2011 SOLUTIONS

1. $$\frac{13-x}{x^2+4x-5} = \frac{13-x}{(x+5)(x-1)} = \frac{A}{(x+5)} + \frac{B}{x-1}$$

$$13-x = A(x-1) + B(x+5)$$

let $x=-5$ $\qquad 18 = A(-6) \Rightarrow A=-3$

let $x=1$ $\qquad 12 = B(6) \Rightarrow B=3$

$$\frac{13-x}{x^2+4x-5} = \frac{2}{x-1} - \frac{3}{x+5}$$

$$\int \frac{13-x}{x^2+4x-5}\,dx = \int \left[\frac{2}{x-1} - \frac{3}{x+5}\right]$$

$$= 2\ln|x-1| - 3\ln|x+5| + c.$$

$$= \ln|x-1|^2 - \ln|x+5|^3 + c$$

$$= \underline{\underline{\ln\left(\frac{|x-1|^2}{|x+5|^3}\right) + c}}$$

2. $\left(\frac{1}{2}x-3\right)^4 = 1\cdot\left(\frac{1}{2}x\right)^4 + 4\cdot\left(\frac{1}{2}x\right)^3\cdot(-3)^1 + 6\cdot\left(\frac{1}{2}x\right)^2\cdot(-3)^2$

1, 4, 6, 4, 1

$$+ 4\cdot\left(\frac{1}{2}x\right)^1(-3)^3 + 1\cdot(-3)^4$$

$$= \underline{\frac{1}{16}x^4 - \frac{3}{2}x^3 + \frac{27}{2}x^2 - 54x + 81}$$

3(a) $y + e^y = x^2$

$$\frac{dy}{dx} + e^y\frac{dy}{dx} = 2x.$$

$$\frac{dy}{dx}\left(1+e^y\right) = 2x$$

$$\underline{\frac{dy}{dx} = \frac{2x}{1+e^y}}$$

(b) $f(x) = \sin x \cos^3 x$

$$f'(x) = \cos^3 x\cdot\cos x + \sin x\cdot 3\cos^2 x\cdot(-\sin x)$$

$$= \cos^4 x - 3\sin^2 x\cos^2 x$$

$$= \underline{\cos^2 x\left(\cos^2 x - 3\sin^2 x\right)}$$

4(a) $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{vmatrix} = 0$

$$\Rightarrow -2\lambda - 40 - 3\lambda = 0$$

$$-5\lambda - 40 = 0$$

$$\underline{\underline{\lambda = -8}}$$

(b) $A' = \begin{pmatrix} 2 & 3\alpha+2\beta & -1 \\ 2\alpha-\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$

$3\alpha + 2\beta = -5$ ①

$2\alpha - \beta = -1$ ② $\times 2$

$4\alpha - 2\beta = -2$ ③

① + ③ $7\alpha = -7$

$$\underline{\underline{\alpha = -1}}$$

Sub $\alpha = -1$ into ②

$$-2 - \beta = -1$$

$$\underline{\underline{\beta = -1.}}$$

5. $\sqrt{1+x} = (1+x)^{1/2}$

$f(x) = (1+x)^{1/2}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\quad f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$ $\quad f'''(0) = 3/8$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$

$$\underline{f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3}$$

$$f(x^2) = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6$$

$$\underline{f(x^2) = \sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6}$$

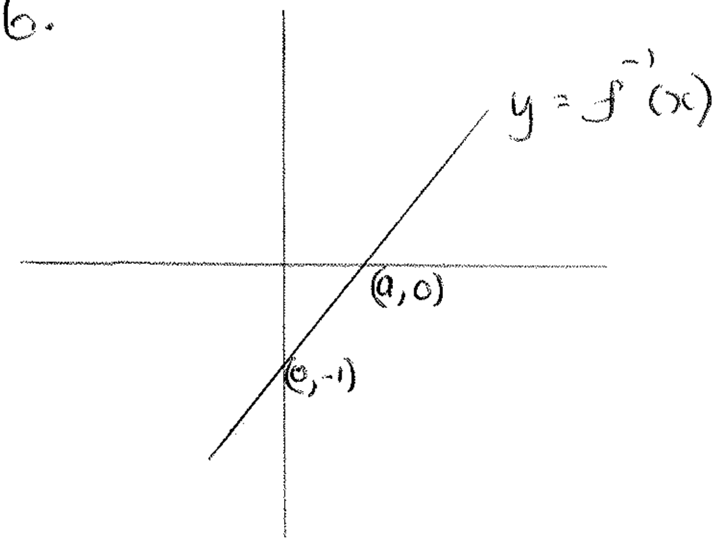
$$\sqrt{(1+x)(1+x^2)} = \sqrt{1+x} \times \sqrt{1+x^2}$$

