^2}}dx$

More Partial Fractions

Expressing a rational function as a sum of partial fractions was covered in Unit 1.
We will apply these techniques to integrating rational functions.

Type 1 Examples: Integrate

a) $\int \frac{x+16}{2x^2x-6}dx$

Let $\frac{x+16}{2x^2x-6} = \frac{A}{2x-3} + \frac{B}{x+2}$

$$x + 16 = A(x+2) + B(2x-3)$$

Set $x = -2$ $\quad 14 = -7B \;\Rightarrow\; B = -2$

Set $x = \frac{3}{2}$ $\quad \frac{35}{2} = \frac{7}{2}A \;\Rightarrow\; A = 5$

The integral becomes

$$\int \left(\frac{5}{2x-3} - \frac{2}{x+2}\right)dx = \frac{5}{2}ln|2x-3| - 2ln|x+2| + C$$

$$= ln(2x-3)^{\frac{5}{2}} - ln(x+2)^2 + lnK$$

$$= ln\frac{K(2x-3)^{\frac{5}{2}}}{(x+2)^2}$$

C and lnK are both constants. It can be useful to write the solution as a single logarithm.

b) $\int \frac{2x^3+7x^2-2x-2}{2x^2+x-6}dx$

By long division $\frac{2x^3+7x^2-2x-2}{2x^2+x-6} = x + 3 + \frac{x+16}{2x^2+x-6}$

As above $\frac{x+16}{2x^2+x-6} = \frac{5}{2x-3} - \frac{2}{x+2}$

The integral becomes

$$\int \left(x + 3 + \frac{5}{2x-3} - \frac{2}{x+2}\right)dx = \frac{1}{2}x^2 + 3x + \frac{5}{2}ln|2x-3| - 2ln|x+2| + C$$

Exercise 13: Integrate

a) $\int \frac{x+8}{(x+2)(x+4)}dx$ b) $\int \frac{x^2}{x^2-4}dx$ c) $\int \frac{x^2-6x-7}{(x-1)(x-2)(x+3)}dx$

d) $\int \frac{x^2-2x-13}{x^2-2x-3}dx$ e) $\int \frac{6x^2+20x-8}{2x^2+5x-3}dx$ f) $\int_2^3 \frac{2}{(x+1)(x-1)}dx$

Type 2 Example: Integrate

$$\int \frac{-8x^2+14x-15}{(2x-1)^2(x+2)}dx$$

Let $\frac{-8x^2+14x-15}{(2x-1)^2(x+2)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x+2}$

$$-8x^2 + 14x - 15 = A(2x-1)(x+2) + B(x+2) + C(2x-1)^2$$

Set $x = \frac{1}{2}$ $\quad -10 = \frac{5}{2}B \quad \Rightarrow \quad B = -4$

Set $x = -2$ $\quad -75 = 25C \quad \Rightarrow \quad C = -3$

Set $x = 0$ $\quad -15 = -2A + 2B + C$

$$A = 2$$

The integral becomes

$$\int \left(\frac{2}{2x-1} - \frac{4}{(2x-1)^2} - \frac{3}{x+2}\right) dx = \int \left(\frac{2}{2x-1} - 4(2x-1)^{-2} - \frac{3}{x+2}\right) dx$$

$$= ln|2x-1| + \frac{4(2x-1)^{-1}}{2} - 3ln|x+2| + C$$

$$= ln|2x-1| + \frac{2}{(2x-1)} - 3ln|x+2| + C$$

Exercise 14: Integrate

a) $\int \frac{3x^2+x+1}{x(x+1)^2}dx$ b) $\int \frac{x^2-2x+10}{(x+2)(x-1)^2}dx$ c) $\int \frac{25}{(x-2)^2(x+1)}dx$

d) $\int \frac{5x+2}{x^2(x+1)}dx$ e) $\int_1^2 \frac{1}{x^2(x+1)}dx$

Type 3 Examples: Integrate

a) $\int \frac{x-1}{(x+1)(x^2+1)}dx$

Let $\frac{x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$$x - 1 = A(x^2+1) + (Bx+C)(x+1)$$

Set $x = -1$ $\quad -2 = 2A \quad \Rightarrow \quad A = -1$

Set $x = 0$ $\quad -1 = A + C \quad \Rightarrow \quad C = 0$

Set $x = 1$ $\quad 0 = 2A + 2B + 2C \quad \Rightarrow \quad B = 1$

The integral becomes

$$\int \left(\frac{-1}{x+1} + \frac{x}{x^2+1}\right) dx = -ln|x+1| + \frac{1}{2}ln(x^2+1) + C$$

b) $\int \frac{1}{(x-2)(x^2+1)}\,dx$

Let $\frac{1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$

$$1 = A(x^2+1) + (Bx+C)(x-2)$$

Set $x = 2$ $\quad 1 = 5A \;\Rightarrow\; A = \frac{1}{5}$

Set $x = 0$ $\quad 1 = A - 2C \;\Rightarrow\; C = -\frac{2}{5}$

Set $x = 1$ $\quad 1 = 2A - B - C \;\Rightarrow\; B = -\frac{1}{5}$

The integral becomes

$$\int \left(\frac{\frac{1}{5}}{x-2} + \frac{-\frac{1}{5}x - \frac{2}{5}}{x^2+1} \right) dx = \frac{1}{5}\int \left(\frac{1}{x-2} + \frac{-x-2}{x^2+1} \right) dx$$

$$= \frac{1}{5}\int \left(\frac{1}{x-2} - \frac{x}{x^2+1} - \frac{2}{x^2+1} \right) dx$$

$$= \frac{1}{5}\left(ln|x-2| - \frac{1}{2} ln(x^2+1) - 2tan^{-1}x \right) + C$$

$$= \frac{1}{5}\, ln|x-2| - \frac{1}{10} ln(x^2+1) - \frac{2}{5} tan^{-1}x + C$$

Exercise 15: Integrate

a) $\int \frac{3x+1}{(x-1)(x^2+1)}\,dx$ b) $\int \frac{3x^2+92x}{(x+6)(x^2+1)}\,dx$ c) $\int \frac{x}{x^4-1}\,dx$

d) $\int \frac{x}{(x+1)(x^2+4)}\,dx$ e) $\int_3^4 \frac{2x+1}{(x-2)(x^2+1)}\,dx$

Integration by Parts

We have already used a formula differentiate a product of two functions $u(x)$ and $v(x)$ i.e. $(uv)' = u'v + uv'$.
Integrating both sides of the above formulae with respect to x gives

$$\int (uv)'dx = \int u'vdx + \int uv'dx$$

i.e. $$uv = \int u'vdx + \int uv'dx.$$

Rearranging gives

$$\int uv'dx = uv - \int u'vdx.$$

This formula enables us to integrate a product of two functions and the method is known as integration by parts.
It is important to pick which function is u and which function is v', care must be taken to ensure that $u'v$ is simple enough to integrate.

Examples: Integrate

a) $\int xcosxdx$

Let $u = x$ and $v' = cosx$

$u' = 1$ and $v = sinx$

$$\int xcosxdx = xsinx - \int sinxdx$$
$$= xsinx + cosx + C$$

Note if we had set it up like this
$u = cosx$ and $v' = x$
$u' = sinx$ and $v = \frac{1}{2}x^2$
The integral becomes

$$\int xcosxdx = \frac{1}{2}x^2cosx - \int \frac{1}{2}x^2sinxdx$$

This time the integral is no easier than the original.

b) $\int xlnxdx$

Let $u = lnx$ and $v' = x$

$u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$

$$\int xlnxdx = \frac{1}{2}x^2lnx - \int \frac{1}{2}x^2 \times \frac{1}{x}dx$$
$$= \frac{1}{2}x^2lnx - \int \frac{1}{2}xdx$$
$$= \frac{1}{2}x^2lnx - \frac{1}{4}x^2 + C$$

c) $\int_2^3 xe^x dx$

Let $u = x$ and $v' = e^x$

$u' = 1$ and $v = e^x$

$\int_2^3 xe^x dx = [xe^x - \int e^x dx]_2^3$

$= [xe^x - e^x]_2^3$

$= 3e^3 - e^3 - (2e^2 - e^2)$

$= 2e^3 - e^2$

Exercise 16: Integrate

a) $\int xsinxdx$ b) $\int xsin3xdx$ c) $\int \sqrt{x}lnxdx$ d) $\int \frac{1}{x^3} lnxdx$

e) $\int xe^x dx$ f) $\int xcos4xdx$ g) $\int_0^{\pi} xcosxdx$

Repeated Applications of Integration by Parts

Examples: Integrate

a) $\int x^2 sinxdx$

Let $u = x^2$ and $v' = sinx$

$u' = 2x$ and $v = -cosx$

$\int x^2 sinxdx = -x^2 cosx + \int 2xcosxdx$

Let $u = 2x$ and $v' = cosx$

$u' = 2$ and $v = sinx$

$\int x^2 sinxdx = -x^2 cosx + 2xsinx - \int 2sinxdx$

$= -x^2 cosx + 2xsinx + 2cosx + C$

b) $\int x^2e^{3x}dx$

Let $u = x^2$ and $v' = e^{3x}$

$u' = 2x$ and $v = \frac{1}{3}e^{3x}$

$$\int x^2e^{3x}dx = \frac{1}{3}x^2e^{3x} - \int \frac{2}{3}xe^{3x}dx$$

Let $u = \frac{2}{3}x$ and $v' = e^{3x}$

$u' = \frac{2}{3}$ and $v = \frac{1}{3}e^{3x}$

$$\int x^2e^{3x}dx = \frac{1}{3}x^2e^{3x} - \left[\frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x}dx\right]$$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \int \frac{2}{9}e^{3x}dx$$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

Exercise 17: Integrate

a) $\int x^2e^{x}dx$ b) $\int x^2sin3xdx$ c) $\int x^2e^{2x}dx$ d) $\int x^2cos2xdx$

e) $\int x^2e^{-x}dx$ f) $\int (5x^3 + 3x)e^{x}dx$ g) $\int_0^1 x^2e^{x}dx$

Integration by Parts Using a Dummy Function

Functions like $lnx, sin^{-1}x, cos^{-1}x, tan^{-1}x$ do not have a standard integral but have a standard derivative. In order to integrate them, we introduce a "dummy" function, namely the number 1.

