St Andrew's Academy

Department of Mathematics

Advanced Higher

Course Textbook

UNIT 1

Unit 1

N7

Partial Fractions

MAC 1.1

Given the expression $\frac{2}{x-1}+\frac{3}{x+1}+\frac{4}{2x-3}$ we can express it as a single fraction.

$$\frac{2}{x-1}+\frac{3}{x+1}+\frac{4}{2x-3}=\frac{2(x+1)(2x-3)+3(x-1)(2x-3)+4(x-1)(x+1)}{(x-1)(x+1)(2x-3)}$$

$$=\frac{14x^2-17x-1}{(x-1)(x+1)(2x-3)}$$

$\frac{14x^2-17x-1}{(x-1)(x+1)(2x-3)}$ is called a rational function.

$\frac{2}{x-1}+\frac{3}{x+1}+\frac{4}{2x-3}$ are called the corresponding partial fractions.

The focus of this chapter is to find the partial fractions when given the rational function. There are a number of different types to consider.

In general terms we shall refer to the rational function as $\frac{f(x)}{g(x)}$, where f(x) and g(x) are polynomial functions.

Before we begin the process, the degree of f(x) must be lower than the degree of g(x). This may require a long division to be carried out before the partial fractions process is completed.

Long Division

The process for long division of polynomials is the same as the process for numbers

Example

$x^3 \div (x^2-4)$

$$\begin{array}{r|l} & \qquad\qquad x \\ \hline x^2+0x-4 & x^3+0x^2+0x+0 \\ & x^3+0x^2-4x \\ \hline & \qquad\qquad 4x \end{array}$$

Therefore $\frac{x^3}{x^2-4}=x+\frac{4x}{x^2-4}$.

Exercise 1: By using long division write the following rational fractions in the same way the example above.

a) $\frac{x^3+3x^2}{x^2+2}$ b) $\frac{5x^4-4x^3+x^2}{x^2-4}$ c) $\frac{x^6-x^4+x^2-x-1}{x^2(x-1)}$

Partial Fractions Type 1

In this type of partial fractions the degree of f(x) is less than g(x) and the factors of g(x) are linear and different.

Example

Find the partial fractions for

a) $\frac{x+16}{2x^2+x-6}$

$\frac{x+16}{2x^2+x-6} = \frac{x+16}{(2x-3)(x+2)}$ by factorising the denominator.

Let $\frac{x+16}{(2x-3)(x+2)} = \frac{A}{2x-3} + \frac{B}{x+2}$

Multiply both sides by $(2x-3)(x+2)$ to remove the denominator.

$$x + 6 = A(x+2) + B(2x-3)$$

Substitute $x = \frac{3}{2}$ $\quad \frac{35}{2} = \frac{7}{2}A \quad \Rightarrow \quad A = 5$

Substitute $x = -2$ $\quad 14 = -7B \quad \Rightarrow \quad B = -2$

Hence $\frac{x+16}{2x^2+x-6} = \frac{5}{2x-3} - \frac{2}{x+2}$

b) $\frac{2x^2+4x}{(x^2-1)(2x+1)}$

$\frac{2x^2+4x}{(x^2-1)(2x+1)} = \frac{2x^2+4x}{(x-1)(x+1)(2x+1)}$

Let $\frac{2x^2+4x}{(x-1)(x+1)(2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+1}$

Multiply both sides by $(x-1)(x+1)(2x+1)$

$2x^2 + 4x = A(x+1)(2x+1) + B(x-1)(2x+1) + C(x-1)(x+1)$

Substitute $x = 1$ $\quad 6 = 6A \quad \Rightarrow \quad A = 1$

Substitute $x = -1$ $\quad -2 = 2B \quad \Rightarrow \quad B = -1$

Substitute $x = -\frac{1}{2}$ $\quad -\frac{3}{2} = -\frac{3}{4}C \quad \Rightarrow \quad C = 2$

Hence $\frac{2x^2+4x}{(x^2-1)(2x+1)} = \frac{1}{x-1} - \frac{1}{x+1} + \frac{2}{2x+1}$

Exercise 2:

1) Find the partial fractions for the following rational functions

a) $\frac{4x-9}{(x-2)(x-3)}$ b) $\frac{3-8x}{x(1-x)}$ c) $\frac{x+24}{x^2-x-12}$ d) $\frac{2(3x+4)}{x^2+4x}$

2) Divide out first and then find partial fractions for

a) $\frac{x^2+6x-13}{x^2+x-12}$ b) $\frac{x^3}{(x+4)(x-1)}$

Partial Fractions Type 2

In this type of partial fractions the degree of f(x) is less than g(x) and the factors of g(x) are linear and repeated.

Example: Find the partial fractions for $\frac{-8x^2+14x-15}{(2x-1)^2(x+2)}$

Let $\frac{-8x^2+14x-15}{(2x-1)^2(x+2)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x+2}$

Multiply both sides by $(2x-1)^2(x+2)$

$-8x^2 + 14x - 15 = A(2x-1)(x+2) + B(x+2) + C(2x-1)^2$

Substitute $x = \frac{1}{2}$ $\quad -10 = \frac{5}{2}B \quad \Rightarrow \quad B = -4$

Substitute $x = -2$ $\quad -75 = 25C \quad \Rightarrow \quad C = -3$

We have used all the values of x from the factors, this will happen when there are repeated factors. Pick any other value for x and use the values for B and C already found.

Substitute $x = 0$ $\quad -15 = -2A + 2B + C$

$-15 = -2A + 8 - 3 \quad \Rightarrow \quad A = 2$

Hence $\frac{-8x^2+14x-15}{(2x-1)^2(x+2)} = \frac{2}{2x-1} - \frac{4}{(2x-1)^2} - \frac{3}{x+2}$

Exercise 3

1) Find partial fractions for the following rational functions:

a) $\frac{3x^2+1}{x(x+1)^2}$ b) $\frac{3x^2+2}{x(x-1)^2}$ c) $\frac{x^2-2x+10}{(x+2)(x-1)^2}$ d) $\frac{7x^2-48x+75}{(2x-3)(x-4)^2}$

2) Express this improper rational function as the sum of a polynomial and partial fractions $\frac{x^6-x^4+x^2-x-1}{x^2(x-1)}$

Partial Fractions Type 3

In this type of partial fractions the degree of f(x) is less than g(x) where g(x) has a linear factor and an irreducible quadratic.

Example: Find the partial fractions for $\frac{5x-7}{(x+3)(x^2+2)}$

Let $\frac{5x-7}{(x+3)(x^2+2)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+2}$

Multiply both sides by $(x+3)(x^2+2)$

$$5x - 7 = A(x^2+2) + (Bx+C)(x+3)$$

Substitute $x = -3 \quad -22 = 11A \quad \Longrightarrow \quad A = -2$

We have used all the factors of x from the factors. Pick any other values of x that you have not used.

Substitute $x = 0 \quad -7 = 2A + 3C \quad \Rightarrow \quad -7 = -4 + 3C \quad \Rightarrow \quad C = -1$

Substitute $x = 1 \quad -2 = 3A + 4(B + C) \quad \Rightarrow \quad -2 = 4B - 10 \quad \Rightarrow \quad B = 2$

Hence $\frac{5x-7}{(x+3)(x^2+2)} = \frac{-2}{x+3} + \frac{2x-1}{x^2+2}$

Exercise 4 Find the partial fractions for

a) $\frac{8x+8}{(x-2)(x^2+4)}$ b) $\frac{7x^2+4x}{(x+2)(x^2+1)}$ c) $\frac{7x^2+17x+80}{(x+5)(x^2+9)}$

d) $\frac{x^3+2x^2+61}{(x+3)^2(x^2+4)}$ e) $\frac{x^4+1}{x(x^2+2)}$ f) $\frac{x^2}{(x-1)^2}$

Exercise 5 Find the partial fractions for:

a) $\frac{7}{(2x-3)(x+2)}$ b) $\frac{10+6x-3x^2}{(2x-1)(x+3)^2}$ c) $\frac{2x-3}{(x^2-1)}$

d) $\frac{3x^2-1}{(2x-1)^2(x-1)}$ e) $\frac{2x^2-7}{(x-3)(2x+5)}$ f) $\frac{2x^3+11}{(x^2+4)(x-3)}$

g) $\frac{3x+7}{(1+2x)(x^2-x+2)}$ h) $\frac{3x^3+6x^2-11x+1}{(1+x)^3(x-2)}$ i) $\frac{x^3+2x^2+61}{(x+3)^2(x^2+4)}$

Binomial Theorem and Complex Numbers

Pascal's Triangle

The expansions of $(x+y)^n$ for n=1,2,3,4,5 are shown below. The coefficients form an array known as Pascal's Triangle.

$(x+y)^1 = x+y$					1		1				
$(x+y)^2 = x^2+2xy+y^2$				1		2		1			
$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$			1		3		3		1		
$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$		1		4		6		4		1	
$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$	1		5		10		10		5		1

Note:
The symmetry on Pascal's Triangle
The next line of the coefficients can be deduced from the previous line.
As the powers of x in the expansion decrease by one, the powers of y increase by one.

We can now write down the expansion of $(x+y)^6$. The next line of Pascal's Triangle will give the coefficients: 1 6 15 20 15 6 1.
$(x+y)^6 = x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$

Examples: Expand $\left(2a-\frac{1}{b}\right)^5$
The coefficients are 1 5 10 10 5 1, from Pascal's Triangle. Replace the x with 2a and the y with $-\frac{1}{b}$ to get

$$\left(2a-\frac{1}{b}\right)^5 = (2a)^5 + 5(2a)^4\left(-\frac{1}{b}\right) + 10(2a)^3\left(-\frac{1}{b}\right)^2 + 10(2a)^2\left(-\frac{1}{b}\right)^3 + 2(2a)\left(-\frac{1}{b}\right)^4 + \left(-\frac{1}{b}\right)^5$$

$$= 32a^5 - \frac{80a^4}{b} + \frac{80a^3}{b^2} - \frac{40a^2}{b^3} + \frac{10a}{b^4} - \frac{1}{b^5}$$

Exercise 1: Expand

a) $(x+5)^3$ b) $(a+2)^4$ c) $(m-3)^5$ d) $(5-y)^4$

e) $(2x+3)^3$ f) $(3p+1)^4$ g) $(2x+y)^5$ h) $(2k-5)^3$

i) $(3x-2y)^4$ j) $(x^2+2)^3$ k) $(y^3-2)^4$ l) $\left(x+\frac{1}{x}\right)^3$

m) $\left(x+\frac{2}{x}\right)^4$ n) $\left(2x-\frac{3}{x}\right)^3$ o) $\left(x^2-\frac{2}{x}\right)^5$ p) $(2x^2+1)^4$

q) $(3x^2-2)^3$ r) $\left(2x^3-\frac{2}{x}\right)^4$ s) $\left(x^2-\frac{1}{x^2}\right)^3$ t) $\left(\frac{x^2}{y}-\frac{y}{x}\right)^4$

The Factorial

n! (read as n factorial) is defined as:

$$n! = n(n-1)(n-2)(n-3)(n-4)...4\times3\times2\times1 \text{ for } n\in\mathbf{N}.$$

Note that 0! =1

Notation

n choose r

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!} \quad \text{for } n\in\mathbf{N} \text{ and } 0\leq r\leq n$$

$\binom{n}{r}$ calculates the number of ways of choosing r objects from n objects.

Example: Joe holds 5 playing cards, the 2, 3, 4, 5 and 6 of clubs.
How many ways does he have of putting down 2 cards?

List: 2,3 2,4 2,5 2,6 3,4 3,5 3,6 4,5 4,6 5,6 10 ways

OR

$$\binom{5}{2} = \frac{5!}{2!\,(5-2)!} = \frac{5!}{2!\,3!} = \frac{5\times4\times3!}{2\times1\times3!} = 10$$

However, problems such as these are not part of the AH course.

Exercise 2:

1) Find the values of a) $\binom{7}{3}$ b) $\binom{10}{7}$

2) Find the values of $\binom{5}{0}$, $\binom{5}{1}$, $\binom{5}{2}$, $\binom{5}{3}$, $\binom{5}{4}$, $\binom{5}{5}$.
Compare these to Pascal's Triangle.

3) Use the definition to show that a) $\binom{n}{0}$=1 b) $\binom{n}{1} = n$

It appears from the above exercise that Pascal's triangle could be re-written as

n=1 $\binom{1}{0}$ $\binom{1}{1}$

n=2 $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

n=3 $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

n=4 $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$ etc

From the symmetry of Pascal's triangle we can see that:

$\binom{3}{0} = \binom{3}{3}$, $\binom{4}{1} = \binom{4}{3}$, $\binom{5}{1} = \binom{5}{4}$, $\binom{7}{2} = \binom{7}{5}$ etc.

From the ability to produce the next line we can see that:

$\binom{3}{2} + \binom{3}{3} = \binom{4}{3}$, $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$, $\binom{7}{3} + \binom{7}{4} = \binom{8}{4}$ etc.

Exercise 3:

1) Use the symmetry property to find to find another value of $\binom{n}{r}$ equivalent to: a) $\binom{14}{10}$ b) $\binom{20}{15}$ c) $\binom{n}{n}$

2) Prove this property i.e. show that $\binom{n}{r} = \binom{n}{n-r}$

3) Use the process for finding the next line to write the following in the form $\binom{n}{r}$: a) $\binom{10}{4} + \binom{10}{5}$, b) $\binom{20}{6} + \binom{20}{7}$.

4) Prove this property i.e. show that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

You are required to know the results in questions 2 and 4.

The Binomial Theorem

This states that for x,y∊**R** and n∊**N**

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 \ldots + \binom{n}{r} x^{n-r}y^r + \cdots + \binom{n}{n} y^n$$

The General Term is $\binom{n}{r} x^{n-r}y^r$.

Examples:

1) Use the Binomial Theorem to expand $(1-x)^4$.

$(1-x)^4=[1+(-x)]^4$

$=\binom{4}{0} + \binom{4}{1}(-x) + \binom{4}{2}(-x)^2 + \binom{4}{3}(-x)^3 + \binom{4}{4}(-x)^4$

$=1 - 4x + 6x^2 - 4x^3 + x^4$

2) Use the Binomial Theorem to expand $\left(x + \frac{1}{x}\right)^{16}$ as far as the fourth term.

$\binom{16}{0} x^{16} + \binom{16}{1} x^{15}\left(\frac{1}{x}\right) + \binom{16}{2} x^{14}\left(\frac{1}{x}\right)^2 + \binom{16}{3} x^{13}\left(\frac{1}{x}\right)^3$

$= x^{16} + 16x^{14} + 120x^{12} + 560x^{10}$

3) Use the Binomial Theorem to find an accurate value of 2.1^4

$$(2+0.1)^4 = \binom{4}{0}2^4 + \binom{4}{1}2^3 \times 0.1 + \binom{4}{2}2^2 \times 0.1^2 + \binom{4}{3}2 \times 0.1^3 + \binom{4}{4}0.1^4$$

= 16 + 3.2 + 0.24 + 0.008 + 0.0001

= 19.4481

Similarly $1.9^4 = (2-0.1)^4$ and this can be expanded to find an accurate value of 13.0321.

Sigma Notation

The Binomial Theorem can be written as

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$\sum$ is read as: "the sum of".

Exercise 4:

1) Expand the following using the Binomial Expansion:

a) $(3+x)^3$ b) $(5+2x)^3$ c) $(2-x)^4$ d) $(x+2y)^5$

e) $(1+3x)^3$ f) $(2x-3y)^4$ g) $\left(2x+\frac{3}{x}\right)^5$ h) $\left(x^2-\frac{5}{x}\right)^4$

2) Expand $(3+x)^5$ and use your expansion to find a) 3.1^5 b) 2.9^5

3) Expand $(2+x)^7$ in ascending powers of x up to x^3.
Hence evaluate 2.1^7 correct to 6 significant figures.

4) Use the Binomial Theorem to evaluate 1.01^4 correct to 5 decimal places.

Looking for Individual Terms

Examples:

1) Find the value of n for which $\binom{n}{2} = 55$.

$\frac{n!}{2!(n-2)!} = 55$

$\frac{n(n-1)(n-2)(n-3)....3\times2\times1}{2\times1\times(n-2)(n-3)...3\times2\times1} = 55$

$\frac{n(n-1)}{2} = 55$

$n^2 - n - 110 = 0$

(n-11)(n+10)=0

n=11 or n=-10

But n is a whole number, so n=11.

2) Find the term in x^7 in $\left(x + \frac{2}{x}\right)^9$

The general term is $\binom{9}{r} x^{9-r} (2x^{-1})^r$

$= \binom{9}{r} x^{9-r} 2^r x^{-r}$

$= \binom{9}{r} 2^r x^{9-2r}$

The term in x^7 occurs when 9-2r=7

r=1

The term is $\binom{9}{1} 2^1 x^7 = 18x^7$

3) Find the term independent of x in $\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9$

The general term is $\binom{9}{r} \left(\frac{4x^2}{3}\right)^{9-r} \left(\frac{-3x^{-1}}{2}\right)^r$

$= \binom{9}{r} \frac{4^{9-r}}{3^{9-r}} \times \frac{(-3)^r}{2^r} x^{18-2r} x^{-r}$

$= \binom{9}{r} \frac{4^{9-r}}{3^{9-r}} \times \frac{(-3)^r}{2^r} x^{18-3r}$

The term independent of x is given by 18-3r=0

r=6

The term is $\binom{9}{6} \frac{4^3}{3^3} \times \frac{(-3)^6}{2^6} = 2268$

4) Find the sixth term in the expansion of $(2x-3y)^9$.

Term 1 $\binom{9}{0}(2x)^9$ Term 2 $\binom{9}{1}(2x)^8(-3y)^1$ .....

Term 6 $\binom{9}{5}(2x)^4(-3y)^5 = \frac{9!}{5!4!} \times 2^4x^4 \times (-3)^5y^5 = -489888x^4y^5$

Exercise 5:

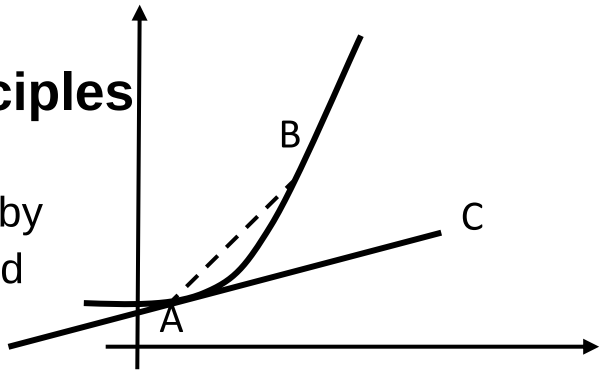
1) Find the value of n for which a) $\binom{n}{2} = 21$ b) $\binom{n}{3} = 10$
2) Find the term in x^3 in $(2x-3)^5$
3) Find the term independent of x for $\left(x^2 + \frac{1}{x}\right)^3$
4) Find the coefficient of x^3 in the expansion of $(3x-2)^{12}$
5) Find the coefficient of x^9 and the term independent of x in the expansion of $\left(\frac{1}{x^2} - x\right)^{18}$
6) Find the fourth term in the expansion of $(a-3x)^{10}$ when written in ascending powers of x.
7) Find the coefficient of x^2y^4 in the expansion of $(x+2y)^6$
8) Find the coefficient of x^9 in the expansion of $(1+3x^3)^4$
9) Find the coefficient of x^2 in the expansion of $(x+\frac{1}{x^2})^5$
10) Find the coefficient of x^4y^3 in the expansion of $(x-y)^7$
11) Find the coefficient of x^2y^2 in the expansion of $(2x-y)^4$
12) Find the coefficient of x^8 in the expansion of $(1+x^2)^8$
13) Find the coefficient of y^5 in the expansion of $(y-\frac{1}{y})^5$
14) Find the constant term in the expansion of $(3x+\frac{1}{x})^{12}$
15) Find the constant term in the expansion of $(\frac{a}{2} - \frac{1}{3a})^4$

Differentiation

To complete this section well, you must know the purpose of differentiation, the applications of differentiation and how to differentiate all the different types of functions from Higher i.e. polynomials, trigonometric functions and composite functions.