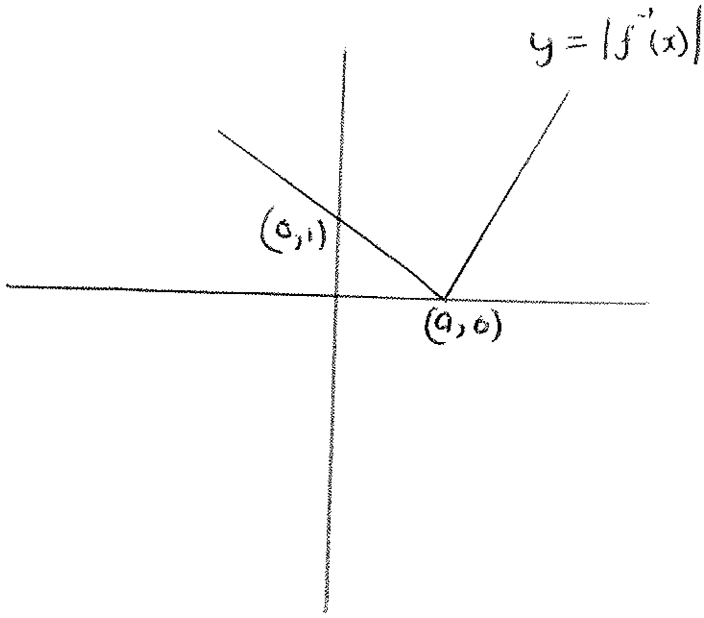
$$= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3\right)\left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6\right)$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x^3 + \ldots\ldots$$

$$= \underline{1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots\ldots\ldots}$$

6.





7. $$y = \frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$$

$$\ln y = \ln\left(\frac{e^{\sin x}(2+x)^3}{(1-x)^{1/2}}\right)$$

$$\ln y = \ln(e^{\sin x}) + \ln((2+x))^3 - \ln((1-x))^{1/2}$$

$$\ln y = \sin x + 3\ln(2+x) - \tfrac{1}{2}\ln(1-x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x + \frac{3}{2+x} + \frac{1}{2(1-x)}$$

$$\frac{dy}{dx} = y\left(\cos x + \frac{3}{2+x} + \frac{1}{2(1-x)}\right)$$

at $x = 0$, gradient of curve $= \dfrac{dy}{dx} = \dfrac{e^0 \cdot 2^3}{1}\left(1 + 3/2 + 1/2\right)$

$= \underline{24}$

8. $$\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2.$$

$$= \frac{n^2}{4}(n+1)^2 - \left(\frac{n}{2}(n+1)\right)^2$$

$$= \frac{n^2}{4}(n+1)^2 - \frac{n^2}{4}(n+1)^2$$

$$= \underline{\underline{0}} \Rightarrow \underline{\underline{\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2}}$$