Examples: Integrate

a) $\int lnxdx = \int 1 \times lnxdx$

Let $u = lnx$ and $v' = 1$

$u' = \frac{1}{x}$ and $v = x$

$$\int lnxdx = xlnx - \int x \times \frac{1}{x}dx$$

$$= xlnx - \int 1dx$$

$$= xlnx - x + C$$

b) $\int sin^{-1}xdx = \int 1 \times sin^{-1}xdx$

Let $u = sin^{-1}x$ and $v' = 1$

$u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$

$\int sin^{-1}xdx = xsin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$

For $\int \frac{x}{\sqrt{1-x^2}}dx$ use a substitution $t = 1 - x^2$

$dt = -2x$

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int -\frac{1}{2t^{\frac{1}{2}}}dt$$
$$= -\frac{1}{2}\int t^{-\frac{1}{2}}dt$$
$$= -\frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= -t^{\frac{1}{2}} + C$$
$$= -\sqrt{1-x^2} + C$$

$$\int sin^{-1}xdx = xsin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$$
$$= xsin^{-1}x + \sqrt{1-x^2} + C$$

Exercise 18: Integrate

a) $\int tan^{-1}xdx$ b) $\int sin^{-1}3xdx$ c) $\int tan^{-1}2xdx$

d) $\int sin^{-1}\frac{1}{2}xdx$ e) $\int cos^{-1}xdx$ f) $\int tan^{-1}\frac{1}{2}xdx$

g) $\int ln2xdx$ h) $\int (lnx)^2dx$ i) $\int_1^e lnxdx$

Integrals that Return to Original Form

There are some integrals which return to the original integral that we started with. We have a special way of dealing with this.

Examples: Integrate

a) $\int e^x sinxdx$

Let $u = e^x$ and $v' = sinx$

$u' = e^x$ and $v = -cosx$

$\int e^x sinxdx = -e^x cosx + \int e^x cosxdx$

Let $u = e^x$ and $v' = cosx$

$u' = e^x$ and $v = sinx$

$\int e^x sinxdx = -e^x cosx + e^x sinx - \int e^x sinxdx$

This is original form.
Treat it like an equation.

$2\int e^x sinxdx = -e^x cosx + e^x sinx$

$\int e^x sinxdx = \frac{1}{2}(-e^x cosx + e^x sinx) + C$

b) $\int e^{2x} cosxdx$

Let $u = e^{2x}$ and $v' = cosx$

$u' = 2e^{2x}$ and $v = sinx$

$\int e^{2x} cosxdx = e^{2x} sinx - \int 2e^{2x} sinxdx$

Let $u = 2e^{2x}$ and $v' = sinx$

$u' = 4e^{2x}$ and $v = -cosx$

$\int e^{2x} cosxdx = e^{2x} sinx - \{-2e^{2x} cosx + \int 4e^{2x} cosxdx\}$

$\int e^{2x} cosxdx = e^{2x} sinx + 2e^{2x} cosx - 4\int e^{2x} cosxdx$

$5\int e^{2x} cosxdx = e^{2x} sinx + 2e^{2x} cosx$

$\int e^{2x} cosxdx = \frac{1}{5}(e^{2x} sinx + 2e^{2x} cosx) + C$

Exercise 19: Integrate

a) $\int e^x sin2xdx$

b) $\int e^{-x} sinxdx$

c) $\int e^{-2x} cos3xdx$

d) $\int e^x cos^2 xdx$

Mixed Integrals

Exercise 20: Integrate

a) $\int \frac{dx}{x^2+16}$ b) $\int \frac{4x}{x^2+16} dx$ c) $\int \frac{dx}{x^2-16}$

d) $\int 4xe^{x^2} dx$ e) $\int x^2 e^{2x} dx$ f) $\int \frac{tan^{-1}x}{1+x^2} dx$

g) $\int \frac{5x-7}{(x+3)(x^2+2)} dx$ h) $\int xsinxdx$ i) $\int sin^2 xcosxdx$

j) $\int \frac{x+1}{x-1} dx$ k) $\int \frac{dx}{(3x-2)^2}$ l) $\int \frac{dx}{4-3x}$

m) $\int \frac{dx}{\sqrt{9-4x^2}}$ n) $\int e^{(3+2x)} dx$ o) $\int tan^{-1} xdx$

p) $\int \frac{x+4}{x^2+16} dx$ q) $\int sec^2 xtan^4 xdx$ r) $\int \frac{x^2}{x+2} dx$

s) $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx, \ let \ \ x = sin\theta$ t) $\int x(x+2)^5 dx, \ let \ u = x+2$

A differential equation is an equation connecting x, y and the differential coefficients $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

For example (1) $y\frac{dy}{dx} + 2xy = x$ and (2) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 4 = sinx$.

The order of the differential equation is the value of the largest differential coefficient i.e. (1) is of the first order and (2) is of second order.

In order to solve a differential equation in the variables x and y, it is necessary to find some function $y = f(x)$ which satisfies the equation.

Let us consider the first order differential equation $\frac{dy}{dx} = 2x + 3$.

We know, by integration, that solutions to this differential equation will be of the form $y = x^2 + 3x + C$.

We say that $y = x^2 + 3x + C$ is the general solution of $\frac{dy}{dx} = 2x + 3$.

If we were to draw the graphs of $y = x^2 + 3x + C$, for various values of C, we would obtain a family of curves, each of which has the property that $\frac{dy}{dx} = 2x + 3$.

A particular solution could be found by choosing values of x and y.

Solving First Order Differential Equations by Variable Separable

Any first order differential equation that can be expressed in the form

$$f(y)\frac{dy}{dx} = g(x)$$

can be solved by separating variables.

Suppose $f(y)\frac{dy}{dx} = g(x)$

Integrating both sides with respect to x

$$\int f(y)\frac{dy}{dx}dx = \int g(x)dx$$

$$\boxed{\int f(y)dy = \int g(x)dx}$$

Examples: Find the general solution of

a) $y\frac{dy}{dx}=\frac{1}{x^2}$

$\int ydy=\int\frac{1}{x^2}dx$

$\frac{y^2}{2}=-\frac{1}{x}+C$

$y^2=-\frac{2}{x}+C$

b) $x^2\frac{dy}{dx}=y+3$

$\int\frac{dy}{y+3}=\int\frac{1}{x^2}dx$

$ln|y+3|=-\frac{1}{x}+C$

c) $(x+2)\frac{dy}{dx}=1$

$\int 1dy=\int\frac{1}{x+2}dx$

$y=ln|x+2|+C$

d) $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$

$\int\frac{dy}{1+y^2}=\int\frac{dy}{1+x^2}$

$tan^{-1}y=tan^{-1}x+C$

Exercise 1: Find the general solution of

a) $(1+x)\frac{dy}{dx}=xy$ b) $\frac{dy}{dx}=x(1-y)^2$ c) $\frac{dy}{dx}=e^xy^2$

d) $x(y-1)\frac{dy}{dx}=2y$ e) $sinxcosy=sinycosx\frac{dy}{dx}$

f) $y-x\frac{dy}{dx}=1+x^2\frac{dy}{dx}$ g) $\frac{dy}{dx}=\frac{2y^2-5}{4xy}$

Examples: Find the particular solution of

a) $\frac{dy}{dx} = x(y-2)$, given $x = 0$ when $y = 5$.

$$\int \frac{dy}{y-2} = \int x dx$$

$$ln|y-2| = \frac{x^2}{2} + C$$

When $x = 0$ and $y = 5$

$$ln3 = 0 + C$$

$$C = ln3$$

$$ln|y-2| = \frac{x^2}{2} + ln3$$

$$ln|y-2| - ln3 = \frac{x^2}{2}$$

$$ln\left|\frac{y-2}{3}\right| = \frac{x^2}{2}$$

$$\frac{y-2}{3} = e^{\frac{x^2}{2}}$$

$$y = 3e^{\frac{x^2}{2}} + 2$$

b) $(1+x^2)\frac{dy}{dx} = 4e^y$, given $x = 0$ when $y = 0$.

$$\int \frac{dy}{e^y} = \int \frac{4}{1+x^2} dx$$

$$-\frac{1}{e^y} = 4tan^{-1}x + C$$

When $x = 0$ and $y = 0$

$$-1 = 0 + C$$

$$C = -1$$

$$-\frac{1}{e^y} = 4tan^{-1}x - 1$$

$$e^y = \frac{1}{1-4tan^{-1}x}$$

$$y = ln\left(\frac{1}{1-4tan^{-1}x}\right)$$

$$y = -ln(1-4tan^{-1}x)$$

Exercise 2: Find the particular solution of

a) $(1 - cos2x)\frac{dy}{dx} = 2sin2x \quad when\ x = \frac{\pi}{4}\ and\ y = 1$

b) $(1 + x^2)\frac{dy}{dx} = 1 + y^2 \quad when\ x = 0\ and\ y = 1$

c) $\frac{dy}{dx} = x(y - 2) \quad when\ x = 0\ and\ y = 5$

d) $\frac{dy}{dx} = \sqrt{1 - y^2} \quad when\ x = \frac{\pi}{6}\ and\ y = 0$

e) $\frac{dy}{dx} = ycosx \quad when\ x = 0\ and\ y = 1$

f) $\frac{dy}{dx} = tanxtany \quad when\ x = \frac{\pi}{4}\ and\ y = \frac{\pi}{4}$

g) $e^x\frac{dy}{dx} = xy^2 \quad when\ y = 1\ and\ x = 0$

Problems Leading to First Order Differential Equations

Examples:

1 The rate at which a radioactive material decays at any instant is proportional to the mass remaining at that instant.
There are 200g of radioactive material to start with.
The half life is 10 days (this means that there will be 100g left after 10 days, 50g left after 20 days, 25g left after 30 days etc)

a) Given that there are x grams of radioactive material left after t days, find a formulae for x in terms of t.

b) Find the mass left after 15 days.

c) Find the time taken for the 200g to decay to 30g.

Solution

a) $\frac{dx}{dt} \propto x$

$$\frac{dx}{dt} = kx$$

$$\int \frac{1}{x}dx = \int kdt$$

$$lnx = kt + C$$

$$x = e^{kt+C}$$

$$x = e^{kt}e^{C}$$

$$x = Ae^{kt}$$

At $t = 0, x = 200$

$200 = Ae^0$

$A = 200$

At $t = 10, x = 100$

$100 = 200e^{10k}$

$0.5 = e^{10k}$

$ln0.5 = 10k$

$k = -0.06931$

$x = 200e^{-0.06931t}$

b) $x = 200e^{-0.06931t}$

$x = 200e^{-0.06931\times 15}$

$x = 70.7g$

c) $x = 200e^{-0.06931t}$

$30 = 200e^{-0.06931t}$

$ln\left(\frac{30}{200}\right) = -0.06931t$

$t = 27.4\ days$

2 At t hours after noon, there are x bacteria in a culture.
The growth of the bacteria is modelled by the differential equation
$\frac{dx}{dt} = \frac{k(n-x)x}{n}$ where n and k are positive constants.

a) Show that the general solution of this equation is $x = \frac{n}{Ae^{-kt}+1}$ where A is an arbitrary constant.

b) Given that there are 200 bacteria at noon, find the relation between n and A.

c) Given also that x tends to 600 as t tends to ∞, find n and A.