Differentiation from First Principles

An approximation of the gradient of the tangent AC the curve at A can be found by taking a second point B, on the curve and calculating the gradient of the chord AB instead.



If A is the point (x,f(x)) and B is the point (x+h,f(x+h)), where h is small, we can calculate the gradient of the chord.

$$m_{AB} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

m_{AB} tends to a limit as h tends to zero.
The limit is denoted by f'(x), the derivative of f(x) and gives the gradient of the tangent AC to the curve y=f(x) at A.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

This is differentiation from first principles.

Example: Find the derivative from first principles of x^2.

$f(x)=x^2$ $\quad$ $f(x+h)=(x+h)^2=x^2+2xh+h^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2xh+h^2}{h}$$

$$f'(x) = \lim_{h\to 0} 2x+h$$

$$f'(x) = 2x$$

Exercise 1: Differentiate the following functions from first principles:

a) $f(x)=x^3$ $\quad$ b) $f(x)=x^2+2x$ $\quad$ c) $f(x) = 3x^2+4x-5$

Standard Derivatives	
f(x)	f'(x)
x^n	nx^{n-1}
$(ax+b)^n$	$an(ax+b)^n$
sinx	Cosx
cosx	-sinx
sin(ax+b)	acos(ax+b)
cos(ax+b)	-sin(ax+b)

Exercise 2:

1 Differentiate the following functions with respect to x

a) $f(x) = x^3 - x^2 + 5x - 6$

b) $f(x) = 3x^2 + 7 - \frac{4}{x}$

c) $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$

d) $f(x) = x^{\frac{3}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

e) $f(x) = \frac{1}{x^2} - \frac{1}{x^3}$

f) $f(x) = \frac{\sqrt{x}}{x^2} + \frac{x^2}{\sqrt{x}}$

g) $f(x) = (4x + 5)^2$

h) $f(x) = (2x^4 - 3)^{\frac{1}{2}}$

i) $f(x) = \frac{3}{\sqrt{4-x^2}}$

j) $f(x) = \frac{4}{\sqrt[3]{x^3+3x}}$

k) $f(x) = cos^3x$

l) $f(x) = \sqrt{sinx}$

2 Find the equation of the tangent of $y = 3x^2 + 2x - 7$ at $x = 2$.

3 Find the coordinates of the stationary points of the curve $y = x^3 - 3x + 2$

New Trigonometric Functions

Secant: $sec\theta = \frac{1}{cos\theta}$

Cosecant: $cosec\theta = \frac{1}{sin\theta}$

Cotangent: $cot\theta = \frac{1}{tan\theta}$

Learn these functions!

The Chain Rule

This process is for differentiating composite functions. Two versions are shown below.

If y is a function of u, and u is a function of x then y is a function of x. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ OR $[f(g(x))]' = f'(g(x)) \times g'(x)$

Example: Differentiate $y = (x^2 + 3x - 5)^5$

Higher – Differentiate the outer function and multiply by the derivative of the outer function.

$$\frac{dy}{dx} = 5(x^2 + 3x - 5)^4(2x + 3)$$

AH – Identify the inner as u and write the outer function in terms of u.

$y = u^5$ and $u = x^2 + 3x - 5$

$$\frac{dy}{du} = 5u^4 \qquad \frac{du}{dx} = 2x + 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 5u^4(2x + 3)$$

$$\frac{dy}{dx} = 5(x^2 + 3x - 5)^4(2x = 3)$$

This more formal method is useful for more complicated functions but otherwise use the original method.

Exercise 3

1 Differentiate the following functions using both routines shown above.

a) $y = (x^2 + 4x - 5)^3$ b) $y = \sqrt{x^3 + 5}$ c) $y = \left(1 + 2\sqrt{x}\right)^4$

d) $y = \frac{3}{\sqrt{4-x^2}}$ e) $y = sin^4 x$ f) $y = cos^3 2x$

2 Show that

a) $\frac{d}{dx} secx = secxtanx$ b) $\frac{d}{dx} cosecx = -cosecxcotx$

The Product Rule

This is used to differentiate a function which is written as a product of two functions. If the functions are f(x) and g(x) then

$(fg)' = f'g + fg'$

Differentiate the first, leave the second add leave the first, differentiate the second.

Example: Differentiate

a) $y = x^2\sin3x$ Here $f(x) = x^2$ and $g(x) = \sin3x$

$\frac{dy}{dx} = 2x\sin3x + x^2 3\cos3x$

$= 2x\sin x + 3x^2\cos3x$

b) $y = x\sqrt{2x-1}$

$= x(2x-1)^{\frac{1}{2}}$ Here f(x)=x and g(x)= $(2x-1)^{\frac{1}{2}}$

$\frac{dy}{dx} = 1 \times (2x-1)^{\frac{1}{2}} + x\frac{1}{2}(2x-1)^{-\frac{1}{2}} \times 2$

$= \frac{(2x-1)^{\frac{1}{2}}}{1} + \frac{x}{(2x-1)^{\frac{1}{2}}}$

$= \frac{2x-1}{(2x-1)^{\frac{1}{2}}} + \frac{x}{(2x-1)^{\frac{1}{2}}}$ Find common denominator.

$= \frac{3x-1}{\sqrt{2x-1}}$ Write answer as a single fraction.

Exercise 4:

1 Differentiate the following functions using the product rule.

a) $y = x^2(x-3)^2$ b) $y = x(2x+3)^4$ c) $y = x\sqrt{x-6}$

d) $y = \sqrt{x}(x-3)^3$ e) $y = (x+1)^2(x-1)^4$ f) $y = x^3\sqrt{3x-1}$

g) $y = xsinx$ h) $y = x^2sinx$ i) $y = sinxcosx$

j) $y = sin2xcos5x$ k) $y = cosx^2sin^3x$

The Quotient Rule

This is used to differentiate a function which is written as a quotient of two functions. If the functions are f(x) and g(x) then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

For the numerator - differentiate the first, leave the second subtract leave the first, differentiate the second.
For the denominator – square the original denominator.

Examples: Differentiate the following functions using the quotient rule

a) $y = \frac{x^2-1}{x^2+1}$

$$\frac{dy}{dx} = \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2}$$

$$= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

b) $y = \frac{2x}{\sqrt{x^2+1}}$

$$\frac{dy}{dx} = \frac{2(x^2+1)^{\frac{1}{2}} - 2x\frac{1}{2}(x^2+1)^{-\frac{1}{2}}2x}{x^2+1}$$

$$= \frac{2(x^2+1)^{\frac{1}{2}} - 2x^2(x^2+1)^{-\frac{1}{2}}}{x^2+1}$$

$$= \frac{2(x^2+1) - 2x^2}{(x^2+1)^{\frac{3}{2}}} = \frac{2}{(x^2+1)^{\frac{3}{2}}}$$

Exercise 5

1 Differentiate using the quotient rule

a) $y = \frac{x^2}{x+3}$ b) $y = \frac{4-x}{x^2}$ c) $y = \frac{4x}{(1-x)^3}$

d) $y = \frac{2x^2}{x-2}$ e) $y = \frac{(1-2x)^3}{x^3}$ f) $y = \frac{\sqrt{x+1}}{x^2}$

2 Differentiate the functions in q1b and e using the “splitting up” method from Higher.

3 Try q1d by dividing $2x^2$ by x-2 first.

4 Show that a) $\frac{d}{dx}tanx = sec^2x$ b) $\frac{d}{dx}cotx = -cosec^2x$

Derivatives of New Functions	
f(x)	f'(x)
tanx	sec^2x
cosecx	-cosecxcotx
secx	secxtanx
cotx	$-cosec^2x$
e^x	e^x
Ln x	$\frac{1}{x}$

Many of these have been proved using the chain rule and quotient rule, now they should be learned as standard derivatives.

Examples: Differentiate

a) $y = tan2x$

$\frac{dy}{dx} = 2sec^2 2x$

b) $y = tan^2 x$

$\frac{dy}{dx} = 2tanxsec^2 x$

c) $y = 3cosec2x$

$\frac{dy}{dx} = 3 \times -2cosec2xcot2x$

$= -6cosec2xcot2x$

d) $y = 2sec3x$

$\frac{dy}{dx} = 2 \times 3sec3xtan3x$

$= 6sec3xtan3x$

e) $y = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

f) $y = e^{x^2}$

$\frac{dy}{dx} = 2xe^{x^2}$

In general: $\boldsymbol{\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}}$

g) $y = \ln(3x+2)$

$\frac{dy}{dx} = 3 \times \frac{1}{3x+2}$

$= \frac{3}{3x+2}$

h) $y = ln4x$

$\frac{dy}{dx} = 4 \times \frac{1}{4x}$

$= \frac{1}{x}$

In general: $\{\boldsymbol{ln(f(x))}\} = \boldsymbol{f'(x)} \times \frac{\boldsymbol{1}}{\boldsymbol{f(x)}}$

i) $y = x^2 e^{3x}$

$f(x) = x^2 \qquad g(x) = e^{3x}$

$f'(x) = 2x \qquad g'(x) = 3e^{3x}$

$\frac{dy}{dx} = 2xe^{3x} + 3x^2e^{3x}$

j) $y = \frac{lnx}{x}$

$f(x) = lnx \qquad g(x) = x$

$f'(x) = \frac{1}{x} \qquad g'(x) = 1$

$\frac{dy}{dx} = \frac{\frac{1}{x}\times x - lnx \times x}{x^2}$

$= \frac{1-lnx}{x^2}$

Exercise 6: Differentiate

a) $y = tan^3 2x$

b) $y = -2cosec^4 x$

c) $y = secxtanx$

d) $y = x^2 cotx$

e) $y = \ln(5x+2)$

f) $y = (x+2)e^{-x}$

g) $y = \frac{e^x}{x+2}$

h) $y = \frac{x^2}{lnx}$

i) $y = ln\sqrt{x^2+1}$

j) $y = xe^{-2x^2}$

k) $y = ln\left(\frac{1+x}{1-x}\right)$

l) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Higher Derivatives

Let $y = x^2 + 3x + 4$

$$\frac{dy}{dx} = 2x + 3$$

Here $\frac{dy}{dx}$ is defined as a function of x and so can be differentiated with respect to x.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2$$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is usually written as $\frac{d^2y}{dx^2}$, $f''(x), or\ y''$ and is called the second derivative of y with respect to x.

This process can be repeated to get the 3rd, 4th ....nth derivative which can be written as $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ ... $\frac{d^ny}{dx^n}$.

The second derivative can be used to find the nature of stationary points.
If $\frac{d^2y}{dx^2} > 0$ at $x = a$ then there is a minimum turning point at $x = a$.
If $\frac{d^2y}{dx^2} < 0$ at $x = a$ then there is a minimum turning point at $x = a$.

Examples

1 Find the second derivative of $f(x) = x^3 - x^2 + 5x - 6$

$$f'(x) = 3x^2 - x + 5$$

$$f''(x) = 6x - 1$$

2 Find the coordinates of the stationary points of the curve $y = 2x^3 - 3x^2 - 36x$

$\frac{dy}{dx} = 6x^2 - 6x - 36$

For stationary points, set $\frac{dy}{dx} = 0$

$6x^2 - 6x - 36 = 0$

$6(x-3)(x+2) = 0$

$x = 3 \qquad x = -2$

$y = 81 \qquad y = 44$

$\frac{d^2y}{dx^2} = 12x - 6$

At (3,81) $\frac{d^2y}{dx^2} > 0$ There is a minimum turning point at (3,81)

At (-2,44) $\frac{d^2y}{dx^2} < 0$ There is a maximum turning point at (3,81)

Exercise 7

1 Find the second derivative of y = $x^3 + 5x^2$.

2 Find the third derivative of f(x) = sin3x.

3 Find $\frac{d^4y}{dx^4}$, when $y = e^{ax}$ and where a is a constant.

Hence, make a conjecture for $\frac{d^ny}{dx^n}$.

4 f(x) = ln(1-x), make a conjecture about the nth derivative.

5 Find the coordinates of the stationary points of the curve $y = x^3 - 3x + 2$

The Derivatives of Inverse Trigonometric Functions

$y = \sin^{-1}x$

$If\ y = \sin^{-1}x, then\ by\ definition, x = siny.$

$\frac{dx}{dy} = cosy \quad \left[assume\ that\ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},\ \frac{dx}{dy} \neq 0\right]$

$\frac{dy}{dx} = \frac{1}{cosy}$

$But\ since\ cos^2y = 1 - sin^2y = 1 - x^2$

$cosy = \pm\sqrt{1 - x^2}$

$Since\ y = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], then\ cosy > 0\ and\ so\ cosy = \sqrt{1 - x^2}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$Result: y = \sin^{-1}x \implies \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$y = \cos^{-1}x$

$If\ y = \cos^{-1}x, then\ by\ definition, x = cosy.$

$\frac{dx}{dy} = -siny \quad \left[assume\ that\ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},\ \frac{dx}{dy} \neq 0\right]$

$\frac{dy}{dx} = -\frac{1}{siny}$

$But\ since\ sin^2y = 1 - cos^2y = 1 - x^2$

$siny = \pm\sqrt{1 - x^2}$

$Since\ y = [0, \pi], then\ siny > 0\ and\ so\ siny = \sqrt{1 - x^2}$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$Result: y = \cos^{-1}x \implies \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$y = tan^{-1}x$

$If\ y = tan^{-1}x, then\ by\ definition, x = tany.$

$\frac{dx}{dy} = sec^2y \quad \left[assume\ that\ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},\ \frac{dx}{dy} \neq 0\right]$

$= 1 + tan^2y$

$= 1 + x^2$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

$$Result: y = tan^{-1}x \Longrightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

Examples: Differentiate

a) $y = sin^{-1}3x$

$y = sin^{-1}u\ \ where\ u = 3x$

$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}\ and\ \frac{du}{dx} = 3$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times 3$

$= \frac{3}{\sqrt{1-9x^2}}$

b) $y = tan^{-1}4x$

$\frac{dy}{dx} = \frac{1}{1+(4x)^2} \times 4$

$= \frac{4}{1+16x^2}$

c) $y = cos^{-1}\left(\frac{x}{2}\right)$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{1}{2}$

$= -\frac{1}{\sqrt{1-\frac{x^2}{4}}} \times \frac{1}{2}$

$= -\frac{1}{\sqrt{\frac{4}{4}-\frac{x^2}{4}}} \times \frac{1}{2}$

$= -\frac{1}{\sqrt{\frac{4-x^2}{4}}} \times \frac{1}{2}$

$= -\frac{1}{\frac{\sqrt{4-x^2}}{2}} \times \frac{1}{2}$

$= -\frac{1}{\sqrt{4-x^2}}$

d) $y = cos^{-1}\left(\frac{1}{1+x}\right)$

$y = cos^{-1}u\ \ where\ \ u = (1+x)^{-1}$

$\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}\ \ and\ \frac{du}{dx} = -(1+x)^{-2}$

$= -\frac{1}{(1+x)^2}$

$u^2 = \frac{1}{(1+x)^2}$

$1 + u^2 = 1 - \frac{1}{(1+x)^2}$

$= \frac{(1+x)^2-1}{(1+x)^2}$

$= \frac{x^2+2x}{(1+x)^2}$

$\sqrt{1+u^2} = \frac{\sqrt{x^2+2x}}{1+x}$

$-\frac{1}{\sqrt{1-u^2}} = \frac{-(1+x)}{\sqrt{x^2+2x}}$

$\frac{dy}{dx} = \frac{-(1+x)}{\sqrt{x^2+2x}} \times -\frac{1}{(1+x)^2} = \frac{1}{(1+x)\sqrt{x^2+2x}}$

Exercise 8: Differentiate

a) $y = sin^{-1}(\sqrt{x})$

b) $y = tan^{-1}(\sqrt{x})$

c) $y = xtan^{-1}x$

d) $y = xtan^{-1}\left(\frac{x}{2}\right)$

e) $y = xsin^{-1}x + \sqrt{1-x^2}$

f) $y = cos^{-1}(2x-1)$

g) $y = sin^{-1}\left(\frac{x-1}{x+1}\right)$

h) $y = tan^{-1}\left(\frac{x-1}{x+1}\right)$

i) $y = tan^{-1}\left(\frac{2x}{\sqrt{1-x^2}}\right)$

j) $y = sin^{-1}\left(\frac{2x}{\sqrt{1-x^2}}\right)$

k) $y = tan^{-1}(secx)$

l) $y = cos(sin^{-1}x)$

m) $y = tan^{-1}(e^x)$

n) $y = sin^{-1}(x^2) - xe^{x^2}$

Implicit Differentiation

Consider a function y defined by the equation: $3x^2 + 7xy + 9y^6 = 6$
It is difficult or even impossible to write y in terms of x.
Such a function, y, is called an implicit differentiation.
These can be differentiated term by term with respect to x, assuming that y is a function of x.

Examples:

1 Differentiate $3x^2 + 7xy + 9y^6 = 6$ with respect to x.

Term 1 $\frac{d}{dx}(3x^2) = 6x$

Term 2 $7xy$ is a product and is differentiated using the product rule

$$\frac{d}{dx}(7xy) = \frac{d}{dx}(7x) \times y + 7x \times \frac{dy}{dx} \quad = 7y + 7x\frac{dy}{dx}$$

Term 3 $9y^2$ is a function of y and y is a function of x

$$\frac{d}{dx}(9y^2) = 18y\frac{dy}{dx}$$

Term 4 6 is a constant and $\frac{d}{dx}(6) = 0$

Therefore differentiating $3x^2 + 7xy + 9y^6 = 6$ gives

$$6x + 7y + 7x\frac{dy}{dx} + 18y\frac{dy}{dx} = 0$$

$$(7x + 18y)\frac{dy}{dx} = -(6x + 7y)$$

$$\frac{dy}{dx} = -\frac{(6x+7y)}{(7x+18y)}$$

2 Find the equation of the tangent at the point $(2,1)$ on the curve $2x^2 - 3xy - y^2 = 1$.

Differentiating gives $4x - 3y - 3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$$-(3x + 2y)\frac{dy}{dx} = -4x + 3y$$

$$\frac{dy}{dx} = \frac{4x-3y}{3x+2y}$$

When $x = 2$ and $y = 1$: $\frac{dy}{dx} = \frac{5}{8}$

The gradient of the tangent at $(2,1)$ is $\frac{5}{8}$

The equation of the tangent is $y - 1 = \frac{5}{8}(x - 2)$

$$5x - 8y = 2$$

3 Find the derivative of $sinx + 2cosy = 1$

$$cosx - 2siny\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{cosx}{2siny}$$

4 Show that $(1,2)$ is a stationary point on the curve $x^2 - xy + y^3 = 7$ and find its nature.

Substitute $(1,2)$ into the curve $LHS = 1^2 - 1 \times 2 + 2^3 = 7 = RHS$, therefore $(1,2)$ lies on the curve.

