2013 ADVANCED HIGHER

1) $$\left(3x - \frac{2}{x^2}\right)^4 = \binom{4}{0}(3x)^4 + \binom{4}{1}(3x)^3\left(-\frac{2}{x^2}\right) + \binom{4}{2}(3x)^2\left(-\frac{2}{x^2}\right)^2$$

$$+ \binom{4}{3}(3x)\left(-\frac{2}{x^2}\right)^3 + \binom{4}{4}\left(-\frac{2}{x^2}\right)^4$$

$$= \underline{\underline{81x^4 - 216x + \frac{216}{x^2} - \frac{96}{x^5} + \frac{16}{x^8}}}$$

2) $$f(x) = \underbrace{e^{\cos x}}_{f} \underbrace{\sin^2 x}_{g}$$

$$f' = e^{\cos x} . - \sin x$$

$$= - \sin x \, e^{\cos x}$$

$$g' = 2 \sin x \cos x$$

$$= \sin 2x$$

$$f'(x) = fg' + f'g = \sin 2x \, e^{\cos x} - \sin^3 x \, e^{\cos x}$$

$$= \underline{\underline{e^{\cos x}\left(\sin 2x - \sin^3 x\right)}}$$

3a) $$A^2 = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 16-2p & 5p \\ -10 & -2p+1 \end{pmatrix}}}$$

b) $$\det A^2 = ad - bc = (16-2p)(-2p+1) - 5p(-10)$$

$$= -32p + 16 + 4p^2 - 2p + 50p$$

$$= 4p^2 + 16p + 16$$

A^2 SINGULAR $\Rightarrow$ $$4p^2 + 16p + 16 = 0$$

$$p^2 + 4p + 4 = 0$$

$$(p+2)(p+2) = 0$$

$$\underline{\underline{p = -2}}$$

c) $B = 3A'$

$$\begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix} = 3\begin{pmatrix} 4 & -2 \\ p & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 12 & -6 \\ 3p & 3 \end{pmatrix}$$

$$\underline{\underline{x = 12}} \qquad 3p = 1$$

$$\underline{\underline{p = \frac{1}{3}}}$$

4 a) $v = e^{3t} + 2e^{t}$

$$a = \frac{dv}{dt} = \underline{\underline{3e^{3t} + 2e^{t}}}$$

b) $$d = \int_0^{\ln 3} v\,dt = \int_0^{\ln 3} e^{3t} + 2e^{t}\,dt = \left[\frac{1}{3}e^{3t} + 2e^{t}\right]_0^{\ln 3}$$

$$= \left(\frac{1}{3}e^{3\ln 3} + 2e^{\ln 3}\right) - \left(\frac{1}{3}e^{0} + 2e^{0}\right)$$

$$= (9 + 6) - \left(\frac{1}{3} + 2\right)$$

$$= \underline{\underline{\frac{38}{3}}}$$

5) $1204 = 1 \times 833 + 371$

$833 = 2 \times 371 + 91$

$371 = 4 \times 91 + 7$

$91 = 13 \times 7 + 0$

$\therefore$ $\underline{\underline{\gcd(1204, 833) = 7}}$

$7 = 371 - 4.91$

$= 371 - 4(833 - 2.371)$

$= 9.371 - 4.833$

$= 9(1204 - 1.833) - 4.833$

$= \underline{\underline{9.1204 - 13.833}}$

6) $\displaystyle\int \frac{\sec^2 3x}{1+\tan 3x}\,dx$

$u = \tan 3x$

$\dfrac{du}{dx} = 3\sec^2 3x$

$dx = \dfrac{du}{3\sec^2 3x}$

$= \displaystyle\int \frac{\cancel{\sec^2 3x}}{1+u} \cdot \frac{du}{3\cancel{\sec^2 3x}}$

$= \dfrac{1}{3}\displaystyle\int \frac{1}{1+u}\,du$

$= \dfrac{1}{3}\ln(1+u) + C$

$= \underline{\underline{\dfrac{1}{3}\ln(1+\tan 3x) + C}}$

7) $z = 1 - \sqrt{3}\,i$

$\bar{z} = 1 + \sqrt{3}\,i$

$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$

$\underline{\underline{\bar{z} = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)}}$

$\underline{\underline{\bar{z}^2 = 4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)}}$