Solution

a) $\frac{dx}{dt} = \frac{k(n-x)x}{n}$

$\int \frac{n}{x(n-x)} dx = \int kdt$

Let $\frac{n}{x(n-x)} = \frac{A}{x} + \frac{B}{n-x}$

$n = A(n-x) + Bx$

$Set\ x = 0 \quad n = An \quad \Rightarrow \quad A = 1$

$Set\ x = n \quad n = Bn \quad \Rightarrow \quad B = 1$

Integral becomes $\int\left(\frac{1}{x}+\frac{1}{n-x}\right)dx=\int kdt$

$$lnx-ln|n-x|=kt+C$$

$$ln\left(\frac{x}{n-x}\right)=kt+C$$

$$\frac{x}{n-x}=e^{kt+C}$$

$$\frac{x}{n-x}=Ae^{kt}$$

$$\frac{n-x}{x}=\ Ae^{-kt}$$

$$Axe^{-kt}=n-x$$

$$Axe^{-kt}+x=n$$

$$x(Ae^{-kt}+1)=n$$

$$x=\frac{n}{Ae^{-kt}+1}$$

b) At $t=0,x=200$ $\quad 200=\frac{n}{A+1}$

$$n=200(A+1)$$

c) As $x\to 600,t\to\infty$ $\quad 600=\frac{n}{0+1}$

$$n=600$$

From above $n=200(A+1)$

$$600=200(A+1)$$

$$A=2$$

3 An infectious disease spreads at a rate which is proportional to the product the number uninfected.

Initially, one half of the population is infected and the rate of spread is such that were it to remain constant, the whole population would become infected after 24 days.

Calculate the proportion of the population which is infected after 12 days.

Solution

Let x be the fraction infected and so $1-x$ is the fraction uninfected.

$\frac{dx}{dt}$ is the rate at which the disease spreads.

Hence $\frac{dx}{dt}\propto x(1-x)$

$$\frac{dx}{dt}=kx(1-x)$$

Initially, $x = \frac{1}{2}$ and $\frac{dx}{dt}$ is equal to the constant rate at which the remaining half would become infected in 24 days.

$$\frac{dx}{dt} = \frac{\frac{1}{2}}{24} = \frac{1}{48}$$

Substituting $x = \frac{1}{2}$ and $\frac{dx}{dt} = \frac{1}{48}$ into $\frac{dx}{dt} = kx(1-x)$

$$\frac{1}{48} = k \times \frac{1}{2} \times \frac{1}{2}$$

$$k = \frac{1}{12}$$

$$\frac{dx}{dt} = \frac{1}{12}x(1-x)$$

$$\int \frac{1}{x(1-x)}dx = \int \frac{1}{12}dt$$

Let $\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$

$$1 = A(1-x) + Bx$$

Set $x = 0 \quad 1 = A$

Set $x = 1 \quad 1 = B$

$$\int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \int \frac{1}{12}dt$$

$$lnx - ln(1-x) = \frac{1}{12}t + C$$

When $t = 0, x = \frac{1}{2} \quad ln\frac{1}{2} - ln\frac{1}{2} = 0 + C$

$$C = 0$$

$$lnx - ln(1-x) = \frac{1}{12}t$$

$$ln\left(\frac{x}{1-x}\right) = \frac{1}{12}t$$

$$\frac{x}{1-x} = e^{\frac{1}{12}t}$$

When $t = 12, \quad \frac{x}{1-x} = e$

$$x = e - ex$$

$$x + ex = e$$

$$x = \frac{e}{1+e} = 0.73$$

73% of the population is infected after 12 days.

Exercise 3

1 The rate at which a radioactive material decays at any instant is proportional to the mass remaining at that instant. Given that there are x grams present after t days, the differential equation in x is modelled by $\frac{dx}{dt} = kx$.
The half life of a radioactive material is roughly 25 days.
Find the time taken for 100g of the material to decay to 20g.

2 A pan of water is heated in a kitchen where the temperature is 15^0. When the milk reaches boiling point it is left to cool and after t minutes the temperature of the milk is ϕ degrees.
The rate of cooling is proportional to $\phi - 15$.
The differential equation modelling this situation is $\frac{d\phi}{dt} = k(\phi - 15)$.

a) Find the general solution of the differential equation.

b) After 10 minutes the temperature of the milk was 50^0.
   (i) Calculate the temperature of the milk 5 minutes after it boiled.
   (ii) The milk is required when the temperature is 45^0. Calculate how long this takes after the milk boiled.

3 The population of a small town was 468 in 1980 and 534 in 1990. Assuming that the rate of increase of the population p, is proportional to p,

a) write down a differential equation representing the above information and find the solution to the equation.

b) calculate the population in 2000.

4 The surface area of a pool is 10000m^2 and is partially covered with weeds. At any instant the weeds are increasing in area at a rate proportional to its area at that instant.

a) If the area of the weed is x m^2 formed in t days, form a differential equation.

b) Initially, the area covered in weeds is 100m^2 and after 7 days, the area is 1000m^2. Show that:

$$ln\left(\frac{x}{100}\right) = \frac{1}{7}tln(10)$$

c) Find the area of the pool not covered by weeds after 10.5 days.

d) Find the time t, when the weeds cover half the surface of the pool.

5 A man is given a drug which causes an initial level of 2mg of the drug per litre of his blood. After t hours there are x mg of the drug per litre and it is known that the rate of decrease is x is proportional to x. After 1 hour, $x = 1.6$.

a) Calculate the value of x after 3 hours.

b) Calculate the time after which $x = 0.5$.

6 A container is shaped so that when the depth of water is x cm, the volume of water in the container is $(x^2 + 3x)$ cm^3. Water is poured into the container so that, when the depth of water is x cm, the rate of increase is $(x^2 + 4)$ cm^3/sec.

a) Show that the differential equation for this situation can be modelled by

$$\frac{dx}{dt} = \frac{x^2+4}{2x+3},$$ where t is the time in seconds.

b) Solve the differential equation to obtain t in terms of x, given that initially the container is empty.

7 The motion of a particle is such that its speed at time t is given by

$$\frac{dx}{dt} = \frac{1}{2}(v - v^2)$$

when $t = 0$ and $v = 0.2$.

Express v in terms of t.

8 Heat is supplied to an electric kettle at a rate of 2000 watts, but heat is lost to the surroundings at a rate of 20 watts for every ^{0}C difference between the kettle and that of the surroundings. One watt causes the temperature of the kettle to rise at a rate of 0.02^{0}C per minute.

If the temperature of the surroundings is 15^{0}C and θ^0C is the temperature of the kettle after t minutes, the differential equation modelling this is

$$\frac{d\theta}{dt} = 40 - \frac{2}{5}(\theta - 15)$$

Assuming that $\theta = 15$ when $t = 0$, how long will it take for the temperature to rise from 15^{0}C to 100^{0}C?

9 The gradient of the tangent to a curve is given by $\frac{dy}{dx} = \frac{1+y^2}{tanx}$.

Find the equation of the curve if it passes through the point $\left(\frac{\pi}{2}, 1\right)$.

Integrating Factor Method

Linear differential equations take the form $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x)$ and $Q(x)$ are functions of x.
To solve this type of equation, we have to multiply all terms by the <u>integrating factor</u>, denoted by $\mu(x)$, which converts the whole left hand side of the equation into the derivative of one function.

For $\quad \frac{dy}{dx} + P(x)y = Q(x)$ multiply all terms by $\mu(x)$

$\Rightarrow \quad \mu(x)\frac{dy}{dx} + P(x)\mu(x)y = \mu(x)Q(x)$

Now $\quad \frac{d}{dx}[\mu(x)y] = \mu(x)\frac{dy}{dx} + \frac{d}{dx}[\mu(x)]y$ using the product rule

So we choose $\mu(x)$ such that $\quad \frac{d}{dx}[\mu(x)] = P(x)\mu(x)$

$$\Rightarrow \quad \frac{1}{\mu(x)}\frac{d}{dx}[\mu(x)] = P(x)$$

$$\Rightarrow \quad \frac{d}{dx}[\ln(\mu(x))] = P(x)$$

$$\Rightarrow \quad \ln\big(\mu(x)\big) = \int P(x)dx$$

$$\Rightarrow \quad \mu(x) = e^{\int P(x)dx}$$

The equation $\mu(x)\frac{dy}{dx} + P(x)\mu(x)y = \mu(x)Q(x)$ becomes

$$\mu(x)\frac{dy}{dx} + \frac{d}{dx}[\mu(x)]y = \mu(x)Q(x)$$

$$\Rightarrow \quad \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

$$\Rightarrow \quad \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

$$\Rightarrow \quad \mu(x)y = \int \mu(x)Q(x)\,dx$$

$$\Rightarrow \quad y = \frac{1}{\mu(x)}\int \mu(x)Q(x)\,dx$$

The solution to $\frac{dy}{dx} + P(x)y = Q(x)$ is

$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)\,dx$ where $\mu(x) = e^{\int P(x)dx}$

Examples:

1) Find the general solution of

a) $\frac{dy}{dx}+\frac{y}{x}=1$

$P(x)=\frac{1}{x},\ Q(x)=1,\ \mu(x)=e^{\int\frac{1}{x}dx}=e^{lnx}=x$

The equation becomes $x\frac{dy}{dx}+y=x$

$$\frac{d}{dx}[xy]=x$$

$$xy=\int xdx$$

$$xy=\frac{x^2}{2}+C$$

$$y=\frac{x}{2}+\frac{C}{x}$$

b) $x\frac{dy}{dx}+(x-2)y=x^3$

$x\frac{dy}{dx}+(x-2)y=x^3$ must be written as $\frac{dy}{dx}+\frac{x-2}{x}y=x^2$

$P(x)=\frac{x-2}{x}=1-\frac{2}{x},\ Q(x)=x^2,$

$\mu(x)=e^{\int\left(1-\frac{2}{x}\right)dx}=e^{x-2lnx}=e^{x-lnx^2}=e^xe^{-lnx^2}=e^xe^{ln\frac{1}{x^2}}=\frac{1}{x^2}e^x$