Differentiating with respect to x gives

$$2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(3y^2 - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{3y^2-x}$$

At $(1,2)$, $\frac{dy}{dx} = 0$, hence $(1,2)$ is a stationary point on the curve.

To find $\frac{d^2y}{dx^2}$, differentiate $2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

Term 3 $-x\frac{dy}{dx}$ is the product of $-x$ and $\frac{dy}{dx}$,

the derivative is $-\frac{dy}{dx} - x\frac{d^2y}{dx^2}$

Term 4 $3y^2\frac{dy}{dx}$ is the product of $3y^2$ and $\frac{dy}{dx}$,

the derivative is $6y\frac{dy}{dx}\frac{dy}{dx} + 3y^2\frac{d^2y}{dx^2} = 6y\left(\frac{dy}{dx}\right)^2 + 3y^2\frac{d^2y}{dx^2}$

The derivative of $2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$ is

$$2 - \frac{dy}{dx} - \frac{dy}{dx} - x\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 3y^2\frac{d^2y}{dx^2} = 0$$

Substituting $(1,2)$, $\frac{dy}{dx} = 0$ into the above line gives

$$2 - \frac{d^2y}{dx^2} + 12\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = -\frac{2}{11} < 0$$

Therefore $(1,2)$ is a maximum turning point.

Exercise 9:

1 Find $\frac{dy}{dx}$ for the following

a) $x^2 - y^2 = 0$

b) $y^2 = 2x + 2y$

c) $xy^2 = 9$

d) $4x^2 - y^3 + 2x + 3y = 0$

e) $x^3y + xy^3 = x - y$

f) $sinxcosy = 1$

g) $e^{xy} = 2$

h) $e^x lny = x$

2 Find the equations of the tangents at the points on the following curves

a) $x^3 - 2y^3 = 3xy$ at $(2,1)$

b) $x^2y^2 = x^2 + 5y^2$ at $\left(3, \frac{3}{2}\right)$

c) $y(x+y)^2 = 3(x^3 - 5)$ at $(2,1)$

3 For the curve $xy(x+y) = 84$, find $\frac{dy}{dx}$ at $(3,4)$.

4 Find the gradient of the curve $x^2 + 3xy + y^2 = x + y + 8$ at $(1,2)$.

5 Find the 2nd derivative of

a) $x^2 - y^2 = 0$

b) $y^2 = 2x + 2y$

c) $xy^2 = 9$

d) $4x^2 - y^3 + 2x + 3y = 0$

6 For the curve, $4x^2 + y^3 = 2x + 7y$, find the values of:

$\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1,2)$.

7 Show that $(-1,3)$ and $(0,0)$ are stationary points on the curve $3x^2 + 2xy - 5y^2 = 16y = 0$ and find the nature of each.

Related Rates of Change

When one variable, x, is a function of another, t, then

$$\frac{dx}{dt} = \frac{1}{\frac{dt}{dx}}$$

When y is a function of x, say $y = f(x)$, and both y and x are functions of a third variable, t, such that $x = x(t)$ and $y = y(t)$ then we have, by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Examples:

1 Air is blown into a spherical balloon at a rate of $160cm^3/sec$.
Find the rate of increase of the radius when the radius is $5cm$.

$V_{sphere} = \frac{4}{3}\pi r^3$ from this we can obtain $\frac{dV}{dr} = 4\pi r^2$

We are given $\frac{dV}{dt} = 160$.

We require $\frac{dr}{dt}$.

The equation connecting the derivatives is

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$160 = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{160}{4\pi r^2}$

When $r = 5 \implies \frac{dr}{dt} = \frac{160}{4\pi(5)^2} = 0.51cm/s$

2 If $G = (3m + 2)^3$ find $\frac{dm}{dt}$ when $m = 1$, given that $\frac{dG}{dt} = 3$.

From $G = (3m + 2)^3$ we can find $\frac{dG}{dm} = 9(3m + 2)^2$

The equation connecting the derivatives is

$\frac{dG}{dt} = \frac{dG}{dm} \times \frac{dm}{dt}$

$3 = 9(3m + 2)^2 \times \frac{dm}{dt}$

$\frac{dm}{dt} = \frac{3}{9(3m+2)^2} = \frac{1}{3(3m+2)^2}$

When $m = 1, \frac{dm}{dt} = \frac{1}{3(3\times1+2)^2} = \frac{1}{75}$

3 A cylinder, of radius r and height h, is closed at both ends. Its total surface area is $15units^2$. Find an expression for $\frac{dr}{dh}$.

The surface area is given by $A = 2\pi r^2 + 2\pi rh$ so
$2\pi r^2 + 2\pi rh = 15$ and r is a function of h.

Implicit differentiation gives

$4\pi r\frac{dr}{dh} + 2\pi h\frac{dr}{dh} + 2\pi r = 0$

$\frac{dr}{dh} = \frac{-2\pi r}{4\pi r+2\pi h} = \frac{-r}{2r+h}$

Exercise 10:

1 If $P = (2m+3)^4$, find $\frac{dm}{dt}$ when $m = 1$, given that $\frac{dP}{dt} = 2$.

2 If $r = \frac{1+p}{1+p^2}$, find $\frac{dp}{dt}$ when $p = 1$, given that $\frac{dr}{dt} = 14$.

3 The radius of a circular oil slick is increasing at a rate of 0.2metres/second. Find the rate at which the area is increasing when the radius is 10m.

4 A rectangle has dimensions $x$cm and $y$cm. Both x and y are changing but in such a way that the area of a rectangle remains constant at 40cm^2.

a) Show that $\frac{dy}{dt} = \frac{40}{x^2} \times \frac{dx}{dt}$

b) If x increases at a rate of 0.2cm/sec, find the rate at which y is changing when $x = 8$.

5 The volume of a cylinder is constant at 50cm^3, but both height and radius are changing.

a) Show that $\frac{dh}{dt} = \frac{-100}{\pi r^3} \times \frac{dr}{dt}$

b) At an instant when the radius is 5cm, the height is decreasing by 3cm/sec. Find the rate of change of the radius at this instant.

6 The volume of a sphere is increasing at a rate of 6cm^3/s.
Find the rate at which the surface area is increasing when the radius is 3cm.

Logarithmic Differentiation

The differentiation of complicated functions, particularly those that have powers consisting of functions of x, or contain products and quotients, can be made simpler by taking logarithms to the base e and differentiating implicitly.

Examples: Find the derivative

a) $y = a^x$

Take ln of both sides

$lny = lna^x$

$lny = xlna$

Differentiating implicitly

$\frac{1}{y} \times \frac{dy}{dx} = lna$ (lna is a constant)

$\frac{dy}{dx} = ylna$

$\frac{dy}{dx} = a^x lna$

b) $y = x^x$

Take ln of both sides

$lny = lnx^x$

$lny = xlnx$

Differentiating implicitly

$\frac{1}{y} \times \frac{dy}{dx} = lnx + x \times \frac{1}{x}$

$\frac{1}{y} \times \frac{dy}{dx} = lnx + 1$

$\frac{dy}{dx} = y(lnx + 1)$

$\frac{dy}{dx} = x^x(lnx + 1)$

c) $y = 2^{sinx}$

Take ln of both sides

$lny = ln2^{sinx}$

$lny = sinxln2$

Differentiating implicitly

$\frac{1}{y} \times \frac{dy}{dx} = cosxln2$

$\frac{dy}{dx} = ycosxln2$

$\frac{dy}{dx} = 2^{sinx}\, cosxln2$

d) $y = (sinx)^x$

Take ln of both sides

$lny = ln(sinx)^x$

$lny = xln(sinx)$

Differentiating implicitly

$\frac{1}{y} \times \frac{dy}{dx} = ln(sinx) + x \times \frac{cosx}{sinx}$

$\frac{1}{y} \times \frac{dy}{dx} = ln(sinx) + xcotx$

$\frac{dy}{dx} = (sinx)^x(ln(sinx) + xcotx)$

e) $y = \sqrt{\frac{1+x}{1-x}}$

Take ln of both sides

$lny = ln\sqrt{\frac{1+x}{1-x}}$

$lny = ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$

$lny = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right)$

$lny = \frac{1}{2}\{\ln(1+x) - \ln(1-x)\}$

Differentiating implicitly

$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{2}\left\{\frac{1}{1+x} + \frac{1}{1-x}\right\} = \frac{1}{2}\left\{\frac{2}{(1+x)(1-x)}\right\} = \frac{1}{(1+x)(1-x)}$

$\frac{dy}{dx} = \sqrt{\frac{1+x}{1-x}}\left(\frac{1}{(1+x)(1-x)}\right) = \frac{1}{(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}$

Exercise 11: Differentiate using logarithmic differentiation

a) $y = 10^x$ b) $y = 2^{x^2}$ c) $y = x^{-x}$

d) $y = x^{sinx}$ e) $y = x^{\frac{1}{x}}$ f) $y = x^{lnx}$

g) $y = (lnx)^x$ h) $y = (lnx)^{lnx}$ i) $y = \frac{x^5}{\sqrt{3x+5}}$

j) $y = \frac{x^3(2x-1)^5}{(x+1)^2}$

Using Parameters

It is sometimes convenient to define the co-ordinates of a moving point by means of two equations, expressing x and y separately in terms of a third variable, say t, called a parameter.

For example the position of a ball, relative to the x and y axes, may be plotted at 1 second intervals. Here, clearly the x and y positions of the ball depend on time t.
It can be shown that the path of the ball is given by the equations

$x = at^2$ and $y = 2at$ where a is a constant.

These equations are called parametric equations of the curve, t being the parameter.
By replacing t by various values in the two equations, we obtain the coordinates of various points on the curve.

$P(t): x = at^2, y = 2at$ Hence P is the point $(at^2, 2at)$
$P(0): x = 0, y = 0$ Hence P is the point $(0,0)$
$P(1): x = a, y = 2a$ Hence P is the point $(a, 2a)$
$P(2): x = 4a, y = 4a$ Hence P is the point $(4a, 4a)$
$P(3): x = 9a, y = 6a$ Hence P is the point $(9a, 6a)$
and so on

$x = at^2$ and $y = 2at$ are the parametric equations of a parabola.
It is often useful to eliminate t from the two equations to obtain an equation involving x and y but not t.

This equation is called the constraint equation.

From $y = 2at \implies t = \frac{y}{2a}$

From $x = at^2 \implies x = \frac{ay^2}{4a^2}$

$\implies x = \frac{y^2}{4a}$

$\implies y^2 = 4ax$

which is the general equation of a parabola whose axis of symmetry is the x-axis.

Other common parametric equations are:

Parametric Equation	Constraint Equation	Shape
$x = rcos\theta, y = rsin\theta$	$x^2 + y^2 = r^2$	circle
$x = acos\theta, y = bsin\theta$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	ellipse
$x = asec\theta, y = btan\theta$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	hyperbola
$x = ct, y = \frac{c}{t}$	$xy = c^2$	rectangular hyperbola

Finding the First Derivative of a Parametric Function

If a function is defined as $x = x(t), y = y(t)$ (i.e.x and y are functions of t) then $\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$ since $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ and $\frac{dx}{dt} = \frac{1}{\frac{dt}{dx}}$.

Examples:

1 Find $\frac{dy}{dx}$ on the curve $x = at^2, y = 2at$.

$x = at^2$ $\qquad$ $y = 2at$

$\frac{dx}{dt} = 2at$ $\qquad$ $\frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

2 Find a formula for the gradient of the tangent to the curve whose parametric equations are $x = a(\theta - sin\theta), y = a(1 - cos\theta)$.

$x = a(\theta - sin\theta) \qquad y = a(1 - cos\theta)$

$\frac{dx}{d\theta} = a(1 - cos\theta) \qquad \frac{dy}{d\theta} = asin\theta$

$\frac{dy}{dx} = \frac{asin\theta}{a(1-cos\theta)} = \frac{sin\theta}{1-cos\theta}$

3 Find the coordinates of the points on the curve $x = 1 - t^2, y = t^3 + t$ at which the gradient is 2.

$x = 1 - t^2 \qquad y = t^3 + t$

$\frac{dx}{dt} = -2t \qquad \frac{dy}{dt} = 3t^2 + 1$

$\frac{dy}{dx} = \frac{3t^2+1}{-2t}$

Gradient =2, therefore $\frac{3t^2+1}{-2t} = 2$

$$3t^2 + 1 = -4t$$

$$3t^2 + 4t + 1 = 0$$

$$(t + 1)(3t + 1) = 0$$

$$t = -1 \qquad t = -\frac{1}{3}$$

At $t = -1$: $x = 0, y = -2$. The point is $(0, -2)$.

At $t = -\frac{1}{3}$: $x = \frac{8}{9}, y = -\frac{10}{27}$. The point is $\left(\frac{8}{9}, -\frac{10}{27}\right)$.

4 Find the equation of the tangent to the curve $y = at^2, y = 2at$, at the point $P(t)$.

$x = at^2 \qquad y = 2at$

$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

$P(t) = P(x, y) = (at^2, 2at)$

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty - 2at^2 = x - at^2$$

$$x - ty = -at^2$$

Exercise 12:

1 Find $\frac{dy}{dx}$ in terms of the parameter:

a) $x = t^3 + t^2,\ y = t^2 + t$

b) $x = 4cos\theta,\ y = 3sin\theta$

c) $x = \frac{1}{1+t},\ y = \frac{t}{1-t}$

d) $x = \frac{t-1}{t+1},\ y = \frac{2t-1}{t-2}$

e) $x = (t+1)^2,\ y = t^2 - 1$

f) $x = \frac{t}{1-t},\ y = \frac{t^2}{t+3}$

g) $x = \cos 2\theta,\ y = 4sin\theta$

h) $x = acos^2\theta,\ y = asin^3\theta$

i) $x = e^t cost,\ y = e^t sint$

j) $x = a(t - cost),\ y = a(1 - sint)$

2 Find the equations of the tangents to these curves at the point P(x,y)

a) $x = ct,\ y = \frac{c}{t}$

b) $x = at^2,\ y = at(t^2 - 1)$

c) $x = \frac{a}{2}\left(t + \frac{1}{t}\right),\ y = \frac{a}{2}\left(t - \frac{1}{t}\right)$

d) $x = sec\theta,\ y = tan\theta$

Finding the Second Derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

Examples:

1 For the curve $x = at^2, y = 2at$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$x = at^2$ $\qquad$ $y = 2at$

$\frac{dx}{dt} = 2at$ $\qquad$ $\frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \frac{dy}{dt}\left(\frac{1}{t}\right) \times \frac{1}{2at}$$

$$= -\frac{1}{t^2} \times \frac{1}{2at}$$

$$= -\frac{1}{2at^3}$$

2 For the curve $x = acos\theta,\ y = asin\theta$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$x = acos\theta \qquad\qquad y = asin\theta$

$\frac{dx}{dt} = -asin\theta \qquad\qquad \frac{dy}{dt} = acos\theta$

$\frac{dy}{dx} = \frac{acos\theta}{-asin\theta} = -cot\theta$

$\frac{d^2y}{dx^2} = \frac{dy}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= \frac{dy}{dt}(-cot\theta) \times \frac{1}{-asin\theta}$

$= cosec^2x \times \frac{1}{-asin\theta}$

$= -\frac{1}{sin^3\theta}$

3 Find the turning point on the curve $x = t,\ y = t^3 - 3t$ and determine their nature.

$x = t \qquad\qquad y = t^3 - 3t$

$\frac{dx}{dt} = 1 \qquad\qquad \frac{dy}{dt} = 3t^2 - 3$

$\frac{dy}{dx} = \frac{3t^2-3}{1} = 3t^2 - 3$

For stationary points, set $\frac{dy}{dx} = 0$.

$3t^2 - 3 = 0$

$3(t-1)(t+1) = 0$

$t = 1 \qquad t = -1$

When $t = -1,\ x = -1,\ y = 2,\ (-1,2)$

When $t = 1,\ x = 1,\ y = -2,\ (1,-2)$

$\frac{d^2y}{dx^2} = \frac{dy}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= \frac{dy}{dt}(3t^2 - 3) \times \frac{1}{1}$

$= 6t$

When $t = -1$, $\frac{d^2y}{dx^2}$ is negative, so $(-1,2)$ is a maximum t.p.

When $t = 1$, $\frac{d^2y}{dx^2}$ is positive, so $(1,-2)$ is a minimum t.p.

Exercise 13:

1 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t:

a) $x = \frac{1}{t^2},\ y = 1 + t$

b) $x = (t+1)^2,\ y = t^2 - 1$

c) $x = 4cos\theta,\ y = 3sin\theta$

d) $x = cos^2\theta,\ y = sin\theta$

e) $x = 2cos\theta - cos2\theta,\ y = 2sin\theta + sin2\theta$

2 A curve has parametric equations $x = t - cost,\ y = sint$.
Find the coordinates of the points at which the gradient of the curve is zero.

3 Find the coordinates of the stationary points on the curves and determine their nature:

a) $x = 4 - t^2,\ y = 4t - t^3$

b) $x = (5 - 3t)^2,\ y = 6t - t^2$

c) $x = t^2 + 1,\ y = t(t-3)^2$

4 Given that $x = t - sint,\ y = 1 - cost$ show that $y^2\frac{d^2y}{dx^2} + 1 = 0$.

Velocity

Suppose the point P moves along a curve in the $x - y$ plane and suppose we know its position at any time t is given by $x = f(t)$ and $y = g(t)$,

$\frac{dx}{dt}$ is the velocity in the x direction and $\frac{dy}{dt}$ is the velocity in the y direction.

The magnitude of the velocity is therefore $|v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

which gives the instantaneous speed of the particle at P.

Examples:

1 The position of a golf ball, t seconds after being hit, is given by $x = 10t,\ y = 30t - 5t^2$.

Find the speed of the golf ball when it first hits the ground.

The golf ball first hits the ground when $y = 0$.

$30t - 5t^2 = 0$

$5t(6 - t) = 0$

~~$t = 0$~~ $t = 6$

n/a

$\frac{dx}{dt} = 10$ $\frac{dy}{dt} = 30 - 10t$

At $t = 6$ $\frac{dx}{dt} = 10$ $\frac{dy}{dt} = -30$

The speed first hits the ground is

$$|v| = \sqrt{10^2 + (-30)^2} = 31.6m/s$$

2 A cannon ball is fired and its position t seconds later, is given by $x = 10t,\ y = 2 + 9t - 5t^2$.

The target is 2 metres above the ground.

a) Find how far away the target should be if the cannon ball is to hit it.

b) What is the speed of the cannon ball when it hits the target?

a) When $y = 2,\ 2 + 9t - 5t^2 = 2$

$$9t - 5t^2 = 0$$

$$t(9 - 5t) = 0$$

~~$t = 0$~~ $t = \frac{9}{5}$

n/a

When $t = \frac{9}{5},\ x = 10 \times \frac{9}{5} = 18$

The target should be placed with its centre 18 metres horizontally from the cannon and at a height of 2 metres.

b) $\frac{dx}{dt} = 10$ $\frac{dy}{dt} = 9 - 10t$

At $t = \frac{9}{5}$ $\frac{dx}{dt} = 10$ $\frac{dy}{dt} = -9$

The speed first hits the ground is

$$|v| = \sqrt{10^2 + (-9)^2} = 13.5m/s$$

Exercise 14:

1 At time t, the position of a moving point is given by

$x = t(2 - t),\ y = t(3 - t)$

Find the speed when $t = 0$ and $t = 2$.