8) $\displaystyle\int \underset{g}{x^2}\,\underset{f'}{\cos 3x}\,dx$

$f = \dfrac{1}{3}\sin 3x$

$g' = 2x$

$= fg - \displaystyle\int fg'\,dx$

$= \dfrac{1}{3}x^2\sin 3x - \dfrac{2}{3}\displaystyle\int \underset{g}{x}\,\underset{f'}{\sin 3x}$

$f = -\dfrac{1}{3}\cos 3x$

$g' = 1$

$= \dfrac{1}{3}x^2\sin 3x - \dfrac{2}{3}\left(-\dfrac{1}{3}x\cos 3x + \dfrac{1}{3}\displaystyle\int \cos 3x\right)$

$= \dfrac{1}{3}x^2\sin 3x + \dfrac{2}{9}x\cos 3x - \dfrac{2}{9}\left(\dfrac{1}{3}\sin 3x\right) + C$

$$= \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x \cos 3x - \frac{2}{27} \sin 3x + C$$

9) $\underline{n=1}$ $\quad$ LHS $= \sum_{r=1}^{1} (4r^3 + 3r^2 + r)$ $\qquad$ RHS $= 1(1+1)^3$

$$= 4(1)^3 + 3(1)^2 + 1 \qquad\qquad = 8$$

$$= 8$$

$\therefore$ LHS = RHS so true for $n=1$.

Assume true for $\underline{n=k}$: $\sum_{r=1}^{k}(4r^3+3r^2+r) = k(k+1)^3$

$\underline{n = k+1}$

LHS $= \sum_{r=1}^{k+1} (4r^3 + 3r^2 + r)$ $\qquad$ RHS $= (k+1)(k+2)^3$

$$= \sum_{r=1}^{k} (4r^3 + 3r^2 + r) + 4(k+1)^3 + 3(k+1)^2 + (k+1)$$

$$= k(k+1)^3 + 4(k+1)^3 + 3(k+1)^2 + (k+1)$$

$$= (k+1)\left[k(k+1)^2 + 4(k+1)^2 + 3(k+1) + 1\right]$$

$$= (k+1)\left[k^3 + 2k^2 + k + 4k^2 + 8k + 4 + 3k + 3 + 1\right]$$

$$= (k+1)(k^3 + 6k^2 + 12k + 8)$$

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 12 & 8 \\ & & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

$$= (k+1)(k+2)(k^2 + 4k + 4)$$

$$= (k+1)(k+2)(k+2)(k+2)$$

$$= (k+1)(k+2)^3$$

$\therefore$ LHS = RHS so true for $n = k+1$.

Since true for $n=1$ and $n = k+1$ when $n=k$ then by induction it is true for all positive integers n.

10a) $|z+i| = 1 \implies |x+iy+i| = 1$

$$\implies |x+(y+1)i| = 1$$

$$\implies \sqrt{x^2+(y+1)^2} = 1$$

$$\implies x^2+(y+1)^2 = 1$$

$\therefore$ CIRCLE WITH CENTRE $(0,-1)$ AND RADIUS 1.

b) $|z-1| = |z+5| \implies |x+iy-1| = |x+iy+5|$

$$\implies |x-1+iy| = |x+5+iy|$$

$$\implies \sqrt{(x-1)^2+y^2} = \sqrt{(x+5)^2+y^2}$$

$$\implies (x-1)^2+\cancel{y^2} = (x+5)^2+\cancel{y^2}$$

$$\implies \cancel{x^2}-2x+1 = \cancel{x^2}+10x+25$$

$$\implies -2x+1 = 10x+25$$

$$\implies 12x = -24$$

$$\implies x = -2$$

$\therefore$ VERTICAL LINE WHICH CUTS x-AXIS AT -2.

11) $x^2+4xy+y^2+11 = 0$

$$2x+4y+4x\frac{dy}{dx}+2y\frac{dy}{dx} = 0 \quad *$$

$$(4x+2y)\frac{dy}{dx} = -2x-4y$$

$$\frac{dy}{dx} = \frac{-2x-4y}{4x+2y}$$

$$\frac{dy}{dx} = \frac{-2(-2) - 4(3)}{4(-2) + 2(3)} = \frac{-8}{-2} = \underline{\underline{4}}$$

From $*$:

$$2 + 4\frac{dy}{dx} + 4\frac{dy}{dx} + 4x\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$$

$$2 + 4(4) + 4(4) + 4(-2)\frac{d^2y}{dx^2} + 2(4)^2 + 2(3)\frac{d^2y}{dx^2} = 0$$

$$-2\frac{d^2y}{dx^2} + 66 = 0$$

$$\frac{d^2y}{dx^2} = \frac{-66}{-2} = \underline{\underline{33}}$$

12) A IS TRUE.