The equation becomes $\frac{1}{x^2}e^x\frac{dy}{dx}+e^x\frac{x-2}{x^3}y=e^x$

$$\frac{d}{dx}\left[\frac{1}{x^2}e^xy\right]=e^x$$

$$\frac{1}{x^2}e^xy=\int e^xdx$$

$$\frac{1}{x^2}e^xy=e^x+\ C$$

$$y=x^2+\frac{Cx^2}{e^x}$$

c) $\frac{dy}{dx}-2y=6e^{-x}$

$P(x)=-2,\ Q(x)=6e^{-x},\ \mu(x)=e^{\int(-2)dx}=e^{-2x}$

The equation becomes $e^{-2x}\frac{dy}{dx}-2e^{-2x}y=6e^{-3x}$

$$\frac{d}{dx}[e^{-2x}y]=6e^{-3x}$$

$$e^{-2x}y=\int 6e^{-3x}dx$$

$$e^{-2x}y = \frac{6e^{-3x}}{-3} + C$$

$$y = \frac{6e^{-3x}}{-3e^{-2x}} + \frac{C}{e^{-2x}}$$

$$y = \frac{-2}{e^x} + Ce^{2x}$$

d) $(1+x^2)\frac{dy}{dx} - xy = x(1+x^2)$

$(1+x^2)\frac{dy}{dx} - xy = x(1+x^2)$must be written as $\frac{dy}{dx} - \frac{x}{(1+x^2)}y = x$

$P(x) = -\frac{x}{(1+x^2)},\ Q(x) = x,$

$\mu(x) = e^{\int\left(-\frac{x}{(1+x^2)}\right)dx} = e^{-\frac{1}{2}\ln(1+x^2)} = e^{\ln(1+x^2)^{-\frac{1}{2}}} = \frac{1}{\sqrt{1+x^2}}$

The equation becomes $\frac{1}{\sqrt{1+x^2}}\frac{dy}{dx} - \frac{1}{\sqrt{1+x^2}}\frac{x}{(1+x^2)}y = \frac{1}{\sqrt{1+x^2}}x$

$$\frac{d}{dx}\left[\frac{1}{\sqrt{1+x^2}}y\right] = \frac{x}{\sqrt{1+x^2}}$$

$\frac{1}{\sqrt{1+x^2}}y = \int \frac{x}{\sqrt{1+x^2}}dx$ by substitution

$$\frac{1}{\sqrt{1+x^2}}y = \sqrt{1+x^2} + C$$

$$y = 1 + x^2 + C\sqrt{1+x^2}$$

2 Find the particular solution of $\frac{dy}{dx} + y = 2x + 4$ given y =1 when $x = 0$

$P(x) = 1,\ Q(x) = 2x + 4,\ \mu(x) = e^{\int 1dx} = e^x$

The equation becomes $e^x\frac{dy}{dx} + e^x y = (2x+4)e^x$

$$\frac{d}{dx}[e^x y] = (2x+4)e^x$$

$$e^x y = \int(2x+4)e^x dx$$

Using integration by parts $e^x y = (2x+4)e^x - \int 2e^x + C$

$$e^x y = (2x+4)e^x - 2e^x + C$$

$$y = 2x + 4 - 2 + \frac{C}{e^x}$$

$$y = 2x + 2 + \frac{C}{e^x}$$

When $x = 0$ and $y = 1$ $1 = 2\times 0 + 2 + \frac{C}{e^0} \Rightarrow C = -1$

$$y = 2x + 2 - \frac{1}{e^x}$$

Exercise 4:

1 Find the general solutions of these linear differential equations

a) $(x+1)\frac{dy}{dx} - y = (x+1)^2$

b) $\frac{dy}{dx} - ytanx = sinxcosx$

c) $tanx\frac{dy}{dx} + 2y = xcosecx$

d) $\frac{dy}{dx} + \frac{2y}{1-x^2} = 1 - x$

e) $x(x+1)\frac{dy}{dx} - y = x^3e^x$

f) $\frac{dy}{dx} + y = 5cos2x$

g) $(1-x)\frac{dy}{dx} + xy = (1-x)^2e^{-x}$

h) $\frac{dy}{dx} + \frac{x+1}{x}y = e^{-x}$

i) $x(x+1)\frac{dy}{dx} + y = x(x+1)^2e^{-x}$

j) $\frac{dy}{dx} + ycotx = cosx$

2 Find the particular solutions of these linear differential equations

a) $x\frac{dy}{dx} + 2y = x^3$ when $x = 1$ and $y = 2$

b) $(1+x)\frac{dy}{dx} + 2y = x^2$ when $x = 0$ and $y = 0$

c) $sinx\frac{dy}{dx} - ycosx = 1$ when $x = \frac{\pi}{2}$ and $y = 3$

d) $x(x+1)\frac{dy}{dx} + y = (x+1)^2e^x$ when $x = 1$ and $y = 0$

e) $x\frac{dy}{dx} = y + x^2(sinx + cosx)$ when $x = \frac{\pi}{2}$ and $y = 0$

Simple Second Order Differential Equations

To solve equations of the form $\frac{d^2y}{dx^2} = f(x)$ use two successive integrations.

Example Solve $\frac{d^2y}{dx^2} = \frac{1}{x^2}$

$\frac{dy}{dx} = \int \frac{1}{x^2}dx$

$\frac{dy}{dx} = -\frac{1}{x} + C$

$y = \int \left(-\frac{1}{x} + C\right)dx$

$y = -lnx + Cx + D$ where C and D are arbitrary constants.

Linear Second Order Differential Equations

Second order linear equations have the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b and c are constants.
We look for solutions of the form $y = Ae^{Dx}$.

Homogenous Second Order Differential Equations

Let us first consider the case when $f(x) = 0$ i.e. $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.
If we try $y = Ae^{Dx}$ as a solution of this equation, then

$$\frac{dy}{dx} = DAe^{Dx} \text{ and } \frac{d^2y}{dx^2} = D^2Ae^{Dx}.$$

Substituting into the equation gives

$$aD^2Ae^{Dx} + bDAe^{Dx} + cAe^{Dx} = 0$$
$$Ae^{Dx}(aD^2 + bD + c) = 0$$

Thus $y = Ae^{Dx}$ is a solution provided $aD^2 + bD + c = 0$ solves to give one of the following:

a) Two distinct real roots ($b^2 - 4ac > 0$).
If $D = D_1$ and D_2, then the general solution is
$$y = Ae^{D_1x} + Be^{D_2x}$$

b) Equal roots ($b^2 - 4ac = 0$).
If $D = D_1$ (twice), then the general solution is
$$y = Ae^{D_1x} + Bxe^{D_1x} = (A + Bx)e^{D_1x}$$

c) Complex roots ($b^2 - 4ac < 0$).
If $D = D_1$ and D_2, where D_1 and D_2 are of the form $p \pm qi$ then the general solution is
$$y = e^{px}(Acosqx + Bsinqx)$$

The equation $aD^2 + bD + c = 0$ is called the Auxiliary Equation (A.E.)

Examples

1 Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

The auxiliary equation is $D^2 + D - 6 = 0$

$$(D+3)(D-2) = 0$$

$$D = -3 \quad D = 2$$

The general solution is $y = Ae^{-3x} + Be^{2x}$

2 Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

The auxiliary equation is $D^2 + 4D + 4 = 0$

$$(D+2)(D+2) = 0$$

$$D = -2$$

The general solution is $y = e^{-2x}(A + Bx)$

3 Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

The auxiliary equation is $D^2 + 2D + 5 = 0$

$$D = \frac{-2 \pm \sqrt{-16}}{2}$$

$$D = -1 \pm 2i$$

The general solution is $y = e^{-x}(Acos2x + Bsin2x)$

Exercise 5:

1 Find the general solutions of these linear differential equations

a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ b) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

c) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 0$ d) $2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 9y = 0$

2 Find the general solutions of these linear differential equations

a) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ b) $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$

c) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ d) $\frac{d^2y}{dx^2} + 2n\frac{dy}{dx} + n^2y = 0$

3 Find the general solutions of these linear differential equations

a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ b) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 0$

c) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$ d) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Non-Homogenous Second Order Differential Equations

Let us first consider the case when $f(x) = 0$ i.e. $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$.
Firstly we find the general solution as above. This is called the complimentary function (C.F.).
We then find the a particular solution of the given equation. This is called the particular integral (P.I.).
The general solution (G.S.) of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ is found by adding the complementary function and the particular integral.
For the particular integral we choose a solution of the same form as $f(x)$.

a) If $f(x)$ is a polynomial function of degree n then try

$$y = C_n x^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1$$

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then substitute into the given equation.
By comparing coefficients we can find $C_n, C_{n-1} \ldots C_1$.

b) If $f(x)$ is an exponential function then try

$$y = Ce^{px}$$

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then substitute into the given equation.
By comparing coefficients we can find C.

c) If $f(x)$ is a trignometric function then try

$$y = C_1 cospx + C_2 sinpx$$

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then substitute into the given equation.
By comparing coefficients we can find C_1 and C_2.