2 At time t, the position of a moving point is given by

$x = t + 1,\ y = t^2 - 1$

Find the speed when $t = 2$.

3 At time t, the position of a moving point is given by

$x = cos2t,\ y = 2sint$

Find the speed when $t = 0$.

4 At time t, the position of a moving point is given by

$x = e^t,\ y = e^{-2t}$

Find the speed when $t = ln3$.

5 At time t, the position of a moving point is given by

$x = sect,\ y = tant$

Find the speed when $t = \frac{\pi}{6}$.

6 At time t, the position of a moving point is given by

$x = \ln(t + 1),\ y = t^2$

Find the speed when $t = 1$.

7 A particle moves so that its position at time t is given by
$x = 4\cos t,\ y = 3 sint$.
Show that its speed is $\sqrt{9 + 7sin^2 t}$.
Hence find the maximum and minimum speeds and the corresponding positions of the particle.

Sequences and Series

Introduction

A set of numbers, stated in a definite order, such that each number can be obtained from the previous number according to some rule is called a sequence.

3, 5, 7, 9, 11, ... is an infinite sequence.
3, 5, 7, 9,11,...........47 is a finite sequence.

An expression for the n^{th} term (u_n) of a sequence is useful since any specific term can be obtained from it
1, 4, 9, 16, 25,......... has an n^{th} term $u_n = n^2$.

The sum of the terms of a sequence is called a series. The series of a sequence is denoted by S_n.

Arithmetic Sequences and Series

An arithmetic sequence is one which each term differs from the previous term by a constant called the common difference (d).

i.e. $u_{n+1} - u_n = d$ or $u_{n+1} = u_n + d$

If the first term is denoted by a and the common difference is d, then

$$u_1 = a$$
$$u_2 = a + d$$
$$u_3 = a + 2d$$
$$u_4 = a + 3d$$
$$\vdots \qquad \vdots$$
$$u_n = a + (n-1)d$$

The n^{th} term of an arithmetic sequence is denoted by the formula:

$$u_n = a + (n-1)d$$

Example:

Find a formula for the n^{th} term of the sequence 8, 11, 14, 17,.....
Hence find the 20^{th} term.

$a = 8, \quad d = 11 - 8 = 3$

$u_n = a + (n-1)d$

$u_n = 8 + (n-1) \times 3$

$u_n = 3n + 5$

$u_{20} = 3 \times 20 + 5 = 65$

The sum of the first n terms of the series is:

$$S_n = a + (a+d) + (a+2d) + (a+3d) \dots (a + (n-1)d)$$

Rewriting these terms in reverse order gives

$$S_n = (a + (n-1)d) + \cdots + (a+3d) + (a+2d) + (a+d) + a$$

Adding corresponding pairs of terms gives

$$2S_n = (2a + (n-1)d) + (2a + (n-1)d) + \cdots + (2a + (n-1)d)$$

$2S_n = n(2a + (n-1)d)$ since there are n terms

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

The sum of the first n terms of an arithmetic series is found using:

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Note: If the series is finite, then

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$= \frac{n}{2}(a + a + (n-1)d)$$

$= \frac{n}{2}(a + l)$ where l is the last term.

Examples:

1 For $2 + 5 + 8 + 11 + ...$, find u_{15} and S_8.

$a = 2 \qquad d = 3$

$u_n = a + (n-1)d$

$u_n = 2 + (n-1) \times 3$

$u_n = 3n - 1$

$u_{15} = 3 \times 15 - 1 = 44$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_8 = \frac{8}{2}(2 \times 2 + (8-1) \times 3) = 100$

2 If the first term is 37 and the common difference is -4, find u_{15} and S_8.

$a = 37 \qquad d = -4$

$u_n = a + (n-1)d$

$u_n = 37 + (n-1) \times -4$

$u_n = -4n + 41$

$u_{15} = -4 \times 15 + 41 = -19$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_8 = \frac{8}{2}(2 \times 37 + (8-1) \times -4) = 184$

3 Find the number of terms in the series $5 + 8 + 11 + 14 + \cdots + 62$.

$a = 5 \qquad d = 3$

$u_n = a + (n-1)d$

$u_n = 5 + (n-1) \times 3$

$u_n = 3n + 2$

$3n + 2 = 62$

$n = 20$ The number of terms is 20.

> We do not know the value of n but do know the n^{th} term.

4 Find the sum of $2 + 4 + 6 + 8 + \cdots + 146$

We must determine the value of n first.

$$a = 2 \qquad d = 2$$
$$u_n = a + (n-1)d$$
$$u_n = 2 + (n-1) \times 2$$
$$u_n = 2n$$
$$2n = 146$$
$$n = 73$$
$$S_n = \frac{n}{2}(a + l)$$
$$S_{73} = \frac{73}{2}(2 + 146)$$
$$S_{73} = 5402$$

5 The second term of an arithmetic sequence is 18 and the fifth term is 21. Find the common difference, the first term and the sum of the first 10 terms.

Second term $a + d = 18$

Fifth term $a + 4d = 21$

Solving the simultaneous equations gives $a = 17, d = 1$.

The common difference is 1.

The first term is 17.

$$S_n = \frac{n}{2}(2a + (n-1)d)$$
$$S_{10} = \frac{10}{2}(2 \times 17 + (10-1) \times 1) = 215$$

Exercise 1:

1 Find the formula for the n^{th} term of each of the following sequences and find the requested term.

a) 3, 11, 19,... u_{19}

b) 8, 5, 2,... u_{15}

c) 7, 6.5, 6,... u_{12}

2 Find the number of terms in each of the following sequences.

a) 2, 4, 6,....46

b) 50, 47, 44,....14

c) 2, -9, -20,....-130

3 State the values of a and d in each of the following series and find the requested S_n.

a) 4+10+16+... S_{12}

b) 15+13+11+... S_{20}

c) 20+13+6+... S_{16}

4 For each of the arithmetic sequence, find S_n indicated.

a) $u_2 = 15 \quad u_5 = 21 \quad S_{10}$

b) $u_4 = 18 \quad d = -5 \quad S_{16}$

c) $u_3 = 7 \quad u_{12} = 61 \quad S_{15}$

5 $S_{10} = 120, \ S_{20} = 840,$ find S_{30}.

6 $u_{15} = 7, \ S_9 = 18,$ find a, d and u_{20}.

7 How many terms of the arithmetic series 28+24+20+........ does it take to give a sum of zero?

8 The sixth term of an arithmetic sequence is twice the third term. If the first term is 3, find d and the tenth term.

9 How many terms of the arithmetic series 1+3+5+.... will give the sum 1521?

Arithmetic Sequences and Series

A geometric sequence is one which the ratio of each term to the previous term is a constant called the common ratio (r).

i.e. $\frac{u_{n+1}}{u_n} = r$ or $u_{n+1} = u_n r$

If the first term is denoted by a and the common ratio is r, then

$u_1 = a$

$u_2 = u_1 r = ar$

$u_3 = u_2 r = ar^2$

$u_n = u_{n-1} r = ar^{(n-1)}$

The n^{th} term of a geometric sequence is denoted by the formula:
$$u_n = ar^{(n-1)}$$

Let S_n denote the sum of n terms, a the first term and r the common ratio.

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{(n-2)} + ar^{(n-1)}$$
$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{(n-2)} + ar^{(n-1)} + ar^n$$
$$S_n - rS_n = a - ar^n$$
$$(1-r)S_n = a - ar^n$$
$$S_n = \frac{a(1-r^n)}{1-r} \quad r \neq 1$$

Note:

If $r > 1$, it is more convenient to use the result in this form $S_n = \frac{a(r^n-1)}{r-1}$

The sum of the first n terms of a geometric series:
$$S_n = \frac{a(1-r^n)}{1-r} \quad r \neq 1$$

Examples:

1 Find u_{10} for the geometric sequence $144, 108, 81, 60.75$ ...

$$a = 144 \quad r = \frac{144}{108} = \frac{3}{4}$$
$$u_n = ar^{(n-1)}$$
$$u_n = 144 \times \left(\frac{3}{4}\right)^{(n-1)}$$
$$u_{10} = 144 \times \left(\frac{3}{4}\right)^9 = 10.812 \ldots$$

2 Find $S_{19} = 3 - 6 + 12 - 24 + \cdots$

$$a = 3 \quad r = \frac{-6}{3} = -2$$
$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S_n = \frac{3(1-(-2)^n)}{1+2}$$
$$S_{19} = \frac{3(1-(-2)^{19})}{3} = 524289$$

3 A geometric series has the first term 27 and common ratio $\frac{4}{3}$. Find the least number of terms the series can have if its sum exceeds 550.

$$a = 27 \qquad r = \frac{4}{3}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{27\left[\left(\frac{4}{3}\right)^n - 1\right]}{\frac{4}{3} - 1}$$

$$S_n = \frac{27\left[\left(\frac{4}{3}\right)^n - 1\right]}{\frac{1}{3}}$$

$$S_n = 81\left[\left(\frac{4}{3}\right)^n - 1\right]$$

$$S_n = 550$$

$$81\left[\left(\frac{4}{3}\right)^n - 1\right] = 550$$

$$\left(\frac{4}{3}\right)^n - 1 = \frac{550}{81}$$

$$\left(\frac{4}{3}\right)^n = \frac{631}{81}$$

$$ln\left(\frac{4}{3}\right)^n = ln\frac{631}{81}$$

$$nln\left(\frac{4}{3}\right) = ln\frac{631}{81}$$

$$n = 7.136..$$

For $S_n > 550 \Rightarrow n > 7.136.. \Rightarrow n \geq 8 \Rightarrow n = 8$

4 Given $u_3 = 32$, $u_6 = 4$. Find a, r and S_8.

$u_3 = ar^2 = 32$ [1]

$u_6 = ar^5 = 4$ [2]

[2]÷[1] $\frac{ar^5}{ar^2} = \frac{4}{32}$

$r^3 = \frac{1}{8}$

$r = \frac{1}{2}$

Substituting $r = \frac{1}{2}$ into u_3 gives

$a\left(\frac{1}{2}\right)^2 = 32$

$a = 128$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_n = \frac{128\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}} = 255$

Exercise 2:

1 Find the common ratio for each of these geometric sequences

a) 1, 3, 9, 27, ... b) 12, 6, 3, 1.5, ...

c) 7, 0.7, 0.07, ... d) 18, 54, 162, ...

e) 2.25, 1.5, 1, ... f) $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, ...

g) 1, -1, 1, -1, ... h) 1, -2, 4, -8, ...

2 Write down the first 4 terms of these geometric sequences

a) $u_n = 3^{(n-1)}$ b) $u_n = 3(-2)^{(n-1)}$ c) $u_n = 6\left(\frac{1}{2}\right)^{(n-1)}$

3 By first finding u_n, find the required term in the following geometric sequences

a) 1, 2, 4, ... u_5 b) 2, 6, 18, ... u_6

c) 4, 12, 36, ... u_6 d) 2, 20, 200, ... u_5

e) 1, -2, 4, ... u_6 f) 6, 3, $\frac{3}{2}$, ... u_7

4 Find the formula for the n^{th} term of these geometric sequences

a) 1, 2, 4, ... b) 3, 6, 12, ... c) 2, -6, 18, ...

d) 9, 3, 1, ... e) 4, 2, 1, ... f) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, ...

5 Find the common ratio and the fifth term of these geometric sequences

a) $a = 6 \ \ u_3 = 24$ b) $a = \ 50 \ \ u_4 = 400$

c) $a = 36 \ \ u_2 = -12$

6 Find the sum of each of the following geometric series and simplify the answer as far as possible

a) $1 + 2 + 4 +$.... to 8 terms b) $2 + 6 + 18 +$.... to 6 terms

c) $2 - 4 + 8 -$.... to 5 terms d) $2 - 6 + 18 -$.... to 5 terms

e) $1 + \frac{1}{2} + \frac{1}{4} +$.... to 6 terms f) $1 + \frac{1}{3} + \frac{1}{9} +$.... to 5 terms

g) $1 + x + x^2 +$.... to n terms h) $1 - y + y^2 -$.... to n terms

7 Find n if

a) $3 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = 363$

b) $2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 510$

The Sum to Infinity of a Geometric Series

Consider the infinite series $8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$....

$S_1 = 8$ $S_2 = 12$ $S_3 = 14$ $S_4 = 15$

$S_5 = 15\frac{1}{2}$ $S_6 = 15\frac{3}{4}$ $S_7 = 15\frac{7}{8}$ $S_8 = 15\frac{15}{16}$

$S_9 = 15\frac{31}{32}$ $S_{10} = 15\frac{63}{64}$ $S_{11} = 15\frac{127}{128}$ $S_{12} = 15\frac{255}{256}$

The sum S_n appears to be approaching a value of 16.
By taking a sufficiently large value of n, we can make S_n as near to 16 as we wish.

We say that S_n tends to a limit of 16 as n approaches infinity and we write

$S_n \to 16 \;\; as \;\; n \to \infty$ (The series tends to 16 as n tends to infinity)

$\lim_{n \to \infty} S_n = 16$

$S_\infty = 16$

In general, $S_n = \frac{a(1-r^n)}{1-r}$, and if $-1 < r < 1$, then $r^n \to 0$ for large values of n.

Therefore, as $n \to \infty, r^n \to 0$ then $S_n = \frac{a(1-r^n)}{1-r}$ becomes $S_\infty = \frac{a}{1-r}$.

The sum to infinity of a geometric series

$$S_\infty = \frac{a}{1-r}$$

if and only if $-1 < r < 1$.

Example: Find the sum to infinity of the geometric series $16 + 12 + 9 + ...$

$a = 16, \quad r = \frac{12}{16} = \frac{3}{4}$

Since $-1 < r < 1$, S_∞ exists. This must be stated

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{16}{1-\frac{3}{4}} = 64$

Exercise 3:

By first finding the common ratio, find the sum to infinity if it exists.

a) $1 + \frac{1}{3} + \frac{1}{9} + ...$ b) $1 + 2 + 4 + ...$ c) $4 + 1 + \frac{1}{4} + ...$

d) $8 + 4 + 2 + ...$ e) $1 - 5 + 25 - ...$ f) $10 - 9 + 8.1 - ..$

g) $1 - \frac{1}{2} + \frac{1}{4} - ...$ h) $2 + \frac{4}{3} + \frac{8}{9} + ...$

Power Series

Suppose that $f(x) = a + bx + cx^2 + dx^3$

Then $f'(x) = b + 2cx + 3dx^2$

$f''(x) = 2c + 6dx$

$f'''(x) = 6d$

All further derivatives are zero.

From above $f(0) = a$

$f'(0) = b$

$f''(0) = 2c \quad \Rightarrow \quad c = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$

$f'''(0) = 6d \quad \Rightarrow \quad d = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$

Hence $f(x) = a + bx + cx^2 + dx^3$ can be written as

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

Any series of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$ is called a power series.

In many cases, the sum of such series becomes bigger and bigger as you add on each successive term, in which case the series is said to diverge. On the other hand, some series are such that, as more and more terms are added, the sum approaches more and more closely to a particular limit, (i.e. a single function), in which case it is said to converge to this limit.

MacLaurin's Theorem

MacLaurin's Theorem states that, under certain circumstances, a function $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(iv)}(0)}{4!}x^4 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

The series can be found if $f^{(n)}(0)$ exists for all values of n.

Some series converge to $f(x)$ for all values of x and other converge to $f(x)$ for a limited range of x.

In the following examples, the range of values of x for which the series is valid will be given, but not justified.

Examples: Use MacLaurin's theorem to expand as a series of ascending powers of x:

a) $f(x) = e^x$

$$f(x) = e^x \qquad f(0) = e^0 = 1$$
$$f'(x) = e^x \qquad f'(0) = e^0 = 1$$
$$f''(x) = e^x \qquad f''(0) = e^0 = 1$$
$$f'''(x) = e^x \qquad f'''(0) = e^0 = 1$$

Hence, $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$ for all $x \in \mathbb{R}$

This can be written as $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

b) $f(x) = \ln(1 + x)$

$$f(x) = \ln(1 + x) \qquad f(0) = \ln(1 + 0) = 0$$
$$f'(x) = \frac{1}{1+x} \qquad f'(0) = \frac{1}{1+0} = 1$$
$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''^{(0)} = -\frac{1}{(1+0)^2} = -1$$
$$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(0) = \frac{2}{(1+0)^3} = 2$$
$$f''''(x) = -\frac{6}{(1+x)^4} \qquad f''''(0) = -\frac{6}{(1+0)^4} = -6$$

Hence, $\ln(1 + x) = 0 + 1x + \frac{(-1)}{2!}x^2 + \frac{2}{3!}x^3 + \frac{(-6)}{4!}x^4 + \cdots$

$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for $-1 < x \leq 1$

c) $f(x) = sinx$ (x in radians)

$$f(x) = sinx \qquad f(0) = sin0 = 0$$
$$f'(x) = cosx \qquad f'(0) = cos0 = 1$$
$$f''(x) = -sinx \qquad f''(0) = -sin0 = 0$$
$$f'''(x) = -cosx \qquad f'''(0) = -cos0 = -1$$
$$f^{(iv)}(x) = sinx \qquad f^{(iv)}(0) = sin0 = 0$$

Hence, $sinx = 0 + 1x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \cdots$

$sinx = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ for all x

d) $f(x) = tan^{-1}x$

$f(x) = tan^{-1}x$ $\qquad f(0) = tan^{-1}0 = 0$

$f'(x) = \frac{1}{1+x^2}$ $\qquad f'(0) = \frac{1}{1+0^2} = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\qquad f''(0) = \frac{-2\times 2}{(1+0^2)^2} = 0$

$f'''(x) = \frac{6x^2-2}{(1+x^2)^3}$ $\qquad f'''(0) = \frac{6\times 0^2-2}{(1+0^2)^3} = -2$

Hence, $tan^{-1}x = 0 + 1x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3 + \cdots$

$tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$... for $-1<x<1$

e) $f(x) = (1 + x)^n$ (The Binomial Theorem)

$f(x) = (1 + x)^n$ $\qquad f(0) = (1 + 0)^n = 1$

$f'(x) = n(1 + x)^{n-1}$ $\qquad f'(0) = n(1 + 0)^{n-1} = n$

$f''(x) = n(n - 1)(1 + x)^{n-2}$ $\qquad f''(0) = n(n - 1)$

$f'''(x) = n(n - 1)(n - 2)(1 + x)^{n-3}$ $\qquad f'''(0) = n(n - 1)(n - 2)$

Hence

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$$

Exercise 4: Expand the following functions in ascending powers of x as far as the power indicated

a) $f(x) = cosx$ as far as x^6

b) $f(x) = tanx$ as far as x^3

c) $f(x) = sin^{-1}x$ as far as x^3

d) $f(x) = \ln(1 - x)$ as far as x^4

e) $f(x) = e^{3x}$ as far as x^4

f) $f(x) = \ln(1 - 2x)$ as far as x^5

g) $f(x) = sin3x$ as far as x^5

h) $f(x) = tan2x$ as far as x^5

i) $f(x) = \ln(2 + x)$ as far as x^3 (hint: $\ln(2 + x) = \ln 2(1 + \frac{x}{2})$)

Using More Than One Expansion

Examples:

a) Expand $f(x) = e^{-2x} sin3x$ in ascending powers of x as far as the x^4 term using the expansion for e^x and $sinx$.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-2x} = 1 + \frac{-2x}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \cdots$$

$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

$$sinx = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$sin3x = \frac{3x}{1!} - \frac{(3x)^3}{3!} + \cdots$$ (ignore higher powers)

$$= 3x - \frac{9}{2}x^3 + \cdots$$

Hence, $f(x) = e^{-2x} sin3x$

$$= \left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots\right)\left(3x - \frac{9}{2}x^3 + \cdots\right)$$

$$= 3x - \frac{9}{2}x^3 - 6x^2 + 9x^4 + 6x^3 - 4x^4 + \cdots$$

$$= 3x - 6x^2 + \frac{3}{2}x^3 + 5x^4$$

b) Expand $f(x) = \ln(cosx)$ in ascending powers of x as far as the term in x^6

using the expansion for $\ln(1+x)$ and $cosx$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

$$cosx = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots \quad \Rightarrow \quad cosx - 1 = -\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots$$

$$\ln(cosx) = \ln(1 + (cosx - 1))$$

$$= (cosx - 1) - \frac{(cosx-1)^2}{2} + \frac{(cosx-1)^3}{3} - \frac{(cosx-1)^4}{4} + \frac{(cosx-1)^5}{5} - \frac{(cosx-1)^6}{6} + \cdots$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6}\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6}\right)^2 + \frac{1}{3}\left(-\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6}\right)^3$$

$$= -\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} - \frac{1}{2}\left(\frac{x^4}{4} - \frac{x^6}{48} - \frac{x^6}{48} + \cdots\right) + \frac{1}{3}\left(-\frac{x^6}{6} + \cdots\right)$$

$$= -\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 - \frac{1}{8}x^4 + \frac{1}{96}x^6 + \frac{1}{96}x^6 - \frac{1}{24}x^6$$

$$= -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6$$

Exercise 5: Expand

a) e^{sinx} as far as the term in x^4

b) $\ln(1 + sinx)$ as far as the term in x^4

c) $e^x sinx$ as far as the term in x^5

d) $\ln(1 + e^x)$ as far as the term in x^4

Introduction to Complex Numbers

We are in a position where we can solve a range of quadratic equations by factorisation and by the quadratic formula.
When the roots are real ($b^2 - 4ac \geq 0$) we obtain one or more solutions.
Some equations however have no real roots ($b^2 - 4ac < 0$).
We can solve $x^2 - 49 = 0$

$$x^2 = 49$$

$$x = -7 \ or \ 7$$

However, $x^2 + 49 = 0$

$$x^2 = -49$$

has no solution in terms of rational or irrational numbers.
In order to solve all quadratic equations we must extend our notion of number and introduce the concept of an imaginary number.

