Advanced Higher Maths 2012 Solutions © Madras College

1a Quotient rule. So $f(x) = \frac{u'v - v'u}{v^2}$ where $u = 3x + 1, u' = 3, v = x^2 + 1, v' = 2x.$

Hence $f'(x) = \frac{3(x^2+1) - 2x(3x+1)}{(x^2+1)^2} = \frac{-3x^2 - 2x + 3}{(x^2+1)^2}$.

1b Product rule. So $g'(x) = u'v + v'u$ where $= \cos^2 x$, $u' = -2\sin x \cos x$

$v = e^{\tan x}$, $v' = \sec^2 x e^{\tan x}$.

Hence $g'(x) = -2\sin x \cos x e^{\tan x} + \cos^2 x \sec^2 x e^{\tan x}$ $= e^{\tan x}(1 - \sin 2x).$

2 a = 2048 and $u_4 = ar^3$ = 256 so $\frac{ar^3}{a} = \frac{256}{2048}$. Hence $r^3 = \frac{1}{8}$ and $r = \frac{1}{2}$.

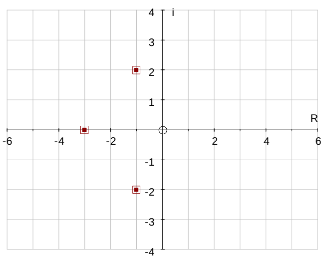
Use $S_n = \frac{a(1-r^n)}{1-r}$.

Thus

$4088 = \frac{2048\left(1 - \frac{1^n}{2}\right)}{1 - \frac{1}{2}}$ and $\frac{1^n}{2} = 1 - \frac{4088}{4096} = \frac{1}{512}$

and n = 9 by inspection or solve using logs.

3 If $(-1+2i)$ is a root then so too is $(-1-2i)$. Hence $(z-(1+2i))$ and $(z-(1-2i))$ are the factors so multiply to give $z^2 + 2z + 5$ then divide into the original polynomial to find the final factor as $z + 3$.



4 $\sum_{r=1}^{9} \binom{9}{r} (2x)^{9-r}(-x^{-2})^r = \sum_{r=1}^{9} \binom{9}{r} (-2)^{9-r} x^{9-3r}$. For the term independent of x we have

$9 - 3r = 0$ so $r = 3$.

We have $\binom{9}{3}(-2)^6 = 84 \times 64 = 5376.$

5 $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$.

We find $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 3 & 1 & 4 \\ 5 & -1 & 2 \end{vmatrix} = 6i + 14j - 8k$.

Using point P and the dot product $\begin{pmatrix} 6 \\ 14 \\ 8 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 10$.

Equation of the plane is $6x + 14y - 8z = 10$.

6

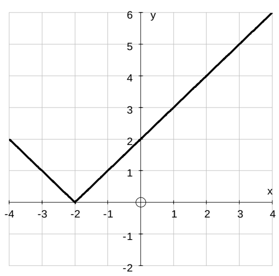
The Maclaurin expansion for e^x is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

Now $1 + e^x = 2 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

$$(1 + e^x)^2 = \left(2 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) \times \left(2 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

$$= 4 + 4x + 3x^2 + \frac{5}{3}x^3.$$

7a



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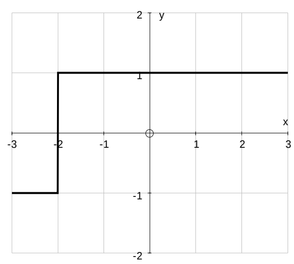

8

$x = 4\sin\theta$ so $dx = 4\cos\theta d\theta$ .The limits change to $\theta = 0, \theta = \frac{\pi}{6}$.

$$\int_0^{\pi/6} \left(16\left(1 - \sin^2\theta\right)\right)^{\frac{1}{2}} 4\cos\theta d\theta$$

$$= 16\int_0^{\pi/6} \cos^2\theta d\theta = \int_0^{\pi/6} (8 + 8\cos 2\theta) d\theta$$

This is $[8\theta + 4\sin 2\theta]_0^{\pi/6} = \frac{4\pi + 6\sqrt{3}}{3}$.

9 $A + A^{-1} = I$

so multiply both sides by *A* which gives $A^2 + I = A$.

Then multiply by *A* again to give $A^3 + A = A^2$ or $A^3 = A^2 - A$.

However $A^2 - A = -I$ so $A^3 = -I$ and so $k = -1.$

10

				0	r3
			7	3	r4
		7	2	5	r1
	7	1	7	6	r2
7	1	2	3	4	

So $1234_{10} = 3412_7$

11a $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

11b $\int u\,dv = uv - \int v\,du$.

$u = \sin^{-1} x$ so $u' = \frac{1}{\sqrt{1-x^2}}$. $v' = \frac{x}{\sqrt{1-x^2}}$

so $v = -\sqrt{1-x^2}$.

We have $-\sqrt{1-x^2}\sin^{-1} x + \int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2}\sin^{-1} x + x + C$.

12 We want to find $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} + \frac{dV}{dh}\frac{dh}{dt}$ where

$\frac{dr}{dt} = -0.02, \frac{dh}{dt} = 0.01, \frac{dV}{dr} = 2\pi rh, \frac{dV}{dh} = \pi r^2$.