Examples:

1 Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x + 4$

The auxiliary equation is $D^2 + 3D + 2 = 0$

$$(D+1)(D+2) = 0$$

$$D = -1 \quad D = -2$$

The complementary function is $y = Ae^{-x} + Be^{-2x}$

For the particular integral try $y = Cx + D$

$$\frac{dy}{dx} = C$$

$$\frac{d^2y}{dx^2} = 0$$

Substitute into the equation gives $0 + 3C + 2(Cx + D) = 4x + 4$

$$2Cx + (3C + 2D) = 4x + 4$$

Comparing coefficients $2C = 4 \quad \Rightarrow \quad C = 2$

$$3C + 2D = 4 \quad \Rightarrow \quad 2D = -2 \quad \Rightarrow \quad D = -1$$

Particular integral is $y = 2x - 1$

The general solution is $y = Ae^{-x} + Be^{-2x} + 2x - 1$

2 Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = x^2 + 4x - 5$

The auxiliary equation is $D^2 - 2D - 3 = 0$

$$(D-3)(D+1) = 0$$

$$D = 3 \quad D = -1$$

The complementary function is $y = Ae^{3x} + Be^{-x}$

For the particular integral try $y = Cx^2 + Dx + E$

$$\frac{dy}{dx} = 2Cx + D$$

$$\frac{d^2y}{dx^2} = 2C$$

Substitute into the equation gives

$$2C - 2(2Cx + D) - 3(Cx^2 + Dx + E) = x^2 + 4x - 5$$

$$-3Cx^2 + (-4C - 3D)x + (2C - 2D - 3E) = x^2 + 4x - 5$$

Comparing coefficients $-3C = 1 \;\Rightarrow\; C = -\frac{1}{3}$

$$-4C - 3D = 4 \;\Rightarrow\; -3D = \frac{8}{3} \;\Rightarrow\; D = -\frac{8}{9}$$

$$2C - 2D - 3E = -5 \;\Rightarrow\; -3E = -\frac{55}{9} \;\Rightarrow\; E = \frac{55}{27}$$

Particular integral is $y = -\frac{1}{3}x^2 - \frac{8}{9}\,x + \frac{55}{27}$

The general solution is $y = Ae^{3x} + Be^{-x} - \frac{1}{3}x^2 - \frac{8}{9}\,x + \frac{55}{27}$

3 Solve $\frac{d^2y}{dx^2} - y = 2e^{3x}$

The auxiliary equation is $D^2 - 1 = 0$

$$(D - 1)(D + 1) = 0$$

$$D = 1 \quad D = -1$$

The complementary function is $y = Ae^{x} + Be^{-x}$

For the particular integral try $y = Ce^{3x}$

$$\frac{dy}{dx} = 3Ce^{3x}$$

$$\frac{d^2y}{dx^2} = 9Ce^{3x}$$

Substitute into the equation gives

$$9Ce^{3x} - Ce^{3x} = 2e^{3x}$$

$$8Ce^{3x} = 2e^{3x}$$

Comparing coefficients $8C = 2 \;\Rightarrow\; C = \frac{1}{4}$

Particular integral is $y = \frac{1}{4}e^{3x}$

The general solution is $y = Ae^{x} + Be^{-x} + \frac{1}{4}e^{3x}$

4 Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 3e^{-2x}$

The auxiliary equation is $D^2 + 2D + 2 = 0$

$$D = \frac{-2 \pm \sqrt{-4}}{2}$$
$$D = -1 \pm i$$

The general solution is $y = e^{-x}(Acosx + Bsinx)$

For the particular integral try $y = Ce^{-2x}$

$$\frac{dy}{dx} = -2Ce^{-2x}$$
$$\frac{d^2y}{dx^2} = 4Ce^{-2x}$$

Substitute into the equation gives

$$4Ce^{-2x} + 2(-2Ce^{-2x}) + 2Ce^{-2x} = 3e^{-2x}$$
$$2Ce^{-2x} = 3e^{-2x}$$

Comparing coefficients $2C = 3 \;\; \Rightarrow \;\; C = \frac{3}{2}$

Particular integral is $y = \frac{3}{2}e^{-2x}$

The general solution is $y = e^{-x}(Acosx + Bsinx) + \frac{3}{2}e^{-2x}$

5 Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 30sin2x$

The auxiliary equation is $D^2 + 6D + 10 = 0$

$$D = \frac{-6 \pm \sqrt{-4}}{2}$$
$$D = -3 \pm i$$

The general solution is $y = e^{-3x}(Acosx + Bsinx)$

For the particular integral try $y = Ccos2x + Dsin2x$

$$\frac{dy}{dx} = -2Csin2x + 2Dcos2x$$
$$\frac{d^2y}{dx^2} = -4Ccos2x - 4Dsin2x$$

Substitute into the equation gives

$$-4C\cos 2x - 4D\sin 2x + 6(-2C\sin 2x + 2D\cos 2x) + 10(C\cos 2x + D\sin 2x) = 30\sin 2x$$

$$(6C + 12D)\cos 2x + (-12C + 6D)\sin 2x = 30\sin 2x$$

Comparing coefficients $6C + 12D = 0$

$$-12C + 6D = 30$$

$\Rightarrow \quad C = -2$ and $D = 1$

Particular integral is $y = -2\cos 2x + \sin 2x$

The general solution is $y = e^{-3x}(A\cos x + B\sin x) - 2\cos 2x + \sin 2x$

6 Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x + \sin x$

The auxiliary equation is $D^2 - 6D + 9 = 0$

$$(D - 3)^3 = 0$$

$D = 3$ twice

The general solution is $y = Ae^{3x} + Bxe^{3x}$

For the particular integral try $y = Ce^x + D\cos x + E\sin x$

$$\frac{dy}{dx} = Ce^x - D\sin x + E\cos x$$

$$\frac{d^2y}{dx^2} = Ce^x - D\cos x - E\sin x$$

Substitute into the equation gives

$$Ce^x - D\cos x - E\sin x - 6(Ce^x - D\sin x + E\cos x) + 9(Ce^x + D\cos x + E\sin x) = e^x + \sin x$$

$$4Ce^x + (8D - 6E)\cos x + (8E + 6D)\sin x = e^x + \sin x$$

Comparing coefficients $4C = 1 \quad \Rightarrow \quad C = \frac{1}{4}$

$$8D - 6E = 0$$

$$6D + 8E = 1$$

$\Rightarrow \quad D = \frac{3}{50}$ and $E = \frac{2}{25}$

Particular integral is $y = \frac{1}{4}e^x + \frac{3}{50}\cos x + \frac{2}{25}\sin x$

The general solution is $y = Ae^{3x} + Bxe^{3x} + \frac{1}{4}e^x + \frac{3}{50}\cos x + \frac{2}{25}\sin x$

7 Solve $\frac{d^2y}{dx^2} - y = 2e^x$

The auxiliary equation is $D^2 - 1 = 0$

$$(D+1)(D-1) = 0$$

$$D = 1 \quad D = -1$$

The general solution is $y = Ae^x + Be^{-x}$

Note: If the terms in the C.F. also appear in the P.I. then an extra x is required in the P.I.

For the particular integral try $y = Cxe^x$

$$\frac{dy}{dx} = Ce^x + Cxe^x$$

$$\frac{d^2y}{dx^2} = 2Ce^x + Cxe^x$$

Substitute into the equation gives

$$2Ce^x + Cxe^x - Cxe^x = 2e^x$$

$$2Ce^x = 2e^x$$

Comparing coefficients $2C = 2 \Rightarrow C = 1$

Particular integral is $y = xe^x$

The general solution is $y = Ae^x + Be^{-x} + xe^x$

Exercise 6:

1 Find the general solution of these linear differential equations

a) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = x^3$ b) $\frac{d^2y}{dx^2} - y = 2 - 5x$

c) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 32x^2$ d) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 1 + x^2$

2 Find the general solution of these linear differential equations

a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}$ b) $4\frac{d^2y}{dx^2} + 13\frac{dy}{dx} + 9y = 7e^{-2x}$

c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x$ d) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{-2x}$

3 Find the general solution of these linear differential equations

a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = sinx$ b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 10cos2x$

c) $4\frac{d^2y}{dx^2} + y = 4sinx$ d) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 26cos3x$

4 Find the general solution of these linear differential equations

a) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x - e^{2x}$ b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^x + e^{-x}$

c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = x - e^x$ d) $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = e^x - 2cos2x$

5 Find the general solution of these linear differential equations

a) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 3e^{-2x}$ b) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2e^x$

6 Find the particular solution of these linear differential equations

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = sinx$ for which $y = 0$, $\frac{dy}{dx} = 0$ when $x = 0$

7 Find the particular solution of these linear differential equations

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 5e^{3x}$ for which $y = 1$, $\frac{dy}{dx} = -6$ when $x = 0$

Answers

Systems of Equations

Ex1:1a) $x = 5, y = -\frac{7}{2}, z = -\frac{7}{2}$ b) x=5, y=3, z=7 c) x=5, y=2, z=1 d) x=3, y=1, z=-2
e) x=0, y=0, z=0 2a) a+b+c=2, 4a+2b+c=7, 9a+3b+c=14 b)a=1, b=2, c=-1 c) $y=x^2 + 2x - 1$
3a) -4g-2f+c=-5, -2g+4f+c=-5, 12g+6f+c=-45 b) g=3, f=1, c=-15; $x^2+y^2-6x+2y-15=0$ c) r=5

Ex2:2a) $x = 3(1+\lambda), y = 10 + 11\lambda, z = \lambda$ b) $x = -\frac{7}{3}\lambda, y = -\frac{1}{3}(3+2\lambda), z = \lambda$
c) $x = \frac{1}{5}(5+3\lambda), y = \frac{1}{5}(4\lambda - 5), z = \lambda$
3 a≠-4; x=2, y=-1, z=0
4 a=-8; $x = \frac{1}{5}(6-\lambda), y = \frac{1}{5}(2\lambda - 7), z = \lambda$
5a) a≠-9 b) a=9, b=7
6a) a=-5, b=-11 b) $x = 3\lambda - 5, y = \lambda, z = 5\lambda - 11$

Ex3 1a) 2x4 b) 3x2 c) 4x2 2) A=C, F=G, H=K 3a) x=4, y=-3 b) x=4, y=-1 c) x=5, y=2
d) x=2, y=-3 4a) $\begin{pmatrix} 3 & 5 \\ 1 & 4 \\ 4 & 0 \\ 2 & 7 \end{pmatrix}$, 4x2 b) $\begin{pmatrix} 2 & 4 & 1 \\ -1 & 8 & -2 \end{pmatrix}$, 2x3 c) $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}$, 2x4
5) Proof 6) x=5, y=-4

Ex4 1a) $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ b) $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ c) $\begin{pmatrix} 9a \\ -2b \end{pmatrix}$ d) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ e) $(3 \quad 9)$ f) $(-3 \quad 5)$ g) $\begin{pmatrix} 3 & 3 \\ 4 & 6 \end{pmatrix}$ h) $\begin{pmatrix} 3 & 1 & 4 \\ -2 & 2 & -4 \end{pmatrix}$
2a) $\begin{pmatrix} 4 & 6 \\ 6 & 1 \end{pmatrix}$ b) $\begin{pmatrix} -1 & 7 \\ 6 & -2 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 9 \\ 7 & -3 \end{pmatrix}$ d) $\begin{pmatrix} 2 & 9 \\ 7 & -3 \end{pmatrix}$, yes it is true 3) A=-B 4) Proof