Suppose there exists a non-zero number i such that $i^2 = -1$.
Then we can solve $x^2 + 49 = 0$

$$x^2 = -49$$

$$x^2 = 49 \times (-1)$$

$$x^2 = 49i^2$$

$$x = -7i \ or \ 7i$$

A number of the form bi, where b is a real number, is called an imaginary number and $x + yi$ where x and y are real is called a complex number.

z is commonly used to denote a complex number and so is $z = x + yi$, then the real part of the complex number z is written as $Re(z) = x$ and the imaginary part is written as $Im(z) = y$.

We can now solve quadratic equations with no real roots.

Example: Find the roots of the equation $x^2 - 2x + 5 = 0$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2 \times 1}$

$x = \frac{2 \pm \sqrt{-16}}{2}$

$x = \frac{2 \pm \sqrt{16i^2}}{2}$

$x = \frac{2 \pm 4i}{2}$

$x = 1 - 2i \ \ or \ \ 1 + 2i$

> **Note**
> If one root of the quadratic equation is $x = a + bi$, then the other root is $x = a - bi$. These are called the complex conjugates of each other.

Proof: To check the roots, substitute into $x^2 - 2x + 5 = 0$.

$x = 1 - 2i \ \Rightarrow \ (1 - 2i)^2 - 2(1 - 2i) + 5$
$= 1 - 4i + 4i^2 - 2 + 4i + 5$
$= 1 - 4i - 4 - 2 + 4i + 5$
$= 0$

$x = 1 + 2i \ \Rightarrow \ (1 + 2i)^2 - 2(1 + 2i) + 5$
$= 1 + 4i + 4i^2 - 2 - 4i + 5$
$= 0$

Notation

$\bar{z}$ (read as z bar) is the complex conjugate of z
i.e. if $z = a + bi$ then $\bar{z} = a - bi$.

Exercise 1:

1 What are the values of:
a) $(2i)^2$ b) $(3i)^2$ c) $(4i)^2$ d) $(-2i)^2$ e) $(-3i)^2$

2 State the complex conjugates of the following complex numbers:
a) 3+2i b) 5-3i c) 8+2i d) -5i

3 Find the roots of the following equations:
a) $x^2 + 4 = 0$ b) $x^2 + 9 = 0$ c) $x^2 + 3 = 0$

4 Using the quadratic formula, solve the following equations:
a) $x^2 - 2x + 2 = 0$ b) $x^2 - 4x + 5 = 0$ c) $x^2 - 4x + 13 = 0$
d) $4x^2 - 4x + 5 = 0$ e) $2x^2 - 2x + 1 = 0$ f) $9x^2 - 6x + 2 = 0$

5 Check that the solutions you found in q3 satisfy the original equation by substitution.

6 Solve the following equations

a) $x^3 - 1 = 0$ b) $x^4 - 1 = 0$

c) $x^3 - x^2 - x - 2 = 0$ d) $(x^2 + 4)(x^2 + 9) = 0$

Add/Subtract/Multiply/Divide

Addition: Add real parts and add imaginary parts

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Subtraction: Add real parts and add imaginary parts

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Multiplication: Multiply using the rules for multiplying brackets

$$(a + bi) \times (c + di) = ac + adi + bci + bdi^2$$
$$= ac + adi + bci - bd$$
$$= (ac - bd) + (ad + bc)i$$

Division: In a similar way to dividing surds use the complex conjugate because

$$(a + bi) \times (a - bi) = a^2 - abi + abi - b^2i^2$$
$$= a^2 + b^2$$

which is a real number.

Examples: Calculate

a) $(3 + i) + (1 + 2i) = 4 + 3i$

b) $(2 - 3i) - (1 + 2i) = 1 - 5i$

c) $(2 - i) \times (3 + 2i) = 6 + 4i - 3i - 2i^2$
$$= 6 + 4i - 3i + 2$$
$$= 8 + i$$

d) $\frac{5+3i}{1-3i} = \frac{5+3i}{1-3i} \times \frac{1+3i}{1+3i}$
$$= \frac{5+15i+3i+9i^2}{1+3i-3i-9i^2}$$
$$= \frac{-4+18i}{10}$$
$$= -\frac{2}{5} + \frac{9}{5}i$$

e) $(1-i)^3$

By Pascal's Triangle

$1^3 + 3 \times 1^2 \times (-i) + 3 \times 1 \times (-i)^2 + (-i)^3$

$= 1 - 3i - 3 + i$

$= -2 - 2i$

or

$(1-i)(1-i)^2$

$= (1-i)(1-2i+i^2)$

$= (1-i)(-2i)$

$= -2i + 2i^2$

$= -2 - 2i$

f) $\sqrt{15+8i}$

Let $\sqrt{15+8i} = x + yi$

$15 + 8i = (x+yi)^2$

$= x^2 - y^2 + 2xyi$

Equating real and imaginary parts gives

$x^2 - y^2 = 15$ (1)

$2xy = 8 \;\Rightarrow\; y = \frac{4}{x}$ (2)

Substituting (2) into (1) gives

$x^2 - \left(\frac{4}{x}\right)^2 = 15$

$x^2 - \frac{16}{x^2} = 15$

$x^4 - 15x^2 - 16 = 0$

$(x^2 - 16)(x^2 + 1) = 0$

$x^2 - 16 = 0 \;\; or \;\; x^2 + 1 = 0$

x is real so $x^2 + 1 = 0$ gives no suitable values

$x = -4 \; or \; 4 \;\Rightarrow\; y = -1 \; or \; 1$

$\sqrt{15+8i} = -4 - i \;\; or \; 4 + i$

Exercise 2:

1 Express each of the following in the form $x + yi$

a) $(3+7i) + (2+i)$ b) $(9-2i) - (3+i)$ c) $(-2+i) + (7-4i)$

d) $(3+2i) + (3-2i)$ e) $(-2+i) - (-2-i)$ f) $(a+bi) + (a-bi)$

g) $(a+bi) - (a-bi)$

2 Express $i^3,\ i^4,\ i^5,\ i^6,\ i^7,\ i^8,\ i^9$ and i^{10} in their simplest form.

3 Simplify

a) $2i \times 4i$ b) $-2i^2$ c) $i(3+2i)$

d) $-i(1-4i)$ e) $(2+i)(3+i)$ f) $(6-5i)(2+3i)$

g) $(2+3i)(2-3i)$ h) $(a+bi)(a-bi)$ i) $(a+bi)(c+di)$

j) $(a+bi)(c-di)$ k) $(1+i)^3$ l) $(1+i)^4$

m) $(1+i)^4(1-i)^5$ n) $(3+i)^2+(3-i)^2$

o) $(cost+isint)^2$ p) $(cosA+isinA)(cosB+isinB)$

4 Simplify and express in the form $x+yi$

a) $\frac{4+i}{i}$ b) $\frac{1}{2+i}$ c) $\frac{2-i}{1-2i}$ d) $\frac{5+i}{5-i}$ e) $\frac{a+bi}{a-bi}$ f) $\frac{a+bi}{c+di}$

g) $\frac{10+5i}{2-i}$ h) $\frac{1}{cosA+isinA}$ i) $\frac{cosA+isinA}{cosA-isinA}$

5 Simplify $(x-1-i)(x-1+i)$

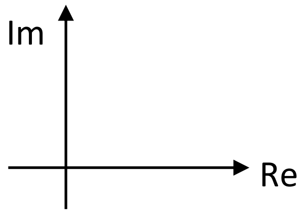
Hence state an equation which has $(1+i)$ and $(1-i)$ as its root.

6 Find the square root of each of these

a) $3-4i$ b) $21-20i$ c) $2i$ d) $-24+10i$

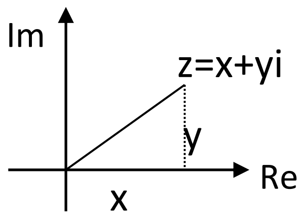
Argand Diagrams

Complex numbers can be represented geometrically using the x and y axes as the Real (Re) and the Imaginary (Im) axes respectively.



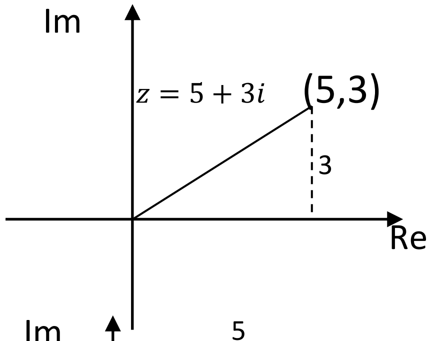
This diagram is known as complex plane.

When we plot points on a complex plane it is called an Argand Diagram.

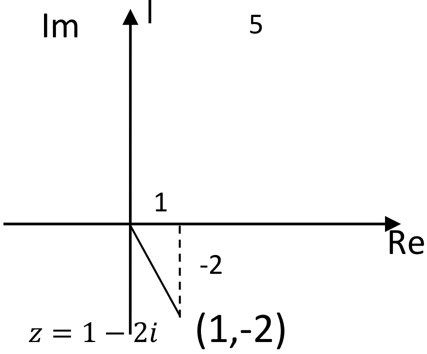


Examples

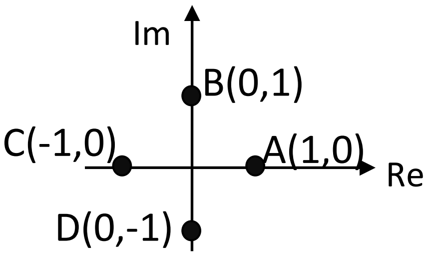
1 Plot $z = 5 + 3i$ on an Argand diagram



2 Plot $z = 1 - 2i$



3 The points A (1,0), B (0,1), C (-1,0) and D (0,-1) are shown below. Write down the corresponding complex numbers represented by these points.



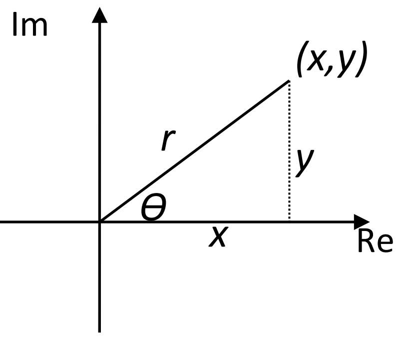
OA: $1 + 0i = 1$
OB: $0 + 1i = i$
OC: $-1 + 0i = -1$
OD: $0 - 1i = -i$

The Modulus and Argument of a Complex Number

On occasion it is convenient to express complex numbers in another form, particularly when we wish to find or illustrate the product, quotient or powers of complex numbers. To do this we must consider the modulus and the argument.

Consider $z = x + yi$

$x = rcos\theta \;\; and \;\; y = rsin\theta$
$r^2 = x^2 + y^2$



A complex number written in the form $z = r(cos\theta + isin\theta)$ is in polar form.
r is the modulus (magnitude) of z, this is written as Mod z or $|z|$.
θ is the argument (amplitude) of z, this is written as arg(z).

$$r = |z| = \sqrt{x^2 + y^2} \text{ and } \theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

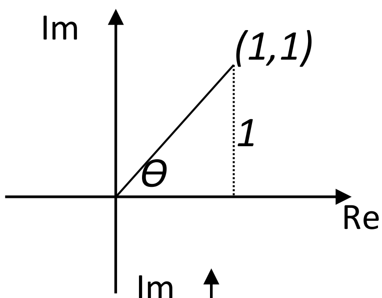
Note: We prefer the value of θ to be $-\pi < \theta \leq \pi$ if working in radians or $-180 < \theta \leq 180$ if working in degrees. We can add or subtract multiples of 2π or 360 to θ and obtain the same complex number.
An Argand diagram is useful for helping to find the argument.

Examples: Find the modulus and the argument of the complex numbers

a) $z = 1 + i$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

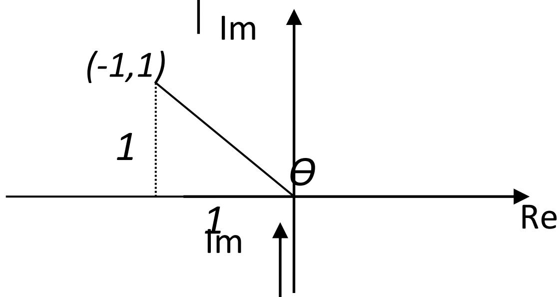
$\arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$ or $45°$



b) $z = -1 + i$

$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\arg(z) = \tan^{-1}\left(\frac{1}{-1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

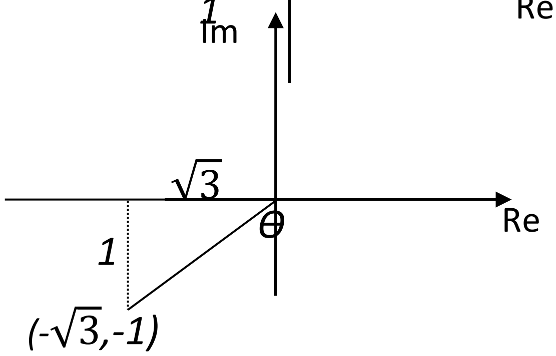


c) $z = -\sqrt{3} - i$

$|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$

$\arg(z) = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = \frac{-5\pi}{6}$

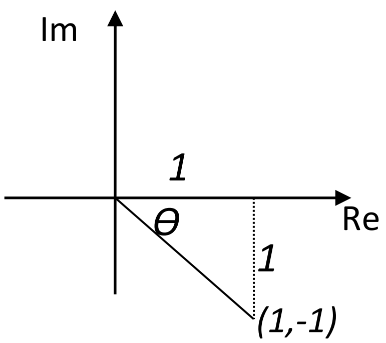
Remember $-\pi < \theta \leq \pi$



d) $z = 1 - i$

$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg(z) = \tan^{-1}\left(\frac{-1}{1}\right) = \frac{-\pi}{4}$



Exercise 3:

1 Find the modulus and the argument of

a) $1+\sqrt{3}i$ b) $2-2i$ c) $-\sqrt{2}-\sqrt{2}i$ d) $2i$ e) 3

f) $-\sqrt{3}+i$ g) $-3i$ h) -5 i) $-3-3i$

2 Given that $z_1 = -3+3\sqrt{3}i$ and $z_2 = \sqrt{3}+i$

a) (i) Find $|z_1|$, $|z_2|$, $|z_1 z_2|$ (ii) Find arg(z_1), arg(z_2), arg($z_1 z_2$)

b) Repeat for $z_1 = 3i$ and $z_2 = \sqrt{2}-\sqrt{2}i$ and note the result.

3 Given that $z_1 = -3+3\sqrt{3}i$ and $z_2 = \sqrt{3}+i$

a) (i) Find $|z_1|$, $|z_2|$, $\left|\frac{z_1}{z_2}\right|$ (ii) Find arg(z_1), arg(z_2), arg$\left(\frac{z_1}{z_2}\right)$

b) Repeat for $z_1 = 3i$ and $z_2 = \sqrt{2}-\sqrt{2}i$ and note the result.

4 Given that $z = 1+i$

a) (i) Find $|z|$, $|iz|$ (ii) Find arg(z), arg(iz)

b) Repeat for $z = -\sqrt{3}-i$ and note the result.

5 Given that $z = 1+i$

a) (i) Find $|z|$, $|\bar{z}|$ (ii) Find arg(z), arg($\bar{z}$)

b) Repeat for $z = -\sqrt{3}-i$ and note the result.

6 Given that $z = 1+i$

a) (i) Find $z\bar{z}$ (ii) Find $|z\bar{z}|$, arg($z\bar{z}$)

b) Repeat for $z = -\sqrt{3}-i$ and note the result.

Summary of results

$|z_1 z_2| = |z_1| \times |z_2|$ $\qquad arg(z_1 z_2) = arg(z_1) + arg(z_2)$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $\qquad arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

$|iz| = |z|$ $\qquad arg(iz) = arg(z) + \frac{\pi}{2}$

$|\bar{z}| = |z|$ $\qquad arg(\bar{z}) = -arg(z)$

$|z\bar{z}| = |z|^2$ $\qquad arg(z\bar{z}) = 0$

The Fundamental Theorem of Algebra

All complex polynomial equations have at least one complex root.

Consequently if a polynomial is of degree n, then there are precisely n roots in the set of complex numbers. Some or all may be real, some or all may be complex.
For polynomials with real coefficients: if $x + yi$ is a root of a polynomial equation then $x - yi$ (the complex conjugate)is also a root.
$(z - (x + yi))$ and $(z - (x - yi))$ are factors of the polynomial.

$$(z-(x+yi))(z-(x-yi)) = z^2 - \big((x+yi)+(x-yi)\big)z + (x+yi)(x-yi)$$

$$= z^2 - 2xz + x^2 + y^2 \quad \text{(a quadratic factor)}$$

Dividing the polynomial by this quadratic factor will reveal the remaining factor(s) which may be real or complex.

Examples

1 Find the roots of the equation $z^3 - 2z^2 - 8z + 21 = 0$

Using synthetic division

-3	1	-2	-8	21
		-3	15	-21
	1	-5	7	0

Since the remainder is 0, then -3 is a root.