Hence $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt} = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right)$.

Substituting in the information given yields

$\frac{dV}{dt} = 0.6\pi(-0.08 + 0.006) = -0.0444\pi\frac{m^3}{s}$.

13 $x = 2t + \frac{1}{2}t^2, x' = 2 + t, x'' = 1 : \ y = \frac{1}{3}t^3 - 3t, y' = t^2 - 3, y'' = 2t$.

So
$\frac{dy}{dx} = \frac{y'}{x'} = \frac{t^2 - 3}{2 + t}$ and $\frac{d^2y}{dx^2} = \frac{x'y'' - x''y'}{[x']^3} = \frac{2t(2+t) - 1.(t^2 - 3)}{(2+t)^3} = \frac{(t+1)(t+3)}{(t+2)^3}$.

For SPs $\frac{dy}{dx} = 0$ so we get $\frac{t^2 - 3}{2 + t} = 0$.

This gives solutions of $= \pm\sqrt{3}$.

Evaluating the second derivative at these values gives $t = \sqrt{3}\ :\ \frac{d^2y}{dx^2} = \frac{+ve \times +ve}{+ve} = +ve$ so a minimum

and for $t = -\sqrt{3}\ :\ \frac{d^2y}{dx^2} = \frac{-ve \times +ve}{+ve} = -ve$ so a maximum.

For the points of inflection $\frac{d^2y}{dx^2} = 0$ so $\frac{(t+1)(t+3)}{(t+2)^3} = 0$.

This is just a quadratic and has two roots $t = -1, t = -3$.

14a

4	0	6	1
2	-2	4	-1
-1	1	λ	2
0	4	-2	3
0	4	4λ+6	9
0	0	4λ+8	6

So $x = \frac{\lambda - 7}{4(\lambda + 2)}$ $y = \frac{3(\lambda + 3)}{4(\lambda + 2)}$ $z = \frac{3}{2(\lambda = 2)}$.

14b When λ=2 the system of equations is inconsistent.

14c When λ = -2.1 the solutions are $x = 22.75, y = -6.75, z = -15$.

This is an example of an ill-conditioned set of equations.

15a $\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

so $1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$.

Choosing $x = 1; -2; 0$ yield $= \frac{1}{9}, B = -\frac{1}{9}\ \ C = -\frac{1}{3}$.

So $\frac{1}{(x-1)(x+2)^2} = \frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$.

15b This is a linear ode so the IF is

$e^{-\int \frac{1}{x-1} dx} = e^{-ln|x-1|} = \frac{1}{x-1}$.

We now have

$\frac{y}{x-1} = \int \frac{1}{(x-1)(x+2)^2} dx = \int \left(\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2} dx \right)$.

This integrates to give

$\frac{1}{9}ln|x-1|-\frac{1}{9}|x+2|+\frac{1}{3(x+2)}+C$. So $y=(x-1)\left[\frac{1}{9}ln\left|\frac{x-1}{x+2}\right|+\frac{1}{3(x+2)}+C\right]$

16a Consider n = 1

$(\cos\theta+i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$

so true for n = 1.
Assume true for n = k and consider n = k+1

$(\cos\theta+i\sin\theta)^k = \cos k\theta + i\sin k\theta$
$(\cos\theta+i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$
$(\cos\theta+i\sin\theta)^{k+1} = (\cos\theta+i\sin\theta)^k(\cos\theta+i\sin\theta)$
$(\cos\theta+i\sin\theta)^{k+1} = (\cos k\theta+i\sin k\theta)(\cos\theta+i\sin\theta)$

Multiply out the rhs and collect real and imaginary parts
$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$.

Use compound angle formulae
$= \cos(k\theta+\theta) + i\sin(k\theta+\theta) = \cos(k+1)\theta + i\sin(k+1)\theta$ as required.

Since true for n = 1 and for k+1 by induction true for all positive integers.

16b Firstly apply de Moivre's theorem proved above so

$$\frac{\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36}+i\sin\frac{\pi}{36}\right)^4} = \frac{\cos\frac{11\pi}{18}+i\sin\frac{11\pi}{18}}{\cos\frac{4\pi}{36}+i\sin\frac{4\pi}{36}}$$.

The denominator simplifies to $\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}$.

Now multiply the top and bottom by the complex conjugate of $\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}$.

$$\frac{\left(\cos\frac{11\pi}{18}+i\sin\frac{11\pi}{18}\right)}{\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)} \times \frac{\left(\cos\frac{11\pi}{18}-i\sin\frac{11\pi}{18}\right)}{\left(\cos\frac{\pi}{9}-i\sin\frac{\pi}{9}\right)}$$.

The dominator simplifies to $\frac{\cos^2 \pi}{9}+\frac{\sin^2 \pi}{9}=1$.

The real part of the numerator is

$\cos\frac{11\pi}{18}\cos\frac{2\pi}{18}+\sin\frac{11\pi}{18}\sin\frac{2\pi}{18}=\cos\left(\frac{11\pi}{18}-\frac{2\pi}{18}\right)=\cos\frac{\pi}{2}=0$ as required.