Ex5 1a) $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ b) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ c) $\begin{pmatrix} -5a \\ 4b \end{pmatrix}$ d) $\begin{pmatrix} 4u \\ -6v \end{pmatrix}$ e) $(1 \quad 1)$ f) $(7 \quad -11)$ g) $\begin{pmatrix} -1 & -3 \\ -4 & -4 \end{pmatrix}$
h) $\begin{pmatrix} 1 & 5 & -2 \\ 12 & 0 & 4 \end{pmatrix}$ 2a) $\begin{pmatrix} -1 & 5 \\ 3 & 5 \end{pmatrix}$ b) $\begin{pmatrix} 6 & 4 \\ 2 & 4 \end{pmatrix}$ c) $\begin{pmatrix} 4 & 7 \\ 2 & 5 \end{pmatrix}$ d) $\begin{pmatrix} 3 & -1 \\ 3 & 3 \end{pmatrix}$ e) $\begin{pmatrix} 7 & -1 \\ -1 & -1 \end{pmatrix}$ f) $\begin{pmatrix} 4 & 0 \\ -4 & -4 \end{pmatrix}$
g) $\begin{pmatrix} 5 & 9 \\ 5 & 9 \end{pmatrix}$ h) $\begin{pmatrix} 7 & -1 \\ -1 & -1 \end{pmatrix}$ 3a) $\begin{pmatrix} -1 & 2 \\ 2 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -4 \\ -1 & -3 \end{pmatrix}$ 4a) $\begin{pmatrix} 4 & 0 \\ 4 & 9 \\ 9 & 12 \end{pmatrix}$ b) $\begin{pmatrix} -1 & -4 \\ 0 & 5 \\ 7 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 0 & -2 \\ 3 & 9 \\ 12 & 7 \end{pmatrix}$
d) $\begin{pmatrix} 0 & -2 \\ 3 & 9 \\ 12 & 7 \end{pmatrix}$

Ex6 1a) $\begin{pmatrix} 6 & 8 \\ -2 & -4 \end{pmatrix}$ b) $\begin{pmatrix} 9 & 12 \\ -3 & -6 \end{pmatrix}$ c) $\begin{pmatrix} -15 & -20 \\ 5 & 10 \end{pmatrix}$ d) $\begin{pmatrix} -3 & -4 \\ 1 & 2 \end{pmatrix}$ 2a) $\begin{pmatrix} 1 & 5 & -2 \\ -2 & -5 & 3 \end{pmatrix}$
b) $\begin{pmatrix} 10 & 6 & 8 \\ 12 & 10 & 6 \end{pmatrix}$ c) $\begin{pmatrix} 6 & 8 & 2 \\ 4 & 0 & 6 \end{pmatrix}$ d) $\begin{pmatrix} 4 & -2 & 6 \\ 8 & 10 & 0 \end{pmatrix}$ e) $\begin{pmatrix} 10 & 6 & 8 \\ 12 & 10 & 6 \end{pmatrix}$ f) $\begin{pmatrix} 18 & 24 & 6 \\ 12 & 0 & 18 \end{pmatrix}$ g) $\begin{pmatrix} 18 & 24 & 6 \\ 12 & 0 & 18 \end{pmatrix}$
h) $\begin{pmatrix} 16 & -8 & 24 \\ 32 & 40 & 0 \end{pmatrix}$ i) $\begin{pmatrix} 16 & -8 & 24 \\ 32 & 40 & 0 \end{pmatrix}$ 3a) $\begin{pmatrix} -2 & -7 \\ 18 & -1 \end{pmatrix}$ b) $\begin{pmatrix} 20 & -15 \\ 7 & 10 \end{pmatrix}$ c) $\begin{pmatrix} 26 & -19 \\ 8 & 13 \end{pmatrix}$ d) $\begin{pmatrix} 44 & -16 \\ -22 & 22 \end{pmatrix}$
4a) $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 3 & 2 \\ -1 & 3 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$ d) $\begin{pmatrix} 4 & -3 \\ -4 & -2 \end{pmatrix}$ 5a) $\begin{pmatrix} 1 & 8 & -18 \\ 5 & -1 & -7 \end{pmatrix}$ b) $\begin{pmatrix} 7 & -6 \\ 1 & -4 \end{pmatrix}$
6) p=-1, q=4, r=3, s=-2

Ex7 1a) (11) b) (26) c) (9) d) (13) e) (15) f) (8x-5y-z) g) $\begin{pmatrix}9\\8\end{pmatrix}$ h) $\begin{pmatrix}10\\4\end{pmatrix}$ i) $\begin{pmatrix}7\\5\end{pmatrix}$ j) $\begin{pmatrix}7\\5\end{pmatrix}$ k) $\begin{pmatrix}-7\\7\end{pmatrix}$ l) $\begin{pmatrix}4\\6\end{pmatrix}$ m) $\begin{pmatrix}1\\22\end{pmatrix}$ n) not possible o) (26) p) $\begin{pmatrix}11\\13\\7\end{pmatrix}$ q) not possible r) $\begin{pmatrix}9\\8\\7\end{pmatrix}$ s) $\begin{pmatrix}1\\0\end{pmatrix}$ 2a) x=4, y=-4 b) x=3, y=-1 c) x=8, y=5 d) x=2, y=-1 3ai) $\begin{pmatrix}8 & 5\\14 & 15\end{pmatrix}$ ii) $\begin{pmatrix}19 & 13\\2 & 4\end{pmatrix}$ AB≠BA bi) $\begin{pmatrix}17\\11\end{pmatrix}$ ii) $\begin{pmatrix}17\\11\end{pmatrix}$ A(BC)=(AB)C c) Proof 4) $\begin{pmatrix}4 & -5\\25 & -1\end{pmatrix}$ $\begin{pmatrix}-13 & -14\\70 & -27\end{pmatrix}$ 5) $\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ $\begin{pmatrix}-4 & 8\\-2 & 4\end{pmatrix}$
6a) $\begin{pmatrix}1 & -2\\-3 & 6\end{pmatrix}$ b) $\begin{pmatrix}1 & -2\\-3 & 6\end{pmatrix}$ c) $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ d) $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ e) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ f) $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$
7) a=1, b=-1, c=-2, d=3 8) p=$\frac{3}{2}$, q=-$\frac{1}{2}$, r=-2, s=1 9) p=5, q=2 10) p=-2, q=13
11) $\begin{pmatrix}5 & 0 & 5\\12 & 1 & 13\\8 & -1 & 7\end{pmatrix}$ $\begin{pmatrix}8 & 10\\2 & 5\end{pmatrix}$ 12) $\begin{pmatrix}4 & 1 & 5\\3 & -1 & 5\\7 & 2 & 8\end{pmatrix}$ $\begin{pmatrix}7 & 4 & 12\\-1 & -2 & -2\\4 & 0 & 6\end{pmatrix}$ 13) $\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$ $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$
14) p=27, q=-54 15) p=5, q=-12

Ex8 1a) 13 b) -5 c) -6 d) 0 2) Proofs 3a) 1 b) cos4ϴ c) ln2ln$\frac{6}{25}$ 4a) 25 b) 14 c) 51 d) 0 e) 1428 f) -320 5) Proofs 6) Proof

Ex9 1) Proofs 2a) $\begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix}$ b) $\begin{pmatrix}5 & -3\\-3 & 2\end{pmatrix}$ c) $\begin{pmatrix}7 & -3\\-9 & 4\end{pmatrix}$ d) $\begin{pmatrix}4 & 5\\7 & 9\end{pmatrix}$ e) $\begin{pmatrix}-4 & 7\\-3 & 5\end{pmatrix}$
f) $\begin{pmatrix}cos\theta & -sin\theta\\sin\theta & cos\theta\end{pmatrix}$

Ex10 1a) $\begin{pmatrix}\frac{3}{2} & -\frac{1}{2}\\-2 & 1\end{pmatrix}$ b) $\begin{pmatrix}-9 & 4\\16 & -7\end{pmatrix}$ c) does not exist d) $\begin{pmatrix}3 & -\frac{7}{3}\\-2 & \frac{5}{3}\end{pmatrix}$ e) $\begin{pmatrix}\frac{1}{2} & 2\\\frac{1}{2} & 1\end{pmatrix}$ f) $\begin{pmatrix}0 & 1\\1 & -1\end{pmatrix}$
2a) $\begin{pmatrix}\frac{1}{2} & -\frac{3}{2}\\0 & 1\end{pmatrix}$ b) $\begin{pmatrix}3 & -5\\-1 & 2\end{pmatrix}$ c) $\begin{pmatrix}\frac{3}{2} & -\frac{19}{2}\\-\frac{1}{2} & \frac{7}{2}\end{pmatrix}$ d) $\begin{pmatrix}3 & -\frac{11}{2}\\-1 & 2\end{pmatrix}$ e) $\begin{pmatrix}\frac{3}{2} & -\frac{19}{2}\\-\frac{1}{2} & \frac{7}{2}\end{pmatrix}$ f) $\begin{pmatrix}3 & -\frac{11}{2}\\-1 & 2\end{pmatrix}$

Ex11 1) $\begin{pmatrix}\frac{9}{8} & -\frac{3}{2} & \frac{29}{8}\\-\frac{1}{8} & \frac{1}{2} & -\frac{13}{8}\\-\frac{3}{2} & \frac{1}{2} & -\frac{7}{8}\end{pmatrix}$ 2) $\begin{pmatrix}11 & -12 & -7\\-8 & 9 & 5\\7 & -8 & -4\end{pmatrix}$ 3) $\begin{pmatrix}0 & \frac{1}{2} & 0\\0 & \frac{1}{2} & -1\\\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\end{pmatrix}$ 4) $\begin{pmatrix}\frac{1}{2} & -\frac{1}{6} & -\frac{1}{6}\\-\frac{1}{6} & \frac{1}{2} & \frac{5}{18}\\-\frac{2}{3} & \frac{7}{9} & \frac{1}{9}\end{pmatrix}$
5) $\begin{pmatrix}-\frac{1}{12} & -\frac{1}{3} & -\frac{1}{6}\\-\frac{1}{6} & -\frac{1}{18} & \frac{5}{18}\\-\frac{2}{3} & \frac{7}{9} & \frac{1}{9}\end{pmatrix}$ 6) $\begin{pmatrix}-\frac{1}{3} & \frac{11}{57} & \frac{2}{19}\\1 & 1\frac{14}{19} & \frac{1}{19}\\-\frac{4}{3} & \frac{65}{57} & -\frac{2}{19}\end{pmatrix}$

Ex12 1) x=8, y=3 2) x=3, y=-2 3) x=4, y=-3 4) x=2, y=-3 5) x=4, y=-1 6) x=3, y=2

Ex13 1) x=5, y=3, z=7 2) x=5, y=2, z=1 3) x=3, y=1, z=-2 4) x=$\frac{7}{6}$, y=-$\frac{4}{3}$, z=0

Ex14 1a) $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ b) $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ c) $\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$ d) $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ e) $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ f) $\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}$ 2) Proofs