$z^3 - 2z^2 - 8z + 21 = (z + 3)(z^2 - 5z + 7)$

$z^2 - 5z + 7 = 0$

Using the quadratic formula

$$z = \frac{5 \pm \sqrt{25-28}}{2} = \frac{5 \pm \sqrt{3}i}{2} = \frac{5}{2} \pm \frac{\sqrt{3}}{2}i$$

The roots are $z = -3,\ \frac{5}{2} + \frac{\sqrt{3}}{2}i,\ \frac{5}{2} - \frac{\sqrt{3}}{2}i.$

2 Verify that $z = 1 + i$ is a root of the equation:

$z^4 - 3z^3 + 5z^2 - 4z + 2 = 0.$

Hence find all the roots of the form $p + qi$.

If $z = 1 + i$ is a roots then

$(1 + i)^4 - 3(1 + i)^3 + 5(1 + i)^2 - 4(1 + i) + 2$ must $= 0$

$(1+i)^2 = 1 + 2i + i^2 = 2i$

$(1+i)^3 = 2i(1+i) = -2 + 2i$

$(1+i)^4 = [(1+i)^2]^2 = (2i)^2 = -4$

$-4 - 3(-2+2i) + 5(2i) - 4(1+i) + 2$

$= -4 + 6 - 6i + 10i - 4 - 4i + 2$

$= 0$ Therefore $z = 1 + i$ is a root.

If $z = 1 + i$ is a root, then $z = 1 - i$ is also a root.

The quadratic factor is $(z - (1+i))(z - (1-i))$

$= z^2 - (1-i)z - (1+i)z + (1+i)(1-i)$

$= z^2 - 2z + 2$

Find the remaining quadratic factor by dividing $z^4 - 3z^3 + 5z^2 - 4z + 2$ by $z^2 - 2z + 2$.

$$
\begin{array}{r|l}
 & \quad z^2 \quad - z \ + 1 \\
\hline
z^2 - 2z + 2 & z^4 - 3z^3 + 5z^2 - 4z + 2 \\
 & z^4 - 2z^3 + 2z^2 \\
\hline
 & \quad -z^3 + 3z^2 - 4z \\
 & \quad -z^3 + 2z^2 - 2z \\
\hline
 & \qquad z^2 - 2z + 2 \\
 & \qquad z^2 - 2z + 2 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$z^4 - 3z^3 + 5z^2 - 4z + 2 = (z^2 - 2z + 2)(z^2 - z + 1)$

$z^2 - z + 1 = 0$

$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

The roots are $z = 1 + i,\ 1 - i,\ \frac{1}{2} + \frac{\sqrt{3}}{2} i,\ \frac{1}{2} - \frac{\sqrt{3}}{2} i$.

Exercise 4:

1 Find all the roots of the equation $z^3 - 11z + 20 = 0$.

2 Verify that $z = 1 + i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$. Hence find all the other roots.

3 Verify that $z = -2 = 3i$ is a root of the equation
$z^4 + 7z^2 - 12z + 130 = 0$.
Hence find all the other roots.

4 Given that $2 - i$ is a root of the equation
$3z^3 - 10z^2 + 7z + 10 = 0$, find all the other roots.

5 Given that $1 - 2i$ is a root of the equation $3z^3 + z + 10 = 0$, find the other roots.

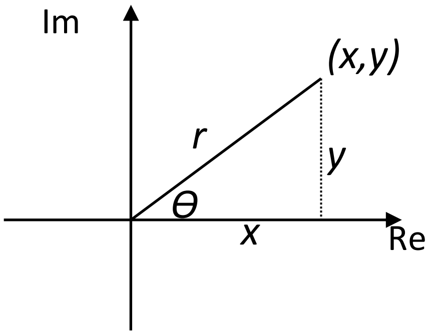
6 Given that $3 + i$ is a root of $z^3 - 3z^2 - 8z + 30 = 0$, find the other roots.

7 Show that $-1 + i$ is a root of $z^4 - 2z^3 - z^2 + 2z + 10 = 0$ and find the other roots.

The Modulus and Argument of a Complex Number

On occasion it is convenient to express complex numbers in another form, particularly when we wish to find or illustrate the product, quotient or powers of complex numbers.

Consider $z = x + yi$



$x = rcos\theta$
$y = rsin\theta$
$r^2 = x^2 + y^2$

$z = x + yi = rcos\theta + risin\theta = r(cos\theta + isin\theta)$

A complex number written in the form $z = r(cos\theta + isin\theta)$ is in polar form.
r is the modulus (magnitude) of z, this is written as Mod z or $|z|$.
θ is the argument (amplitude) of z, this is written as arg(z).

$$r = |z| = \sqrt{x^2 + y^2} \text{ and } \theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

Note that we prefer the value of θ to be $-\pi < \theta \le \pi$ if working in radians or $-180 < \theta \le 180$ if working in degrees. We can add or subtract multiples of 2π or 360 to θ and obtain the same complex number.

An Argand diagram is useful for helping to find the argument.

Examples: Write in polar form

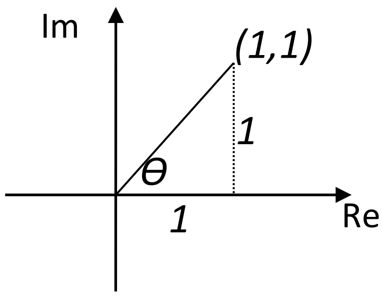
a) $z = 1 + i$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$ or $45°$

$z = \sqrt{2}(cos\frac{\pi}{4} + isin\frac{\pi}{4})$

or $z = \sqrt{2}(cos45° + isin45°)$

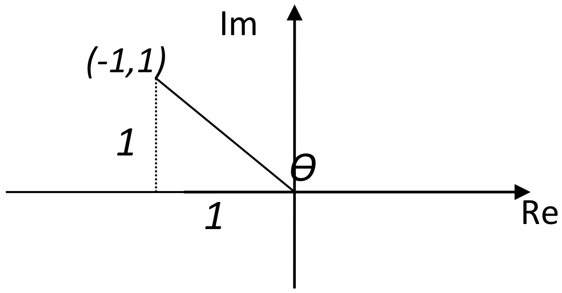


b) $z = -1 + i$

$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\arg(z) = \tan^{-1}\left(\frac{1}{-1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$z = \sqrt{2}(cos\frac{3\pi}{4} + isin\frac{3\pi}{4})$



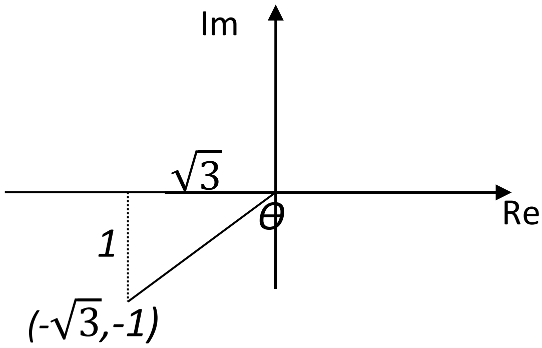
c) $z = -\sqrt{3} - i$

$|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$

$\arg(z) = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = \frac{-5\pi}{6}$

Remember $-\pi < \theta \leq \pi$

$z = 2(cos\frac{-5\pi}{6} + isin\frac{-5\pi}{6})$

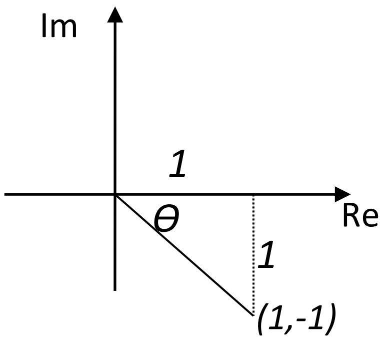


d) $z = 1 - i$

$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg(z) = \tan^{-1}\left(\frac{-1}{1}\right) = \frac{-\pi}{4}$

$z = \sqrt{2}(cos\frac{-\pi}{4} + isin\frac{-\pi}{4})$



Note: $r(cos\theta - isin\theta) = r(\cos(-\theta) + isin(-\theta))$

Exercise 5: Write in polar form

a) $1 + \sqrt{3}i$ b) $2 - 2i$ c) $-\sqrt{2} - \sqrt{2}i$ d) $2i$ e) 3

f) $-\sqrt{3} + i$ g) $-3i$ h) -5 i) $-3 - 3i$

Multiplication and division in polar form

$$z_1 = r_1(cos\alpha + isin\alpha) \qquad z_2 = r_2(cos\beta + isin\beta)$$
$$z_1 z_2 = r_1 r_2(\cos(\alpha+\beta) + isin(\alpha+\beta))$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\alpha-\beta) + isin(\alpha-\beta))$$

Exercise 6:

1 Find the product, $z_1 z_2$, of the following sets of complex numbers

a) $z_1 = 2\left[cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right] \qquad z_2 = 3\left[cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right]$

b) $z_1 = \sqrt{2}\left[cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right] \qquad z_2 = 2\left[cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right]$

c) $z_1 = 3\left[cos\left(\frac{3\pi}{4}\right) + isin\left(\frac{3\pi}{4}\right)\right] \qquad z_2 = 2\left[cos\left(\frac{5\pi}{4}\right) + isin\left(\frac{5\pi}{4}\right)\right]$

d) $z_1 = \frac{1}{2}\left[cos\left(\frac{2\pi}{15}\right) + isin\left(\frac{2\pi}{15}\right)\right] \qquad z_2 = 2\left[cos\left(\frac{13\pi}{15}\right) + isin\left(\frac{13\pi}{15}\right)\right]$

e) $z_1 = 10\left[cos\left(\frac{5\pi}{9}\right) + isin\left(\frac{5\pi}{9}\right)\right] \qquad z_2 = 10\left[cos\left(\frac{13\pi}{9}\right) + isin\left(\frac{13\pi}{9}\right)\right]$

2 Find the quotient, $\frac{z_1}{z_2}$, of the following sets of complex numbers

a) $z_1 = 2\left[cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right] \qquad z_2 = 3\left[cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right]$

b) $z_1 = \sqrt{2}\left[cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right] \qquad z_2 = 2\left[cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right]$

c) $z_1 = 3\left[cos\left(\frac{3\pi}{4}\right) + isin\left(\frac{3\pi}{4}\right)\right] \qquad z_2 = 2\left[cos\left(\frac{5\pi}{4}\right) + isin\left(\frac{5\pi}{4}\right)\right]$

d) $z_1 = \frac{1}{2}\left[cos\left(\frac{2\pi}{15}\right) + isin\left(\frac{2\pi}{15}\right)\right] \qquad z_2 = 2\left[cos\left(\frac{13\pi}{15}\right) + isin\left(\frac{13\pi}{15}\right)\right]$

e) $z_1 = 10\left[cos\left(\frac{5\pi}{9}\right) + isin\left(\frac{5\pi}{9}\right)\right] \qquad z_2 = 10\left[cos\left(\frac{13\pi}{9}\right) + isin\left(\frac{13\pi}{9}\right)\right]$

3 Evaluate

a) $\left[cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right] \times \left[cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right] \times \left[cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right]$

b) $4\left[cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right] \times 2\left[cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right]$

c) $20\left[cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right] \div \left[cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right]$

d) $\left[cos\left(\frac{2\pi}{3}\right) + isin\left(\frac{2\pi}{3}\right)\right] \div 3\left[cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right]$

De Moirve's Theorem

$(cos\theta + isin\theta)^n = \cos(n\theta) + isin(n\theta)$ true for all $n \in \mathbb{R}$

Proof by induction, which we will explore later, will allow us to prove this result for $n \in \mathbb{N}$. A proof for all $n \in \mathbb{R}$ is beyond the scope of the course.

This extends to

$$(r(cos\theta + isin\theta))^n = r^n(\cos(n\theta) + isin(n\theta))$$

Examples

1 Evaluate $\left[4\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)\right]^{\frac{1}{2}}$

$$\left[4\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)\right]^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(cos\frac{\pi}{6} + isin\frac{\pi}{6}\right)$$

$$= \pm 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= \sqrt{3} + i \;\; or \;\; -\sqrt{3} - i$$

2 Express $z = (1 - i)^7$ in the form $x + iy$

$$1 - i = \sqrt{2}\left[cos\left(\frac{-\pi}{4}\right) + isin\left(\frac{-\pi}{4}\right)\right]$$

$$z = \sqrt{2}^7\left[cos\left(\frac{-\pi}{4}\right) + isin\left(\frac{-\pi}{4}\right)\right]^7$$

$$= 8\sqrt{2}\left[cos\left(\frac{-7\pi}{4}\right) + isin\left(\frac{-7\pi}{4}\right)\right]$$

$$= 8\sqrt{2}\left[cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right]$$

$$= 8\sqrt{2}\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = 8 + 8i$$

Note $\frac{-7\pi}{4} + 2\pi = \frac{\pi}{4}$ add 2π to obtain $-\pi < \theta \leq \pi$

Alternative method

$z = [(1 - i)^2]^3(1 - i)$

$= (1 - 2i + i^2)^3(1 - i)$

$= (-2i)^3(1 - i)$

$= -8i^3(1 - i)$

$= 8 + 8i$ or use binomial theorem

3 Express $sin5\theta$ in terms of $sin\theta$

By De Moirve's Theorem

$(cos\theta + isin\theta)^5 = \cos(5\theta) + isin(5\theta)$

By the Binomial Theorem

$(cos\theta + isin\theta)^5$

$= cos^5\theta + 5icos^4\theta sin\theta + 10i^2cos^3\theta sin^2\theta + 10i^3cos^2\theta sin^3\theta$

$\quad + 5i^4cos\theta sin^4\theta + i^5sin^5\theta$

$= cos^5\theta + 5icos^4\theta sin\theta - 10cos^3\theta sin^2\theta - 10icos^2\theta sin^3\theta + 5cos\theta sin^4\theta$

$\quad - isin^5\theta$

Equating the imaginary parts:

$sin5\theta = 5cos^4\theta sin\theta - 10cos^2\theta sin^3\theta - sin^5\theta$

$= 5(1 - sin^2\theta)^2 sin\theta - 10(1 - sin^2\theta)sin^3\theta - sin^5\theta$

$= 5sin\theta - 20sin^3\theta + 16sin^5\theta$

4 If $z = cos\theta + isin\theta$, show that

a) $z^k + \frac{1}{z^k} = 2cosk\theta$ b) $z^k - \frac{1}{z^k} = 2isink\theta$

a) $z^k + \frac{1}{z^k} = (cos\theta + isin\theta)^k + (cos\theta + isin\theta)^{-k}$

$= cosk\theta + isink\theta + \cos(-k\theta) + isin(-k\theta)$

$= cosk\theta + isink\theta + cosk\theta - isink\theta$

$= 2cosk\theta$

b) $z^k - \frac{1}{z^k} = (cos\theta + isin\theta)^k - (cos\theta + isin\theta)^{-k}$

$= cosk\theta + isink\theta - \cos(-k\theta) - isin(-k\theta)$

$= cosk\theta + isink\theta - cosk\theta + isink\theta$

$= 2isink\theta$

You must know these results:

$z^k + \frac{1}{z^k} = 2cosk\theta$

$z^k + \frac{1}{z^k} = 2cosk\theta$

5 Express $cos^3\theta$ in terms of $cos\theta$ and $cos3\theta$

$$2cos\theta = z + \frac{1}{z} \text{ where } z = cos\theta + isin\theta$$

$$(2cos\theta)^3 = \left(z + \frac{1}{z}\right)^3$$

$$= z^3 + 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3$$

$$= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$$

$$= z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right)$$

$$8cos^3\theta = 2cos3\theta + 6cos\theta$$

$$cos^3\theta = \frac{1}{8}(2cos3\theta + 6cos\theta)$$

$$= \frac{1}{4}(cos3\theta + 3cos\theta)$$

6 Assuming the De Moivre is true for natural numbers, prove it holds true for integers

Consider $(cos\theta + isin\theta)^{-k}$ $k \in \mathbb{N}$

$$= \frac{1}{(cos\theta+isin\theta)^k}$$

$$= \frac{1}{cosk\theta+isink\theta}$$

$$= \frac{cosk\theta-isink\theta}{(cosk\theta+isink\theta)(cosk\theta-isink\theta)}$$

$$= \frac{\cos(-k\theta)+isin(-k\theta)}{cos^2k\theta+sin^2k\theta}$$

$$= \cos(-k\theta) + isin(-k\theta)$$

Therefore, De Moivre holds for integers.

Exercise 7

1 Simplify

a) $\left[cos\left(\frac{5\pi}{24}\right) + isin\left(\frac{5\pi}{24}\right)\right]^4$

b) $\left[cos\left(\frac{2\pi}{5}\right) + isin\left(\frac{2\pi}{5}\right)\right]^5$

c) $\frac{[cos2\theta+isin2\theta]^5}{[cos3\theta+isin3\theta]^3}$

d) $[cos\theta + isin\theta]^8[cos\theta - isin\theta]^4$

(note the negative)

2 $z = 1 + \sqrt{3}i$

Find z^7 in polar form with argument in the range $-\pi < \theta \le \pi$, then express it in the form $a + ib$.

3 Simplify $[cos125° + isin125°]^4[cos15° + isin15°]^3$

4 By considering $[cos\theta + isin\theta]^3$ show that

a) $cos3\theta = 4cos^3\theta - 3cos\theta$ b) $sin3\theta = 3sin\theta - 4sin^3\theta$

5 By considering $z = cos\theta + isin\theta$ and expanding $\left(z + \frac{1}{z}\right)^4$ show that

$cos^4\theta = \frac{1}{8}[cos4\theta + 4cos2\theta + 3]$

6 By considering $z = cos\theta + isin\theta$ and expanding $\left(z - \frac{1}{z}\right)^5$ show that

$sin^5\theta = \frac{1}{16}[sin5\theta - 5sin3\theta + 10sin\theta]$

7 By considering $[cos\theta + isin\theta]^4$ show that

$$cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$$

Hence show that

$$cos^4\theta = \frac{1}{8}[cos4\theta + 4cos2\theta + 3]$$

Also show that

$$sin4\theta = 4sin\theta(2cos^3\theta - cos\theta)$$

8 Show that $cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$

More Equations

Examples:

1 Solve $z^3 = -8i$ i.e. find the cube root of $-8i$.

In polar form $-8i = 8\left[cos\left(\frac{-\pi}{2}\right) + isin\left(\frac{-\pi}{2}\right)\right]$

$z^3 = 8\left[cos\left(\frac{-\pi}{2}\right) + isin\left(\frac{-\pi}{2}\right)\right]$

$z = 8^{\frac{1}{3}}\left[cos\left(\frac{-\pi}{2}\right) + isin\left(\frac{-\pi}{2}\right)\right]^{\frac{1}{3}}$

This will only give one solution and we know from the Fundamental Theorem of Algebra that there are three roots.