$$\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2$$

$$= \frac{n^2}{4}(n+1)^2 + \frac{n^2}{4}(n+1)^2$$

$$= \underline{\underline{\frac{n^2}{2}(n+1)^2}}$$

9. $$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$$

$$\int \frac{dy}{1+y} = \int 3(1+x)^{1/2}\, dx$$

$$\ln|1+y| = \frac{3(1+x)^{3/2}}{3/2} + c$$

$$\ln|1+y| = 2(1+x)^{3/2} + c$$

$$e^{\ln|1+y|} = e^{2(1+x)^{3/2} + c}$$

$$1+y = e^{2(1+x)^{3/2}} \cdot e^{c}$$

$$e^{c} = A$$

$$\underline{\underline{y = Ae^{2(1+x)^{3/2}} - 1}}$$

10. $|z-1| = 3 \qquad z = x+iy$

$$|x+iy-1| = 3$$

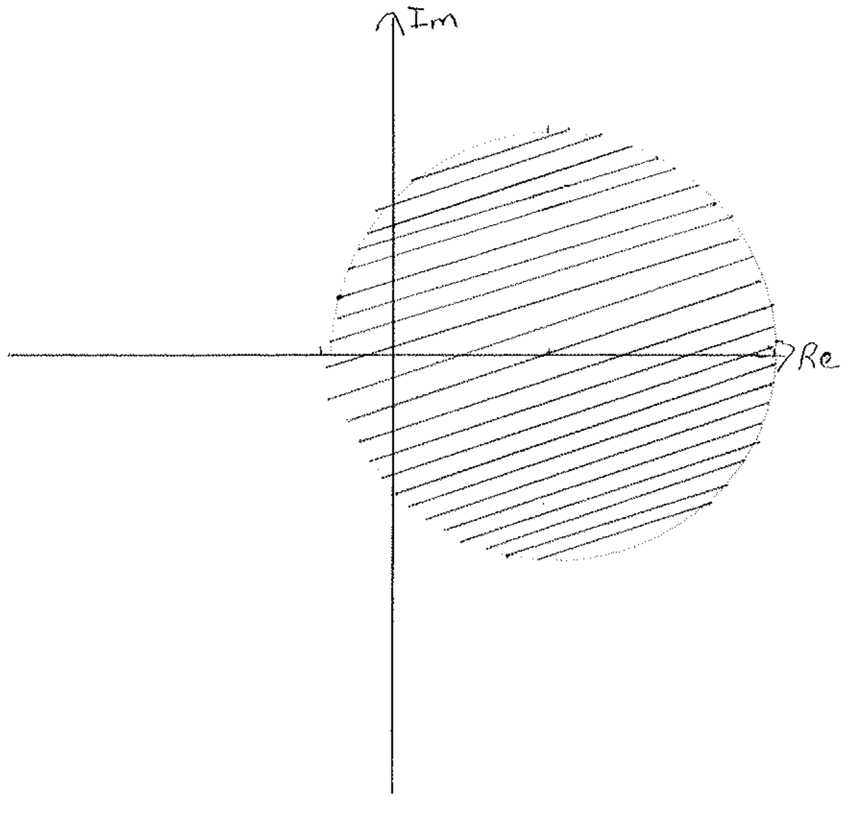
$$|(x-1)+iy| = 3$$

$$\sqrt{(x-1)^2+y^2} = 3$$

$$(x-1)^2+y^2 = 9$$

Equation of a circle centre $(1,0)$
radius 3.

$|z-1| \leq 3$ are the set of points on and inside the circle



11(a) $\displaystyle\int_0^{\pi/4} (\sec x - x)(\sec x + x) \quad dx$

$$\int_0^{\pi/4} (\sec^2 x - x^2)\, dx$$

$$\left[\tan x - \frac{x^3}{3}\right]_0^{\pi/4}$$

$$\left(\tan\frac{\pi}{4} - \frac{(\pi/4)^3}{3}\right) - 0$$

$$\underline{\underline{1 - \frac{\pi^3}{192}}}$$

(b) $\displaystyle\int \frac{x}{\sqrt{1-49x^4}} \quad dx$

$$= \int \frac{x}{\sqrt{1-(7x^2)^2}} \quad dx$$

$u = 7x^2$

$du = 14x\,dx.$

$\frac{1}{14}du = x\,dx.$

$$= \frac{1}{14}\int \frac{1}{\sqrt{1-u^2}} \quad du$$

$$= \frac{1}{14}\sin^{-1} u + C$$

$$= \underline{\underline{\frac{1}{14}\sin^{-1}(7x^2) + C}}$$

12. $8^n + 3^{n-2}$ is divisible by 5

Prove true for $n=2$

$$8^n + 3^{n-2} = 8^2 + 3^{2-2}$$
$$= 65 \text{ is divisible by } 5.$$

$\therefore$ true for $n=2$.