n IS A MULTIPLE OF 9 $\Rightarrow n = 9k \quad (k \in \mathbb{N})$

$\Rightarrow n^2 = (9k)^2 = 81k^2 = 9(9k^2)$

$\Rightarrow n^2 = 9m \quad (m = 9k^2,\ m \in \mathbb{N})$

$\Rightarrow$ $\underline{\underline{n^2 \text{ IS A MULTIPLE OF } 9}}$

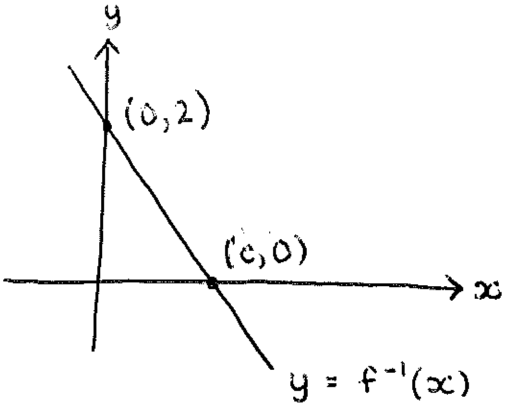
B IS FALSE.

COUNTEREXAMPLE $\underline{\underline{n = 3}}$

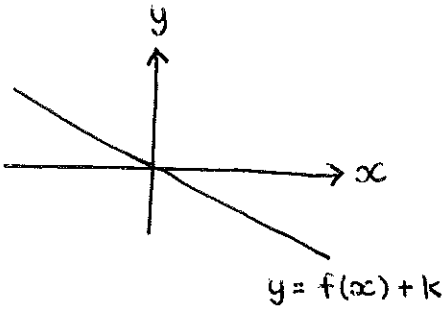
$n^2 = 3^2 = 9$ (MULTIPLE OF 9)

$n = 3$ (NOT A MULTIPLE OF 9)

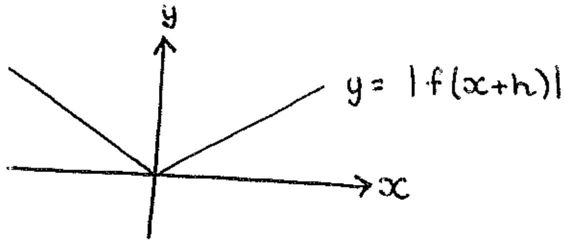
13a)



b) $\underline{\underline{k = -c}}$ (NEEDS TO PASS THROUGH ORIGIN)



c) $\underline{\underline{h = 2}}$



14) $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}$$

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0$$

$$m = 3 \qquad \therefore \quad CF \Rightarrow \quad y = (A + Bx)e^{3x}$$

$$PI \Rightarrow \quad y = Cx^2e^{3x}$$

$$\frac{dy}{dx} = 2Cxe^{3x} + 3Cx^2e^{3x}$$

$$\frac{d^2y}{dx^2} = 2Ce^{3x} + 6Cxe^{3x} + 6Cxe^{3x} + 9Cx^2e^{3x}$$

$$2Ce^{3x} + 12Cxe^{3x} + 9Cx^2e^{3x} - 6(2Cxe^{3x} + 3Cx^2e^{3x}) + 9(Cx^2e^{3x}) = 4e^{3x}$$

$$2Ce^{3x} + \cancel{12Cxe^{3x}} + \cancel{9Cx^2e^{3x}} - \cancel{12Cxe^{3x}} - \cancel{18Cx^2e^{3x}} + \cancel{9Cx^2e^{3x}} = 4e^{3x}$$

$$2C = 4$$

$$C = 2 \qquad \therefore \quad PI \Rightarrow \quad y = 2x^2e^{3x}$$

GENERAL SOLUTION : $y = (A + Bx)e^{3x} + 2x^2e^{3x}$

$$y = (A + Bx + 2x^2)e^{3x}$$

$x = 0,\ y = 1 \implies 1 = Ae^{3(0)} \implies A = 1$

$$y = (2x^2 + Bx + 1)e^{3x}$$

$$\frac{dy}{dx} = (6x^2 + 3Bx + 3)e^{3x} + (4x + B)e^{3x}$$

$x = 0,\ \dfrac{dy}{dx} = -1 \implies -1 = 3e^{3(0)} + Be^{3(0)} \implies B + 3 = -1 \implies B = -4$