Adding multiples of 2π to the argument of a complex number results in the same number so we can write

$$-8i = 8\left[cos\left(\frac{-\pi}{2} + 2k\pi\right) + isin\left(\frac{-\pi}{2} + 2k\pi\right)\right]$$

So $z = 8^{\frac{1}{3}}\left[cos\left(\frac{-\pi}{2} + 2k\pi\right) + isin\left(\frac{-\pi}{2} + 2k\pi\right)\right]^{\frac{1}{3}}$

$$= 2\left[cos\left(\frac{-\pi}{6} + \frac{2k\pi}{3}\right) + isin\left(\frac{-\pi}{6} + \frac{2k\pi}{3}\right)\right] \quad k = 0,1,2$$

$k = 0$: $\quad z = 2\left[cos\left(\frac{-\pi}{6}\right) + isin\left(\frac{-\pi}{6}\right)\right] = \sqrt{3} - i$

$k = 1$: $\quad z = 2\left[cos\left(\frac{-\pi}{6} + \frac{2\pi}{3}\right) + isin\left(\frac{-\pi}{6} + \frac{2\pi}{3}\right)\right]$

$$= 2\left[cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right] = 2i$$

$k = 2$: $\quad z = 2\left[cos\left(\frac{-\pi}{6} + \frac{4\pi}{3}\right) + isin\left(\frac{-\pi}{6} + \frac{4\pi}{3}\right)\right]$

$$= 2\left[cos\left(\frac{7\pi}{6}\right) + isin\left(\frac{7\pi}{6}\right)\right] = -\sqrt{3} - i$$

The solutions to $z^3 = -8i$ are $z = \sqrt{3} - i,\ 2i,\ -\sqrt{3} - i$.

2 Find the fifth roots of unity i.e. solve $z^5 = 1$

$1 = cos0 + isin0$

$z = (cos2k\pi + isin2k\pi)^{\frac{1}{5}}$

$= cos\left(\frac{2k\pi}{5}\right) + isin\left(\frac{2k\pi}{5}\right) \quad k = 0,1,2,3,4$

$z = cos0 + isin0$

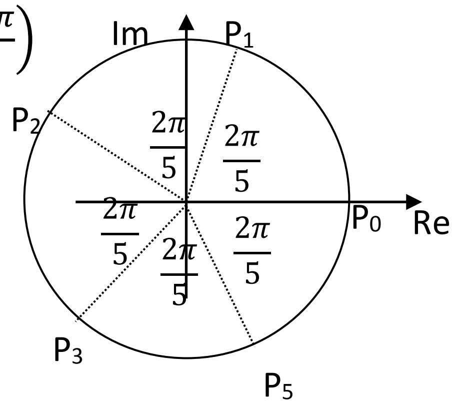
$z = cos\frac{2\pi}{5} + isin\frac{2\pi}{5}$

$z = cos\frac{4\pi}{5} + isin\frac{4\pi}{5}$

$z = cos\frac{6\pi}{5} + isin\frac{6\pi}{5} = cos\left(\frac{-4\pi}{5}\right) + isin\left(\frac{-4\pi}{5}\right)$

$z = cos\frac{8\pi}{5} + isin\frac{8\pi}{5} = cos\left(\frac{-2\pi}{5}\right) + isin\left(\frac{-2\pi}{5}\right)$

The five fifth roots of unity are shown in the Argand Diagram. The roots are equally spaced by $\frac{2\pi}{5}$ radians and lie on the circumference of a circle of radius 1.



Generalisation

The n, nth roots of unity (solutions to $z^n = 1$) can be represented by n equally spaced points on a circle of radius one unit, including the point (1,0). The spacing will be $\frac{2\pi}{n}$.

Exercise 8:

1 Solve a) $z^3 = 4 + 4\sqrt{3}i$ b) $z^3 = 1$
c) $z^4 = -2 + 2i$ d) $z^6 = 1$
e) $z^4 = 1$ f) $z^5 = i$

2 Illustrate the solutions to b) and d) on an Argand diagram.

Geometric Interpretation of Equations and Inequations

The solution set of complex equations and inequations can be represented by sets of points in an Argand diagram

e.g $|z| = a,\ |z - a| = b,\ |z - 1| = |z - i|,\ |z - a| > b$

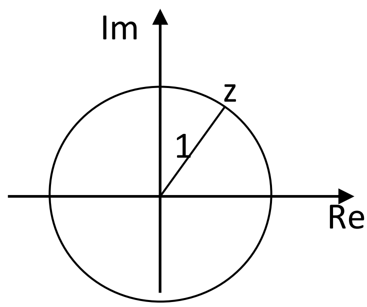
Examples:

1 Solve $|z| = 1$

If $z = x + yi$ and $|z| = 1$ then $\sqrt{x^2 + y^2} = 1$

i.e. $x^2 + y^2 = 1$

All the points representing complex numbers z with modulus 1 lie on the circumference of a circle, centre the origin and radius 1.



2 Solve $|z - 1| = 2$

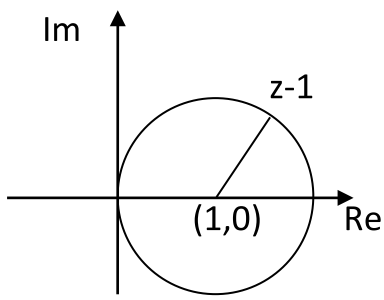
If $z = x + yi$ then $z - 1 = (x - 1) + yi$.

Therefore $|z - 1| = 2$ becomes

$\sqrt{(x-1)^2 + y^2} = 2$

i.e. $(x - 1)^2 + y^2 = 4$

All points representing complex numbers z for which $|z - 1| = 2$ lie on the circumference of a circle, centre (1,0) and radius 2.

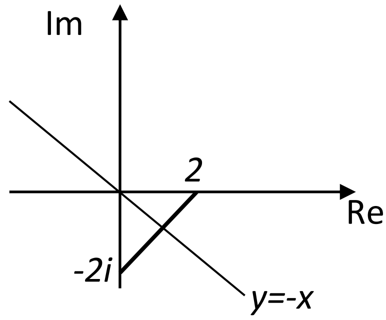


3 Solve $|z-2| = |z+2i|$

If $z = x + yi$ then $z - 2 = (x-2) + yi$ and $z + 2i = x + (y+2)i$.

Therefore $|z-2| = |z+2i|$ becomes

$$\sqrt{(x-2)^2 + y^2} = \sqrt{x^2 + (y+2)^2}$$
$$(x-2)^2 + y^2 = x^2 + (y+2)^2$$
$$x^2 - 4x + 4 + y^2 = x^2 + y^2 + 4y + 4$$
$$y = -x$$



Since 2 is represented by (2,0) and -2i is represented (0,-2), all points representing complex numbers z for which $|z-2| = |z+2i|$ lie on the perpendicular bisector of the line joining (2,0) and (0,-2). The line is $y = -x$.

4 Solve $|z-2| > 4$

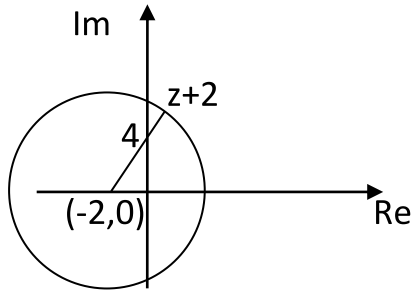
If $z = x + yi$ then $z - 2 = (x-2) + yi$.

Therefore $|z-2| > 4$ becomes

$$\sqrt{(x-2)^2 + y^2} > 4$$

i.e. $(x-2)^2 + y^2 > 16$

All points representing complex numbers z for which $|z-2| > 4$ lie outside the circumference of a circle, centre (-2,0) and radius 4.



Generalisation

$|z - (a+bi)| = c$ will represent a circle with centre (a, b), radius c.

$|z - (a+bi)| = |z - (c+di)|$ will represent the perpendicular bisector to the line joining (a, b) to (c, d).

Exercise 9: Find the set of points z where

a) $|z| = 2$ b) $|z-3| = 5$ c) $|z+3| = |z-4i|$

d) $|z-2| > 3$ e) $|z-(3+2i)| = 4$ f) $|z-1| \leq 4$

g) $|z-2| = |z+1-i|$

Answers

Partial Fractions

Exercise 1

a) $x+3+\frac{-2x-6}{x^2+2}$ b) $5x^2-4x+21+\frac{84-16x}{x^2-4}$ c) $x^3+x^2+\frac{x^2-x-1}{x^2(x-1)}$

Exercise 2

1a) $\frac{1}{x-2}+\frac{3}{x-3}$ b) $\frac{3}{x}-\frac{5}{1-x}$ c) $\frac{4}{x-4}-\frac{3}{x+3}$ d) $\frac{2}{x}+\frac{4}{x+4}$ **2**a) $1+\frac{2}{x-3}+\frac{3}{x+4}$ b) $x-3+\frac{1}{5(x-1)}+\frac{64}{5(x+4)}$

Exercise 3

1a) $\frac{1}{x}+\frac{2}{x+1}-\frac{4}{(x+1)^2}$ b) $\frac{2}{x}+\frac{1}{x-1}+\frac{5}{(x-1)^2}$ c) $\frac{2}{x+2}-\frac{1}{x-1}+\frac{3}{(x-1)^2}$ d) $\frac{3}{2x-3}+\frac{2}{x-4}-\frac{1}{(x-4)^2}$

2 $x^3+x^2+\frac{2}{x}+\frac{1}{x^2}-\frac{1}{x-1}$

Exercise 4

a) $\frac{3}{x-2}-\frac{3x-2}{x^2+4}$ b) $\frac{4}{x+2}+\frac{3x-2}{x^2+1}$ c) $\frac{5}{x+5}+\frac{2x+7}{x^2+9}$ d) $\frac{-2}{x+3}+\frac{4}{(x+3)^2}+\frac{26-5x}{x^2+4}$ e) $x+\frac{1}{2x}-\frac{5x}{2(x^2+2)}$

f) $1+\frac{2}{x-1}+\frac{1}{(x-1)^2}$

Exercise 5

a) $\frac{2}{2x-3}-\frac{1}{x+2}$ b) $\frac{1}{2x-1}-\frac{2}{x+3}+\frac{5}{(x+3)^2}$ c) $\frac{5}{2(x+1)}-\frac{1}{2(x-1)}$ d) $\frac{2}{x-1}-\frac{5}{2(2x-1)}+\frac{1}{2(2x-1)^2}$ e) $1+\frac{1}{x-3}-\frac{1}{2x+5}$

f) $2+\frac{x-5}{x^2+4}+\frac{5}{x-3}$ g) $\frac{2}{1+2x}+\frac{3-x}{x^2-x+2}$ h) $\frac{2}{1+x}+\frac{3}{(1+x)^2}-\frac{5}{(1+x)^3}+\frac{1}{x-2}$ i) $\frac{3}{x+3}+\frac{4}{(x+3)^2}+\frac{1-2x}{x^2+4}$

Binomial Theorem

Ex1 a) $x^3+15x^2+75x+125$ b) $a^4+8a^3+24a^2+32a+16$ c) $m^5-15m^4+90m^3-270m^2+405m-243$

d) $625-500y+150y^2-20y^3+y^4$ e) $8x^3+36x^2+54x+27$ f) $81p^4+108p^3+54p^2+12p+1$

g) $32x^5+80x^4y+80x^3y^2+40x^2y^3+10xy^4+y^5$ h) $8k^3-60k^2+150k-125$

i) $81x^4-216x^3y+216x^2y^2-96xy^3+16y^4$ j) $x^6+6x^4+12x^2+8$ k) $y^{12}-8y^9+24y^6-32y^3+16$

l) $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$ m) $x^4+8x^2+24+\frac{32}{x^2}+\frac{16}{x^4}$ n) $8x^3-36x+\frac{54}{x}-\frac{27}{x^3}$

o) $x^{10}-10x^7+40x^4-80x+\frac{80}{x^2}-\frac{32}{x^5}$ p) $16x^8+32x^6+24x^4+8x^2+1$ q) $27x^6-54x^4+48x^2-8$

r) $16x^{12}-64x^8+96x^4-64+\frac{16}{x^4}$ s) $x^6-3x^2+\frac{3}{x^2}-\frac{1}{x^6}$ t) $\frac{x^8}{y^4}-\frac{4x^5}{y^2}+6x^2-\frac{4y^2}{x}+\frac{y^4}{x^4}$

Ex2 1)a) 35 b) 120 2) 1,5,10,10,5,1 3) Proof

Ex3 1)a) $\binom{14}{4}$ b) $\binom{20}{5}$ c) $\binom{n}{0}=1$ 2) Proof 3)a) $\binom{11}{5}$ b) $\binom{21}{7}$ 4) Proof

Ex4 1)a) $27+27x+9x^2+x^3$ b) $125+150x+60x^2+8x^3$ c) $16-32x+24x^2-8x^3+x^4$

d) $x^5+10x^4y+40x^3y^2+80x^2y^3+80xy^4+32y^5$ e) $1+9x+27x^2+27x^3$

f) $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$ g) $32x^5+240x^3+720x+\frac{1080}{x}+\frac{810}{x^3}+\frac{243}{x^5}$

h) $x^8-20x^5+150x^2-\frac{500}{x}+\frac{625}{x^4}$

2)a) 286.9151 b) 205.11149 3)180.109 4)1.04060

Ex5 1)a) 7 b) 5 2) $720x^3$ 3) 3 4) 3041280

5) -816,18564 6) $-3240a^7x^3$ 7) 240 8)108 9)5 10) -35 11)24 12) 70 13) 1 14) 673596 15) $\frac{1}{6}$

Differentiation

Exercise 1

a) $3x^2$ b) $2x+2$ c) $6x+4$

Exercise 2

1a) $3x^2-2x+5$ b) $6x+\frac{4}{x^2}$ c) $\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x^3}}$ d) $\frac{3}{2}\sqrt{x}-\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x^3}}$ e) $-\frac{2}{x^2}+\frac{3}{x^4}$ f) $-\frac{3}{2x^{\frac{5}{2}}}+\frac{3}{2}x^{\frac{1}{2}}$

g) $8(4x+5)$ h) $\frac{4x^3}{(2x^4-3)^{\frac{1}{2}}}$ i) $\frac{3x}{(4-x^2)^{\frac{3}{2}}}$ j) $\frac{-4(x^2+1)}{(x^3+3x)^{\frac{4}{3}}}$ k) $-3\sin x\cos^2 x$ l) $\frac{\cos x}{2\sqrt{\sin x}}$

2 y=14x-19 **3** Min tp at (1,0) Max tp at (-1,4).

Exercise 3

1a) $(6x+12)(x^2+4x-5)^2$ b) $\frac{3x^2}{2\sqrt{x^3+5}}$ c) $\frac{4(1+2\sqrt{x})^3}{\sqrt{x}}$ d) $\frac{3x}{(4-x^2)^{\frac{3}{2}}}$ e) $4\cos x\sin^3 x$ f) $-6\sin 2x\cos^3 2x$

2 Proofs

Exercise 4

a) $2x(x-3)(2x-2)$ b) $(10x+3)(2x+3)^2$ c) $\frac{3(x-4)}{2\sqrt{x-6}}$ d) $\frac{(x-3)^2(7x-3)}{2\sqrt{x}}$ e) $2(x+1)(3x+1)(x-1)^3$

f) $\frac{x^2(7x-6)}{2\sqrt{x-1}}$ g) $\sin x + x\cos x$ h) $x(2\sin x+x\cos x)$ i) $\cos 2x$ j) $2\cos 2x\cos 5x-5\sin 2x\sin 5x$

k) $-2x\sin x^2\sin^3 x+3\cos x^2\cos x\sin^2 x$

Exercise 5

1a) $\frac{x^2+6x}{(x+3)^2}$ b) $\frac{x-8}{x^3}$ c) $\frac{4(2x+1)}{(1-x)^4}$ d) $\frac{2x(x-4)}{(x-2)^2}$ e) $-\frac{3(1-2x)^2}{x^4}$ f) $-\frac{3x+4}{2x^3\sqrt{x+1}}$

2/3 compare with other answers **4** proof

Exercise 6

a) $6\tan^2 2x\sec^2 2x$ b) $8\operatorname{cosec}^4 x\cot x$ c) $\sec x(\tan^2 x+\sec^2 x)$ d) $x(2\cot x-x\operatorname{cosec}^2 x)$

e) $\frac{5}{5x+2}$ f) $-(x+1)e^{-x}$ g) $\frac{(x+1)e^x}{(x+2)^2}$ h) $\frac{x(2\ln x-1)}{(\ln x)^2}$ i) $\frac{x}{x^2+1}$ j) $e^{-2x^2}(1-4x^2)$ k) $\frac{2}{1-x^2}$ l) $\frac{4}{(e^x+e^{-x})^2}$

Exercise 7

1 $6x+10$ **2** $-27\cos 3x$ **3** $\frac{d^4y}{dx^4}=a^4e^{ax}$ $\frac{d^ny}{dx^4}=a^ne^{ax}$ **4** $f^n(x)=\frac{-(n-1)!}{(1-x)^n}$

5 Min tp at (1,0) Max tp at (-1,4)

Exercise 8

a) $\frac{1}{2\sqrt{x(1-x)}}$ b) $\frac{1}{2(1+x)\sqrt{x}}$ c) $\tan^{-1}x+\frac{x}{1+x^2}$ d) $\tan^{-1}\left(\frac{x}{2}\right)+\frac{2x}{4+x^2}$ e) $\sin^{-1}x$ f) $\frac{-1}{\sqrt{x-x^2}}$ g) $\frac{1}{(x+1)\sqrt{x}}$

h) $\frac{1}{1+x^2}$ i) $\frac{2}{(1+3x^2)\sqrt{1-x^2}}$ j) $\frac{2}{(1-x^2)\sqrt{1-5x^2}}$ k) $\frac{\sec x\tan x}{1+\sec^2 x}=\frac{\sin x}{\cos^2 x+1}$ l) $\frac{-x}{\sqrt{1-x^2}}$ m) $\frac{e^x}{1+e^{2x}}$ n) $\frac{2x}{\sqrt{1-x^2}}-e^{x^2}(2x^2+1)$

Exercise 9

1a) $\frac{x}{y}$ b) $\frac{1}{y-1}$ c) $-\frac{y}{2x}$ d) $\frac{2(4x+1)}{3(y^2-1)}$ e) $\frac{1-3x^2y-y^3}{1+3xy^2+x^3}$ f) $\cot x\cot y$ g) $-\frac{y}{x}$ h) $ye^{-x}-y\ln y$

2a) $3x-4y=2$ b) $5x+8y=27$ c) $2x-y=3$ **3** $-\frac{40}{33}$ **4** $-\frac{7}{6}$

5a) $\frac{x-7}{y^2}$ b) $\frac{2}{(y-1)^2}$ c) $\frac{2y}{x}-\frac{y}{2x^2}$ d) $\frac{8}{3y^2-3}-\frac{6y(8x-2)}{(3y^2-3)^2}$ **6** $2,-\frac{56}{5}$ **7** (-1,3) min (0,0) max

Exercise 10

1 $\frac{1}{108}$ **2** -28 **3** 2π **4**a) proof b) -0.0125cm/s **5**a) proof b) 3.75π **6** 11.8

Exercise 11

a) $10^x ln10$ b) $2^{x^2} \times 2xln2$ c) $-x^{-x}(lnx+1)$ d) $x^{sinx}\left(cosxlnx + \frac{sinx}{x}\right)$ e) $x^{\frac{1}{x}-2}(1-lnx)$

f) $2x^{lnx-1}lnx$ g) $(lnx)^x\left[\ln(lnx) + \frac{1}{lnx}\right]$ h) $\frac{(lnx)^{lnx}}{x} - [\ln(lnx)+1]$ i) $\frac{x^4(27x+50)}{2(3x+5)^{\frac{3}{2}}}$ j) $\frac{3x^2(2x-1)^4(4x^2+5x-1)}{(x+1)^3}$