Assume true for $n=k$, $k \in \mathbb{Z}$, $k \geqslant 2$

$$8^k + 3^{k-2} = 5a \quad \text{for some integer } a$$

PROVE TRUE FOR $n=k+1$

$$8^{k+1} + 3^{k+1-2} = 8 \cdot 8^k + 3 \cdot 3^{k-2}$$
$$= (5+3)8^k + 3 \cdot 3^{k-2}$$
$$= 5 \cdot 8^k + 3 \cdot (8^k + 3^{k-2})$$
$$= 5 \cdot 8^k + 3 \cdot 5a$$
$$= 5(8^k + 3a)$$
$$= 5b, \quad b = 8^k + 3a$$

is divisible by 5

Hence true for $n=k+1$ when it is true for $n=k$

As it is true for $n=2$, by induction it is true

for all $n \in \mathbb{Z}$, $n \geqslant 2$

13. $a, \frac{1}{a}, 1$

$$d = \frac{1}{a} - a = 1 - \frac{1}{a}$$

$$1 - a^2 = a - 1$$

$$a^2 + a - 2 = 0$$

$$(a+2)(a-1) = 0$$

$$\underline{\underline{a = -2}} \text{ for } a < 0$$

$$d = 1 + \frac{1}{2} = \underline{\underline{3/2}}$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$a = -2$
$d = 3/2$
$n = ?$

$$S_n = \frac{n}{2}\left[-4 + (n-1) \times 3/2\right]$$

$$= \frac{n}{2}\left[-4 + \frac{3}{2}n - \frac{3}{2}\right]$$

$$= \frac{n}{2}\left[\frac{3}{2}n - \frac{11}{2}\right]$$

$$S_n = \frac{3}{4}n^2 - \frac{11}{4}n$$

we want $S_n > 1000$

$$\Rightarrow \frac{3}{4}n^2 - \frac{11}{4}n > 1000$$

$$3n^2 - 11n - 4000 > 0$$

First solve $3n^2 - 11n - 4000 = 0$

$$\begin{array}{ccc} 3n^2 & -11n & -4000 = 0 \\ | & | & | \\ a & b & c \end{array}$$

$$n = \frac{11 \pm \sqrt{48121}}{6}$$

$b^2 - 4ac$

$= 121 - (4 \times 3 \times -4000)$

$= 121 - (-48\,000)$

$= 48\,121$

$n = 38.39$ or $n = -34.73$

not valid as $n > 0$

Solve > 0, $n > 38.39$

$n = 39$

14. $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x + 12$$

A.E. $$m^2 - m - 2 = 0$$
$$(m-2)(m+1) = 0$$
$$m = 2 \text{ or } m = -1$$

C.F. $$y = Ae^{2x} + Be^{-x}.$$

P.I. try $$y = ce^x + D$$
$$\frac{dy}{dx} = ce^x$$
$$\frac{d^2y}{dx^2} = ce^x.$$

$$ce^x - ce^x - 2(ce^x + D) = e^x + 12$$
$$-2ce^x - 2D = e^x + 12$$
$$-2c = 1 \qquad -2D = 12$$
$$c = -\frac{1}{2} \qquad D = -6$$

P.I. $$y = -\frac{1}{2}e^x - 6$$

G.S. is CF + PI
$$\underline{y = Ae^{2x} + Be^{-x} - \frac{1}{2}e^x - 6}$$

$x = 0$ when $y = -3/2$

$$-3/2 = Ae^{0} + Be^{0} - \frac{1}{2}e^{0} - 6$$

$$A + B = 5 \quad (1)$$

$x = 0$ when $\frac{dy}{dx} = \frac{1}{2}$ $\qquad \frac{dy}{dx} = 2Ae^{2x} - Be^{-x} - \frac{1}{2}e^{x}$

$$\frac{1}{2} = 2Ae^{0} - Be^{0} - \frac{1}{2}e^{0}$$

$$2A - B = 1 \quad (2)$$

ADD
(1) + (2)