Vectors

Ex1 1a) Number, 8 b) Vector, $5\underline{i}+7\underline{j}+9\underline{k}$ c) Vector, $4\underline{i}+4\underline{k}$ d) Number, 45 2) AB=$\sqrt{38}$ BC=$3\sqrt{6}$ CA=$\sqrt{10}$ cosA=$-\frac{3}{2\sqrt{95}}$ cosB=$\frac{41}{6\sqrt{57}}$ cosC=$\frac{13}{6\sqrt{15}}$ Area=$\frac{1}{2}\sqrt{371}$ 3) Proof 4) $\frac{1}{\sqrt{2}}\underline{i}+\frac{1}{\sqrt{2}}\underline{j}$, $\frac{1}{\sqrt{2}}\underline{i}-\frac{1}{\sqrt{2}}\underline{j}$

5a) $\frac{-3+2\sqrt{2}}{6}\underline{i}+\frac{\sqrt{2}}{6}\underline{j}+\frac{3+2\sqrt{2}}{6}\underline{k}$, $\frac{-3-2\sqrt{2}}{6}\underline{i}-\frac{\sqrt{2}}{6}\underline{j}+\frac{3-2\sqrt{2}}{6}\underline{k}$ b) Proof

Ex2 1a) $-10\underline{i}+7\underline{j}-16\underline{k}$ b) $5\underline{i}-5\underline{k}$ c) -20 d) -15 e) -125 2a) $-11\underline{i}+14\underline{j}+\underline{k}$ b) 2 c) -2, 3) Proof 4) $5\underline{i}+31\underline{j}+14\underline{k}$ 5a) $-3\underline{j}-3\underline{k}$ b) $-3\underline{j}-3\underline{k}$ c) Proof

6) $-\frac{1}{13}\left(3\underline{i}+4\underline{j}+12\underline{k}\right)$, $\frac{1}{13}\left(3\underline{i}+4\underline{j}+12\underline{k}\right)$ 7) $11\frac{1}{2}$ 8) Proof

Ex3 1) $\frac{x-5}{3}=\frac{y+2}{1}=\frac{z-6}{4}$, $\underline{x}=\begin{pmatrix}5\\-2\\6\end{pmatrix}+t\begin{pmatrix}3\\1\\4\end{pmatrix}$ 2) $\frac{x-2}{7}=\frac{y+3}{3}=\frac{z+7}{-2}$, $\underline{x}=\begin{pmatrix}2\\-3\\-7\end{pmatrix}+t\begin{pmatrix}7\\3\\-2\end{pmatrix}$

3a) $\frac{x-3}{2}=\frac{y-2}{-15}=\frac{z+7}{3}$ b) $\frac{x-12}{2}=\frac{y-7}{2}=\frac{z-1}{1}$ c) $\frac{x-3}{3}=\frac{y}{0}=\frac{z}{5}$ d) $\frac{x}{-5}=\frac{y}{2}=\frac{z}{-3}$

Ex4 1a) 3x-2y+7z=18 b) 2x-y-z=9 2a) x-y-2z=5 b) 37x-10y+23z=70 3a) x+3y-2z=-1 b) 35x+21y+15z=105 4a) 3x+12y+17z=43 b) 4x-2y+11z=51 5) 5x+6y-7z=12 6) 4x+3y-z=1

Ex5 1a) 70.5^0 b) 56.3^0 c) 22.5^0 d) 0 2a) 18.0^0 b) 70.2^0 c) 60^0 d) 70.5^0 3a) 28.5^0 b) 22.4^0 c) 22.4^0 d) 11.1^0 4a) (1,2,0) b) (-1,3,4)

5a) $\frac{x-3\frac{1}{2}}{1}=\frac{y+1\frac{1}{2}}{3}=\frac{z}{-1}$ (subtracting $\frac{1}{2}$ from each part gives $\frac{x-4}{1}=\frac{y}{3}=\frac{z+1}{-1}$) b) $\frac{x-\frac{4}{3}}{4}=\frac{y+\frac{1}{3}}{8}=\frac{z}{-3}$

c) $\frac{x-22}{2}=\frac{y+12}{-1}=\frac{z}{-1}$ 6a) (1,4,-9) b) (7,4,9) 7) 7x+y-3z=6 8a) (3,0,1) b) $\frac{x-3}{1}=\frac{y}{4}=\frac{z-1}{1}$

9a) x=2+t, y=-4+2t, z=3-4t b) Proof c) 2x-3y-z=13 10a) Proof b) x=3+2t, y=2+3t z=-4+t c) Proof 11a) Proof b) 4x+2y+z=9, C (3,-1,-1) 12a) x=1+t, y=-1+2t, z=2-3t b) Proof c) (2,1,-1)

Integration

Answers (to save space the +C has not been included in every answer but it remains essential)

Ex1: a) $2x^4$ b) $-\frac{3}{x^2}$ c) $\frac{2}{3}x^{\frac{3}{2}}$ d) $2\sqrt{x}$ e) $x^2-\frac{6}{x}$ f) $2x^{\frac{1}{2}}-2x^{\frac{3}{2}}$ g) $\frac{(3x+4)^5}{15}$ h) $-\frac{(1-2x)^5}{10}$ i) $-\frac{1}{2(2x-3)}$

j) $\frac{\sqrt{4x+1}}{2}$ k) $-\frac{2}{\sqrt{3x+1}}$ l) $-\frac{1}{3}cos3x$ m) $3sin\frac{1}{3}x$ n) $\frac{1}{2}x-\frac{1}{4}sin2x$ o) $\frac{1}{2}x+\frac{1}{2}sinx$

Ex2: a) $\frac{1}{5}e^{5x}$ b) $-\frac{1}{2}e^{-2x}$ c) $6e^{\frac{1}{2}x}$ d) $\frac{1}{3}ln3x$ e) $\ln(x+5)$ f) $\frac{1}{2}\ln(2x-5)$ g) $\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}$

h) $x-e^{-x}$ i) $\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}$ j) $2\ln(3x+2)$ k) $-\frac{3}{2}\ln(1-2x)$ l) $-\frac{9}{7}\ln(5-7x)$ m) $\frac{1}{4}tan4x$

n) $\frac{1}{2}\tan(\pi+2x)$ o) $\frac{7}{2}cot2x$

Ex3: a) $-\frac{11}{24}$ b) $-\frac{5}{12}$ c) $30\frac{4}{5}$ d) $12\frac{2}{3}$ e) $\frac{1}{36}$ f) $11\frac{1}{4}$ g) $\frac{\sqrt{2}}{6}$ h) 1 i) π j) $\frac{1}{3}(1-e^{-6})$ k) $e-1$ l) $2\left(e^{\frac{1}{2}}-1\right)$

m) ln3 n) $\frac{1}{3}ln2.5$ o) ln3

Ex4: a) $\frac{1}{12}(x^2-3)^6$ b) $\frac{1}{9}(x^3-1)^3$ c) $-\frac{1}{3}(1-x^2)^{\frac{3}{2}}$ d) $-\frac{1}{5}sin^5x$ e) $\frac{1}{4(1-x^2)^2}$ f) $\frac{1}{4}ln(1+x^4)$
g) $\frac{1}{2}sec^2x$ h) $\frac{1}{5sin^5x}$ i) $\frac{1}{3}\ln(3e^x-1)$ j) $\ln(tanx)$

Ex5: a) $\frac{1}{5}(3x+2)^5-\frac{1}{2}(3x+2)^4$ b) $\frac{1}{4}(2x+3)^7-\frac{7}{8}(2x+3)^6$ c) $\frac{6}{5}(1+x^2)^{\frac{5}{2}}$ $-2(1+x^2)^{\frac{3}{2}}$
d) $\frac{1}{2}(2x+3)^{\frac{3}{2}}-\frac{9}{2}(2x+3)^{\frac{1}{2}}$

Ex6: a) $-\sqrt{1-x^2}$ b) $2sin^{-1}\frac{x}{2}+x\sqrt{4-x^2}$ c) $-\sqrt{9-x^2}$ d) $2sin^{-1}\frac{x}{2}+\frac{x}{2}\sqrt{4-x^2}$

Ex7: a) $ln3$ b) $6+3ln7$ c) $\frac{31}{160}$

Ex8: 1a) 12 b) $2\frac{1}{4}$ 2a) $20\frac{5}{6}$ b) $3\frac{1}{12}$

Ex9: a) 36 b) $2\frac{2}{3}$ c) $5\frac{1}{3}$ d) 8 e) $4\sqrt{2}$ f) $\frac{16a^2}{3}$

Ex10: a) 9 b) $11\frac{1}{4}$ c) $2\sqrt{3}-2\sqrt{2}$ d) $\frac{2}{3}$ e) $7\frac{1}{2}$ f) $e^2(e^3-1)$

Ex11: 1a) $sin^{-1}\left(\frac{x}{7}\right)+C$ b) $\frac{1}{7}tan^{-1}\left(\frac{x}{7}\right)+C$ c) $sin^{-1}\left(\frac{x}{3}\right)+C$ d) $\frac{1}{10}tan^{-1}\left(\frac{x}{10}\right)+C$ e) $\frac{1}{5}sin^{-1}\left(\frac{5x}{6}\right)+C$
f) $\frac{1}{30}tan^{-1}\left(\frac{5x}{6}\right)+C$ g) $sin^{-1}\left(\frac{x}{2}\right)+C$ h) $\frac{1}{5}tan^{-1}\left(\frac{2x}{5}\right)+C$ 2a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) π d) $\frac{\pi}{12}$

Ex12: a) $\frac{1}{4(1-4x)}+C$ b) $\frac{1}{3}ln\left|\frac{A(1+3x)}{1-3x}\right|+C$ c) $\frac{1}{3}tan^{-1}\left(\frac{x}{3}\right)+C$ d) $-5ln|1-t|+C$ e) $6\sin^{-1}\left(\frac{Q}{6}\right)+C$
f) $lnA(9+x^2)$ g) $\frac{-1}{10(x^5+1)^2}+C$ h) $\frac{1}{2}ln(x^2+9)+\tan^{-1}\left(\frac{x}{3}\right)+C$ i) $ln\left|\frac{At}{4-t}\right|+C$ j) $\frac{-1}{4+x}+C$
k) $\frac{5}{2}ln|2x+3|+C$ l) $C-\sqrt{1-x^2}$