Exercise 12

1a) $\frac{2t+1}{t(3t+2)}$ b) $-\frac{3}{4}cot\theta$ c) $-\frac{(1+t)^2}{(1-t)^2}$ d) $-\frac{3(t+1)^2}{2(t-2)^2}$ e) $\frac{t}{t+1}$ f) $\frac{(t+6)(1+t)^2}{(t+3)^2}$ g) $-cosec\theta$ h) $-\frac{3}{2}sin\theta$

i) $\frac{cost+sint}{cost-sint}$ j) $\frac{cost}{sint-1}$ **2**a) x+t^{2}y=2ct b) (3t^2-1)x-2ty=at^2(t^2+3) c) (t^2+1)x-(t^2-1)y=2at d) xsecθ-ytanθ=1

Exercise 13

1a) $-\frac{1}{2}t^3,\ \frac{3}{4}t^5$ b) $\frac{t}{1+t}, \frac{1}{2(1+t)^3}$ c) $-\frac{3}{4}cott, -\frac{3}{16}cosec^3t$ d) $-\frac{1}{2}cosec\theta, -\frac{1}{4}cosec^3\theta$

e) $\frac{cos\theta+1}{sin\theta}, \frac{-1-cos\theta}{2sin^3\theta(2cos\theta-1)}$ **2** $\left(\frac{\pi}{2},1\right), \left(\frac{3\pi}{2},-1\right)$ **3**a) Max sp $\left(\frac{8}{3}, \frac{16}{9}\sqrt{3}\right)$, Min sp $\left(\frac{8}{3}, -\frac{16}{9}\sqrt{3}\right)$

b) Max sp (16,9) c) Max sp (2,4), Min sp (10,0) **4** Proof

Exercise 14

1 $\sqrt{13}, \sqrt{5}$ **2** $\sqrt{17}$ **3** 2 **4** $\frac{1}{9}\sqrt{733}$ **5** $\frac{1}{3}\sqrt{20}$ **6** $\frac{1}{2}\sqrt{17}$

7 Max of 4 at (0,3) and (10,-3). Min of 3 at (4,0) and (-4,0)

Sequences and Series

Exercise 1

1a) $u_n = 8n-5,\ u_{19} = 147$ b) $u_n = 11-3n,\ u_{15} = -34$ c) $u_n = 7.5-0.5n,\ u_{12} = 1.5$

2a) n=23 b) n=13 c) n=13

3a) a=4, d=6, $S_{12} = 444$ b) a=15, d=-2, $S_{20} = -80$ c) a=20, d=-7, $S_{16} = -520$

4a) a=13, d=2, $S_{10} = 220$ b) a=33, d=-5, $S_{16} = -72$ c) a=-5, d=6, $S_{15} = 555$

5 a=-15, d=6, $S_{30} = 2160$ **6** a=0, d=$\frac{1}{2}$, $u_{20} = \frac{19}{2}$ **7** n=15

8 d=3, $u_{10} = 30$ **9** 39 terms

Exercise 2

1a) $r=3$ b) $r=\frac{1}{2}$ c) $r=\frac{1}{10}$ d) $r=3$ e) $r=\frac{2}{3}$ f) $r=\frac{1}{2}$ g) $r=-1$ h) $r=-2$

2a) 1, 3, 9, 27 b) 3, -6, 12, -24 c) 6, 3, $\frac{3}{2}$, $\frac{3}{4}$

3a) $u_n = 2^{n-1}$, $u_5 = 16$ b) $u_n = 2\times 3^{n-1}$, $u_6 = 486$ c) $u_n = 4\times 3^{n-1}$, $u_6 = 972$

d) $u_n = 2\times 10^{n-1}$, $u_5 = 20000$ e) $u_n = (-2)^{n-1}$, $u_6 = -32$ f) $u_n = 6\times\left(\frac{1}{2}\right)^{n-1}$, $u_7 = \frac{3}{32}$

4a) $u_n = 2^{n-1}$ b) $u_n = 3\times 2^{n-1}$ c) $u_n = 2\times(-3)^{n-1}$ d) $u_n = 9\times\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^{n-3}$

e) $u_n = 4\times\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n-3}$ f) $u_n = \frac{1}{2}\times\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n$

5a) $r=\pm 2,\ u_5 = 96$ b) $r=2,\ u_5 = 800$ c) $r=-\frac{1}{3},\ u_5 = \frac{4}{9}$

6a) 255 b) 728 c) 22 d) 122 e) $1\frac{31}{32}$ f) $1\frac{40}{81}$ g) $\frac{1-x^n}{1-x}$ h) $\frac{1-(-y)^n}{1+y}$ **7**a) n=5 b) n=8

Exercise 3

a) $r=\frac{1}{3}, S_\infty = \frac{3}{2}$ b) $r=2, S_\infty$ does not exist c) $r=\frac{1}{4}, S_\infty = \frac{4}{3}$ d) $r=\frac{1}{2}, S_\infty = 16$

e) $r=-5, S_\infty$ does not exist f) $r=-\frac{9}{10}, S_\infty = \frac{100}{19}$ g) $r=-\frac{1}{2}, S_\infty = \frac{2}{3}$ h) $r=\frac{2}{3}, S_\infty = 3$

Exercise 4

a) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$ b) $x+\frac{1}{3}x^3$ c) $x+\frac{1}{6}x^3$ d) $x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4$

e) $1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\frac{27}{8}x^4$ f) $1-2x-2x^2-4x^3-8x^4-16x^5$ g) $3x-\frac{9}{2}x^3+\frac{81}{40}x^5$

i) $2x+\frac{8}{3}x^3+\frac{64}{15}x^5$ j) $ln2+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{24}x^3$

Exercise 5: a) $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4$ b) $x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4$ c) $x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5$

d) $ln2+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{192}x^4$

Complex Numbers

Ex1 1a) -4 b) -9 c) -16 d) -4 e) -9 2a) $3-2i$ b) $5+3i$ c) $8-2i$ d) $5i$ 3a) $\pm 2i$ b) $\pm 3i$ c) $\pm\sqrt{3}i$
4a) $1\pm i$ b) $2\pm i$ c) $2\pm 3i$ d) $-1\pm i$ e) $\frac{1}{2}\pm i$ f) $-3\pm i$ g) $\frac{1}{2}\pm\frac{1}{2}i$ h) $\frac{1}{3}\pm\frac{1}{3}i$ 4 Proofs 5a) 1 b) $\pm 1, \pm i$ c) $2, -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ d) $\pm 2i, \pm 3i$

Ex2 1a) $5+8i$ b) $6-3i$ c) $5-3i$ d) 6 e) $2i$ f) $2a$ g) $2bi$ 2) $i^3=-i, i^4=1, i^5=i, i^6=-1, i^7=-i,$
$i^8=1, i^9=i, i^{10}=-1,$ 3a) -8 b) 2 c) -2+3i d) -4-i e) 5+5i f) 27+8i g) 13 h) a^2+b^2
i) (ac-bd)+(bc+ad)i j) (ac+bd)+(bc-ad)i k) -2+2i l) -4 m) 16-16i n) 16 o) cos2t+isin2t
p) cos(A+B)+isin(A+B) 4a) 1-4i b) $\frac{2}{5}-\frac{1}{5}i$ c) $\frac{4}{5}+\frac{3}{5}i$ d) $\frac{12}{13}+\frac{5}{13}i$ e) $\frac{a^2-b^2}{a^2+b^2}+\frac{2ab}{a^2+b^2}i$ f) $\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$
g) 3+4i h) cosA-isinA i) cos2A+isin2A 5 x^2-2x+2 and x^2-2x+2=0 6a) $\pm(-2+i)$ b) $\pm(5-2i)$
c) $\pm(1+i)$ d) $\pm(1+5i)$

Ex3 1a) $2\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$ b) $2\sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right]$ c) $2\left[\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right]$
d) $2\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$ e) $3[\cos(0)+i\sin(0)]$ f) $2\left[\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right]$ g) $3\left[\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right]$
h) $5[\cos(\pi)+i\sin(\pi)]$ i) $3\sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right]$

2a)(i) $|z_1|=6$ $|z_2|=2$ $|z_1z_2|=12$ (ii) arg(z_1)=$\frac{2\pi}{3}$ arg(z_2)=$\frac{\pi}{6}$ arg(z_1z_2)=$\frac{5\pi}{6}$
b)(i) $|z_1|=3$ $|z_2|=2$ $|z_1z_2|=6$ (ii) arg(z_1)=$\frac{\pi}{2}$ arg(z_2)=$\frac{-\pi}{4}$ arg(z_1z_2)=$\frac{\pi}{4}$
Therefore $|z_1z_2|=|z_1|\times|z_2|$ arg(z_1z_2)= arg(z_1)+ arg(z_2)
3a) (i) Find $|z_1|=6$ $|z_2|=2$ $\left|\frac{z_1}{z_2}\right|=3$ (ii) Find arg(z_1)=$\frac{2\pi}{3}$ arg(z_2) = $\frac{\pi}{6}$ arg$\left(\frac{z_1}{z_2}\right)$= $\frac{\pi}{2}$
b) (i) Find $|z_1|=3$ $|z_2|=2$ $\left|\frac{z_1}{z_2}\right|=\frac{3}{2}$ (ii) Find arg(z_1)=$\frac{\pi}{2}$ arg(z_2) = $\frac{-\pi}{4}$ arg$\left(\frac{z_1}{z_2}\right)$= $\frac{3\pi}{4}$
Therefore $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$ arg$\left(\frac{z_1}{z_2}\right)$ = arg(z_1)-arg(z_2)
4a)(i) $|z|=\sqrt{2}$, $|iz|=\sqrt{2}$ (ii) arg(z)=$\frac{\pi}{4}$, arg(iz)=$\frac{3\pi}{4}$ b)(i) $|z|=2$, $|iz|=2$ (ii) arg(z)=$\frac{-5\pi}{6}$, arg(iz)=$\frac{-\pi}{3}$
Therefore $|iz|=|z|$ and arg(iz)= arg(z)+$\frac{\pi}{2}$ or multiplying by i rotates z by $\frac{\pi}{2}$ about the origin
5a)(i) $|z|=\sqrt{2}$, $|\bar{z}|=\sqrt{2}$ (ii) arg(z)=$\frac{\pi}{4}$, arg($\bar{z}$)=$\frac{-\pi}{4}$ b)(i) $|z|=2$, $|\bar{z}|=2$ (ii) arg(z)=$\frac{-5\pi}{6}$, arg($\bar{z}$)=$\frac{5\pi}{6}$
Therefore $|\bar{z}|=|z|$ and arg($\bar{z}$)= -arg(z) or $\bar{z}$ is a reflection of z in the real axis.
6a)(i) $z\bar{z}=2$ (ii) $|z\bar{z}|=2$ $\arg(z\bar{z})=0$ b)(i) $z\bar{z}=4$ (ii) $|z\bar{z}|=4$ $\arg(z\bar{z})=0$
Therefore $z\bar{z}$ is always real, $|z\bar{z}|=|z|^2$ and $\arg(z\bar{z})=0$

Ex4 a) $-4, 2\pm i$ b) $1\pm i, -1\pm 2i$ c) $-2\pm 3i, 2\pm\sqrt{6}i$ d) $-\frac{2}{3}, 2\pm i$ e) $-2, 1\pm 2i$ f) $-3,$ $3\pm i$
g) $-1\pm i, 2\pm i$

Ex5 1a) $2\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$ b) $2\sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right]$ c) $2\left[\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right]$
d) $2\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$ e) $3[\cos(0)+i\sin(0)]$ f) $2\left[\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right]$ g) $3\left[\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right]$
h) $5[\cos(\pi)+i\sin(\pi)]$ i) $3\sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right]$
2a)(i) $|z_1|=6$ $|z_2|=2$ $|z_1z_2|=12$ (ii) arg(z_1)=$\frac{2\pi}{3}$ arg(z_2)=$\frac{\pi}{6}$ arg(z_1z_2)=$\frac{5\pi}{6}$
b)(i) $|z_1|=3$ $|z_2|=2$ $|z_1z_2|=6$ (ii) arg(z_1)=$\frac{\pi}{2}$ arg(z_2)=$\frac{-\pi}{4}$ arg(z_1z_2)=$\frac{\pi}{4}$
Therefore $|z_1z_2|=|z_1|\times|z_2|$ arg(z_1z_2)= arg(z_1)+ arg(z_2)
3a) (i) Find $|z_1|=6$ $|z_2|=2$ $\left|\frac{z_1}{z_2}\right|=3$ (ii) Find arg(z_1)=$\frac{2\pi}{3}$ arg(z_2) =$\frac{\pi}{6}$ arg$\left(\frac{z_1}{z_2}\right)$=$\frac{\pi}{2}$
b) (i) Find $|z_1|=3$ $|z_2|=2$ $\left|\frac{z_1}{z_2}\right|=\frac{3}{2}$ (ii) Find arg(z_1)=$\frac{\pi}{2}$ arg(z_2) =$\frac{-\pi}{4}$ arg$\left(\frac{z_1}{z_2}\right)$=$\frac{3\pi}{4}$
Therefore $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$ arg$\left(\frac{z_1}{z_2}\right)$ = arg(z_1)-arg(z_2)
4a)(i) $|z|=\sqrt{2},$ $|iz|=\sqrt{2}$ (ii) arg(z)=$\frac{\pi}{4}$, arg(iz)=$\frac{3\pi}{4}$ b)(i) $|z|=2,$ $|iz|=2$ (ii) arg(z)=$\frac{-5\pi}{6}$, arg(iz)=$\frac{-\pi}{3}$
Therefore $|iz|=|z|$ and arg(iz)= arg(z)+$\frac{\pi}{2}$ or multiplying by i rotates z by $\frac{\pi}{2}$ about the origin
5a)(i) $|z|=\sqrt{2},$ $|\bar{z}|=\sqrt{2}$ (ii) arg(z)=$\frac{\pi}{4}$, arg($\bar{z}$)=$\frac{-\pi}{4}$ b)(i) $|z|=2,$ $|\bar{z}|=2$ (ii) arg(z)=$\frac{-5\pi}{6}$, arg($\bar{z}$)=$\frac{5\pi}{6}$
Therefore $|\bar{z}|=|z|$ and arg($\bar{z}$)= -arg(z) or $\bar{z}$ is a reflection of z in the real axis.
6a)(i) $z\bar{z}=2$ (ii) $|z\bar{z}|=2$ $\arg(z\bar{z})=0$ b)(i) $z\bar{z}=4$ (ii) $|z\bar{z}|=4$ $\arg(z\bar{z})=0$
Therefore $z\bar{z}$ is always real, $|z\bar{z}|=|z|^2$ and $\arg(z\bar{z})=0$
Ex6 1a) $6\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$ b) $2\sqrt{2}\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$ c) $6[\cos 0+i\sin 0]$ d) $cos\pi+isin\pi$
e) $100[\cos 0+i\sin 0]$ 2a) $\frac{2}{3}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$ b) $\frac{\sqrt{2}}{2}\left[\cos\left(\frac{-\pi}{6}\right)+i\sin\left(\frac{-\pi}{6}\right)\right]$
c) $\frac{3}{2}\left[\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right]$ d) $4\left[\cos\left(\frac{11\pi}{15}\right)+i\sin\left(\frac{11\pi}{15}\right)\right]$ e) $\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)$
3a) $\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{6}\right)$ b) $8\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$ c) $20\left[\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right]$ d) $\frac{1}{3}\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$
Ex7 1a) $\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ b) $\cos 2\pi+i\sin 2\pi=1$ c) $\cos\theta+i\sin\theta$
d) $\cos 4\theta+i\sin 4\theta$ 2) $128\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]=64-i64\sqrt{3}$ 3) $cos175°+isin175°$
4) $(\cos\theta+i\sin\theta)^3=\cos 3\theta+i\sin 3\theta$
$(\cos\theta+i\sin\theta)^3=cos^3\theta+3cos^2\theta(isin\theta)+3cos\theta(isin\theta)^2+(isin\theta)^3$
So $\cos 3\theta=cos^3\theta-3cos\theta sin^2\theta$ and $\sin 3\theta=3cos^2\theta sin\theta-sin^3\theta$
Now substitute $sin^2\theta=1-cos^2\theta$ and $cos^2\theta=1-sin^2\theta$ etc
5) $\left(z+\frac{1}{z}\right)^4=(2cos\theta)^4$ $\left(z+\frac{1}{z}\right)^4=z^4+4z^3\left(\frac{1}{z}\right)+6z^2\left(\frac{1}{z}\right)^2+4z\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^4$
$=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$
$16cos^4\theta=2cos4\theta+8cos2\theta+6$ etc
6) $\left(z-\frac{1}{z}\right)^5=(2isin\theta)^5$ $\left(z-\frac{1}{z}\right)^5=z^5+5z^4\left(\frac{-1}{z}\right)+10z^3\left(\frac{-1}{z}\right)^3+10z^2\left(\frac{-1}{z}\right)^3+5z\left(\frac{-1}{z}\right)^4+\left(\frac{-1}{z}\right)^5$
$=\left(z^5-\frac{1}{z^5}\right)+5\left(z^3-\frac{1}{z^3}\right)+10\left(z-\frac{1}{z}\right)$
$32isin\theta=2isin5\theta-10isin3\theta+20isin\theta$
7) $(\cos\theta+i\sin\theta)^4=\cos 4\theta+i\sin 4\theta$
$(\cos\theta+i\sin\theta)^4=cos^4\theta+4cos^3\theta(isin\theta)+6cos^2\theta(isin\theta)^2+4cos\theta(isin\theta)^3+(isin\theta)^4$
$\cos 4\theta=8\,cos^4\theta-8cos^2\theta+1$
$8cos^4\theta=\cos 4\theta+8cos^2\theta-1$ then substitute $cos^2\theta=\frac{1}{2}(1+cos2\theta)$ to obtain result

Ex8a) $2\left[\cos\left(\frac{\pi}{9}\right)+i\sin\left(\frac{\pi}{9}\right)\right],\ 2\left[\cos\left(\frac{7\pi}{9}\right)+i\sin\left(\frac{7\pi}{9}\right)\right],\ 2\left[\cos\left(\frac{-5\pi}{9}\right)+i\sin\left(\frac{-5\pi}{9}\right)\right]$

b) $1,-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i$ c) $8^{\frac{1}{8}}\left[cos\left(\frac{3\pi}{16}+\frac{2k\pi}{4}\right)+isin\left(\frac{3\pi}{16}+\frac{2k\pi}{4}\right)\right]\ \ k=0,1,2,3$

d) $1,-1,\ \frac{1}{2}\left(1\pm\sqrt{3}i\right),\ \frac{1}{2}\left(-1\pm\sqrt{3}i\right)$ e) $\frac{\sqrt{2}}{2}(1\pm i),\frac{\sqrt{2}}{2}(-1\pm i)$

f) $cos\left(\frac{\pi}{10}+\frac{2k\pi}{5}\right)+isin\left(\frac{\pi}{10}+\frac{2k\pi}{5}\right)$ k=0,1,2,3,4

Ex9 a) $x^2+y^2=4$, a circle centre O, radius 2

b) $(x-3)^2+y^2=25$, a circle centre (3,0), radius 5

c) 6x-8y=7, the perpendicular bisector of (-3,0) and (0,4)

d) $(x-2)^2+y^2>9$, outside the circle centre (2,0), radius 3

e) $(x-3)^2+(y-2)^2=16$, a circle of centre (3,2), radius 4

f) $(x-1)^2+y^2\leq 16$, on or inside the circle centre (1,0), radius 4

g) y=3x-1, the perpendicular bisector of (2,0) and (-1,1)