PARTICULAR SOLUTION : $\underline{\underline{y = (2x^2 - 4x + 1)e^{3x}}}$

15a) $\overrightarrow{AB} = \underline{b} - \underline{a}$ $\quad\quad$ $\overrightarrow{AC} = \underline{c} - \underline{a}$

$$= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \quad\quad = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad\quad = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\underline{n}_1 = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \underline{i}\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 2\underline{i} - 2\underline{j} + \underline{k}$$

$k = \underline{n}_1 . \underline{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} . \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = 5 \quad\quad \therefore\ \underline{\underline{2x - 2y + z = 5}}$

b) $k = \underline{n}_2 \cdot \underline{a} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = 4 \quad \therefore \quad \underline{\underline{-y + z = 4}}$

c) $\underline{n}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad \underline{n}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \cos\theta = \dfrac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}$

$\underline{n}_1 \cdot \underline{n}_2 = 3 \qquad \cos\theta = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$|\underline{n}_1| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

$\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \underline{\underline{\dfrac{\pi}{4}}}$

$|\underline{n}_2| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}$

16) $$\frac{dP}{dt} = P(1000 - P)$$

$$\int \frac{1}{P(1000-P)}\, dP = \int dt$$

$$\frac{1}{P(1000-P)} = \frac{A}{P} + \frac{B}{(1000-P)} = \frac{A(1000-P) + BP}{P(1000-P)}$$

$P = 0 \implies 1000A = 1 \qquad P = 1000 \implies 1000B = 1$

$A = \dfrac{1}{1000} \qquad B = \dfrac{1}{1000}$

$$\frac{1}{1000}\int \frac{1}{P} + \frac{1}{(1000-P)}\, dP = \int dt$$

$$\ln P - \ln(1000 - P) = 1000t + C$$

$$\underline{\underline{\ln \frac{P}{(1000-P)} = 1000t + C}}$$

$$\exp\left(\ln \frac{P}{(1000-P)}\right) = \exp(1000t + c)$$

$$\frac{P}{(1000-P)} = De^{1000t}$$

$$P = 1000De^{1000t} - PDe^{1000t}$$

$$P + PDe^{1000t} = 1000De^{1000t}$$

$$P\left(De^{1000t} + 1\right) = 1000De^{1000t}$$

$$P = \frac{1000De^{1000t}}{De^{1000t} + 1} \qquad (\div e^{1000t})$$

$$\underline{\underline{P = \frac{1000D}{D + e^{-1000t}}}}$$

$$P(0) = 200 \implies 200 = \frac{1000D}{D + e^{-1000(0)}}$$

$$200 = \frac{1000D}{D+1}$$

$$200D + 200 = 1000D$$

$$800D = 200$$

$$D = \frac{1}{4} \qquad \therefore \quad P = \frac{250}{\frac{1}{4} + e^{-1000t}}$$

$$P(t) = 900 \implies 900 = \frac{250}{\frac{1}{4} + e^{-1000t}}$$

$$225 + 900e^{-1000t} = 250$$

$$900e^{-1000t} = 25$$

$$e^{-1000t} = \frac{1}{36}$$

$$\ln e^{-1000t} = \ln \frac{1}{36}$$

$$-1000t = \ln \frac{1}{36}$$

$$t = \frac{\ln \frac{1}{36}}{-1000}$$

$$\underline{\underline{t = 0.00358}}$$

17) $a = 1$

$r = x$

$$S_\infty = \frac{a}{1-r} = \underline{\underline{\frac{1}{1-x}}}$$

$a = 1$

$r = -x$

$$S_\infty = \frac{a}{1-r} = \underline{\underline{\frac{1}{1+x}}}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \Rightarrow -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad (1)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad (2)$$

(2) + (1) $$\ln(1+x) - \ln(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^3}{5} + \dots$$

$$\underline{\underline{\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)}}$$

$$\frac{1+x}{1-x} = 2 \implies 1+x = 2-2x \implies 3x = 1 \implies x = \frac{1}{3}$$

$$\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \ln\left(\frac{4/3}{2/3}\right) = \ln 2 \qquad \left(\text{SUBSTITUTE IN } x = \frac{1}{3}\right)$$

$$\ln 2 = 2\left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \dots\right)$$

$$= \underline{\underline{0.693}}$$