Ex13: a) $3\ln(x+2)-2\ln(x+4)+C$ b) $x-\ln(x+2)+\ln(x-2)+C$
c) $3\ln(x-1)-3\ln(x-2)+\ln(x+3)+C$ d) $x-\frac{5}{2}\ln(x-3)+\frac{5}{2}\ln(x+1)+C$
e) $3x-\frac{1}{2}\ln(2x-1)+2\ln(x+3)+C$ f) $\ln 1.5$

Ex14: a) $\ln x+2\ln(x+1)+\frac{3}{x+1}+C$ b) $2\ln(x+2)-\ln(x-1)-\frac{3}{x-1}+C$
c) $\ln(x+2)-\ln(2x-1)-\frac{5}{2x-1}+C$ d) $\frac{1}{3}\ln(x-2)-\frac{1}{3}\ln(x+1)-\frac{4}{x-2}+C$ e) $\frac{1}{2}+\ln\left(\frac{3}{4}\right)$

Ex15: a) $2\ln(x-1)-\ln(x^2+1)+\tan^{-1}x+C$ b) $\frac{15}{2}\ln(x^2+1)-12\ln(x+6)+2\tan^{-1}x+C$
c) $\frac{1}{4}\ln(x+1)+\frac{1}{4}\ln(x-1)-\frac{1}{4}\ln(x^2+1)+C$ d) $\frac{1}{10}\ln(x^2+4)-\frac{1}{5}\ln(x+1)+\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right)+C$
e) $\frac{1}{2}ln\frac{40}{17}$

Ex16: a) $-xcosx+sinx+C$ b) $-\frac{x}{3}cos3x+\frac{1}{9}sin3x+C$ c) $\frac{2}{3}x^{\frac{3}{2}}(lnx-\frac{2}{3})+C$
d) $-\frac{1}{2x^2}lnx-\frac{1}{4x^2}+C$ e) xe^x-e^x+C f) $\frac{1}{4}xsin4x+\frac{1}{16}cos4x+C$ g) -2

Ex17: a) $x^2e^x - 2xe^x + 2e^x + C$ b) $-\frac{1}{3}x^2cos3x + \frac{2}{9}xsin3x + \frac{2}{27}cos3x + C$
c) $\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$ d) $\frac{1}{2}x^2sin2x + \frac{1}{2}xcos2x - \frac{1}{4}sin2x + C$
e) $-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$ f) $(5x^3 + 3x)e^x - (15x^2 + 3)e^x + 30xe^x - 30e^x + C$ g) $e - 2$
Ex18: a) $x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C$ b) $x\sin^{-1}3x - \frac{1}{3}\sqrt{1 - 9x^2} + C$
c) $x\tan^{-1}2x - \frac{1}{4}ln(1 + 4x^2) + C$ d) $x\sin^{-1}\frac{1}{2}x + \sqrt{4 - x^2} + C$ e) $x\cos^{-1}x - \sqrt{1 - x^2} + C$
f) $x\tan^{-1}\frac{1}{2}x - \ln(4 + x^2) + C$ g) $xln2x - x + C$ h) $x(lnx)^2 - 2xlnx + 2x + C$ i) 1

Ex19: a) $\frac{1}{5}e^xsin2x - \frac{2}{5}e^xcos2x + C$ b) $-\frac{1}{2}e^{-x}cosx - \frac{1}{2}e^{-x}sinx + C$
c) $\frac{3}{13}e^{-2x}sin3x - \frac{2}{13}e^{-2x}cos3x + C$ d) $e^xcos^2x + \frac{1}{5}e^xsin2x - \frac{2}{5}e^xcos2x + C$

Ex20: a) $\frac{1}{4}\tan^{-1}\frac{x}{4} + C$ b) $2\ln(x^2 + 6) + C$ c) $\frac{1}{8}ln\left|\frac{k(x-4)}{x+4}\right| + C$ d) $2e^{x^2} + C$
e) $\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$ f) $\frac{1}{2}(\tan^{-1}x)^2 + C$ g) $ln\left|\frac{A(x^2+2)}{(x-3)^2}\right| + C$ h) $-xcosx + sinx + C$
i) $\frac{1}{3}sin^3x + C$ j) $x + 2ln|x - 1| + C$ k) $\frac{-1}{3(3x-2)} + C$ l) $-\frac{1}{3}ln|4 - 3x| + C$ m) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$
n) $\frac{1}{2}e^{(3+2x)} + C$ o) $x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C$ p) $\frac{1}{2}ln(x^2 + 16) + \tan^{-1}\frac{x}{4} + C$ q) $\frac{1}{5}\tan^{-1}x + C$
r) $\frac{1}{2}x^2 - 2x + 4ln|x + 2| + C$ s) $\frac{\pi}{12} - \frac{\sqrt{3}}{8}$ t) $\frac{1}{7}(x + 2)^7 - \frac{1}{3}(x + 2)^6 + C$

Differential Equations

Ex1: a) $y = \frac{Ae^x}{x+1}$ b) $y = 1 - \frac{2}{x^2+C}$ c) $y = \frac{-1}{e^x+C}$ d) $\frac{e^y}{y} = cx^2$ or $y = \frac{e^y}{Cx^2}$ e) $\frac{cosx}{cosy} = C$
f) $y = \frac{(c+1)x+1}{x+1}$ g) $2y^2 - 5 = Cx$

Ex2: a) $y = \ln(1 - cos2x) + 1$ b) $\tan^{-1}y = \tan^{-1}x + \frac{\pi}{4}$ c) $y = 3e^{\frac{1}{2}x^2} + 2$ d) $y = sin\left(x - \frac{\pi}{6}\right)$
e) $y = e^{sinx}$ f) $siny = \frac{1}{2cosx}$ g) $y = \frac{e^x}{x+1}$

Ex3: 1) 58days 2a) $\theta = Ae^{-kt} + 15$ b)i) 69.5^0 ii) 11.7mins 3a) $p = 468e^{0.0132t}$ b) 609
4a) $\frac{dx}{dt} = kx$ b) Proof c) 6837m^2 d) 12days 5a) 1.024mg b) 6.22hrs
6) Proof and $t = ln\left(\frac{x^2+4}{4}\right) + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right)$ 7) $v = \frac{1}{1+4e^{-\frac{1}{2}t}}$ 8) 4.74mins 9) $\tan^{-1}y = ln|\sin x| + \frac{\pi}{4}$

Ex4 1a) $y = (x + 1)(x + C)$ b) $y = -\frac{1}{3}cos^2x + Csecx$ c) $y = cosecx(x + cotx + Ccosecx)$
d) $y = \left(\frac{1-x}{1+x}\right)(\frac{1}{2}x^2 + x + C)$ e) $y = \left(\frac{x}{x+1}\right)(xe^x - e^x + C)$ f) $y = (2sin2x + cos2x) + Ce^{-x}$
g) $y = Ce^x(1 - x) - \frac{1}{2}e^{-x})\ (1 - x)$ h) $y = e^{-x}\left(\frac{x}{2} + \frac{c}{x}\right)$ i) $y = \frac{C(x+1)}{x} - \frac{(x+1)^2}{x}e^{-x}$
j) $y = Ccosecx - \frac{1}{4}\frac{cos2x}{sinx}$ 2a) $y = \frac{1}{5x^2}(x^5 - 9)$ b) $y = \frac{x^3(3x+4)}{12(1+x)^2}$ c) $y = 4sinx - cosecxcotx$
d) $y = \left(\frac{x+1}{x}\right)(e^x - e)$ e) $y = x^2(sinx - cosx) + x(sinx + cosx) - x\left(\frac{\pi}{2} + 1\right)$

Ex5 1a) $y = Ae^{2x} + Be^{x}$ b) $y = Ae^{3x} + Be^{x}$ c) $y = Ae^{2x} + Be^{3x}$ d) $y = Ae^{\frac{3}{2}x} + Be^{3x}$

2a) $y = (A + Bx)e^{-3x}$ b) $y = (A + Bx)e^{-\frac{1}{2}x}$ c) $y = (A + Bx)e^{3x}$ d) $y = (A + Bx)e^{-nx}$

3a) $y = e^{-x}(Acosx + Bsinx)$ b) $y = e^{-2x}(Acos2x + Bsin2x)$ c) $y = e^{-3x}(Acos2x + Bsin2x)$

d) $y = e^{-\frac{1}{2}x}(Acos\frac{\sqrt{3}}{2}x + Bsin\frac{\sqrt{3}}{2}x)$

Ex6 1a) $y = Ae^{x} + Be^{3x} + \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{26}{9}x + \frac{80}{27}$ b) $y = Ae^{x} + Be^{-x} + 5x - 2$

c) $y = Ae^{-x} + Be^{-4x} + 8x^2 - 20x + 21$ d) $y = e^{-x}(Acosx + Bsinx) + \frac{1}{2}x^2 - x + 1$

2a) $y = Ae^{-3x} + Be^{x} + 2e^{2x}$ b) $y = Ae^{-x} + Be^{-\frac{9}{4}x} - 7e^{-2x}$ c) $y = Ae^{-x} + Be^{2x} - \frac{1}{2}e^{x}$

d) $y = (A + Bx)e^{2x} + \frac{1}{8}e^{-2x}$ 3a) $y = Ae^{x} + Be^{2x} + \frac{3}{10}cosx + \frac{1}{10}sinx$

b) $y = (A + Bx)e^{x} - \frac{6}{5}cos2x - \frac{8}{5}sinx$ c) $y = Acos\frac{1}{2}x + Bsin\frac{1}{2}x - \frac{4}{3}sinx$

d) $y = e^{-4x}(Acos3x + Bsin3x) + \frac{1}{3}cos3x + \frac{3}{4}sin3x$ 4a) $y = (A + Bx)e^{-2x} + \frac{1}{4}x - \frac{1}{4} - \frac{1}{16}e^{2x}$

b) $y = Ae^{-3x} + Be^{2x} - \frac{1}{4}e^{x} - \frac{1}{6}e^{-x}$ c) $y = e^{-2x}(Acos2x + Bsin2x) + \frac{1}{8}x - \frac{1}{16} - \frac{1}{13}e^{x}$

d) $y = (A + Bx)e^{-\frac{1}{2}x} + \frac{1}{9}e^{x} + \frac{30}{289}cos2x - \frac{16}{289}sin2x$ 5a) $y = Ae^{-2x} + Be^{-3x} + 3xe^{-2x}$

b) $y = Ae^{x} + Be^{3x} - xe^{x}$ 6) $y = e^{-x}\left(\frac{2}{5}cosx + \frac{6}{5}sinx\right) - \frac{2}{5}cosx + \frac{1}{5}sinx$ 7) $y = 2e^{-2x} - e^{3x} + xe^{3x}$