$$3A = 6$$
$$A = 2$$

$$A + B = 5$$
$$2 + B = 5$$
$$B = 3.$$

Particular solution is

$$y = 2e^{2x} + 3e^{-x} - \frac{1}{2}e^{x} - 6$$

15. (a) L_1 $\quad \dfrac{x-1}{k} = \dfrac{y}{-1} = \dfrac{z+3}{1} = u$

Parametric form

$x = ku + 1$

$y = -u$

$z = u - 3$

L_2 $\quad \dfrac{x-4}{1} = \dfrac{y+3}{1} = \dfrac{z+3}{2} = t$

Parametric form

$x = t + 4$

$y = t - 3$

$z = 2t - 3$

L_1 and L_2 intersect use $y=y$, $z=z$

$y=y$: $\quad -u = t - 3$

$z=z$: $\quad u - 3 = 2t - 3$

ADD $\quad -3 = 3t - 6$

$3t = 3$

$\underline{t = 1}$

$-u = t - 3$

$-u = -2 \Rightarrow \underline{u = 2}$

Lines intersect so $t=1$ and $u=2$ will also work for x

$$x = t + 4$$
$$= 1 + 4$$
$$= 5$$

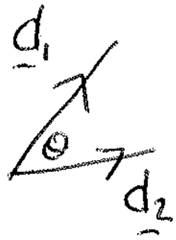
now $x = ku + 1.$

$$5 = 2k + 1$$
$$4 = 2k$$
$$k = 2$$

P.O.I. $x=5$ $\quad y = -u = -2$ $\quad z = u - 3 = -1$

$$\underline{\underline{(5, -2, -1).}}$$

(b)



$$\cos\theta = \frac{\underline{d}_1 \cdot \underline{d}_2}{|\underline{d}_1|\,|\underline{d}_2|}$$

$\underline{d}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ $\quad \underline{d}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\quad = \dfrac{3}{\sqrt{6} \times \sqrt{6}}$

$\underline{d}_1 \cdot \underline{d}_2 = 2 - 1 + 2 = 3$ $\quad \cos\theta = 1/2$

$|\underline{d}_1| = \sqrt{6}$ $\quad \underline{\underline{\theta = 60^0}}$

$|\underline{d}_2| = \sqrt{6}$

16.

(a) $$I_n = \int_0^1 \frac{1}{(1+x^2)^n} \, dx$$

$$= \int_0^1 1 \times \frac{1}{(1+x^2)^n} \, dx \qquad u = \frac{1}{(1+x^2)^n} \qquad v' = 1$$

$$u' = \frac{-2nx}{(1+x^2)^{n+1}} \qquad v = x$$

$$I_n = \left[\frac{x}{(1+x^2)^n}\right]_0^1 - \int \frac{-2nx^2}{(1+x^2)^{n+1}} \, dx \qquad uv - \int u'v$$

$$I_n = \left(\frac{1}{2^n}\right) - (0) + 2n \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} \, dx$$

$$= \frac{1}{2^n} + 2n \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} \, dx.$$

(b) $$\frac{x^2}{(1+x^2)^{n+1}} = \frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}}.$$

$$x^2 = A(1+x^2) + B$$

Equate x^2

$1 = A$

let $x = 0$ $\quad 0 = A + B$

$B = -1$

$$\frac{x^2}{(1+x^2)^{n+1}} = \frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}} \qquad *$$

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx.$$

from *

$$I_n = \frac{1}{2^n} + 2n\left[\underbrace{\int_0^1 \frac{1}{(1+x^2)^n}\,dx}_{I_n} - \underbrace{\int_0^1 \frac{1}{(1+x^2)^{n+1}}\,dx}_{I_{n+1}}\right]$$

$$I_n = \frac{1}{2^n} + 2n\,I_n - 2n\,I_{n+1}$$

$$2n\,I_{n+1} = \frac{1}{2^n} + 2n\,I_n - I_n$$

$$2n\,I_{n+1} = \frac{1}{2^n} + (2n-1)\,I_n$$

$$I_{n+1} = \frac{1}{2n}\left(\frac{1}{2^n} + (2n-1)\,I_n\right)$$

$$I_{n+1} = \frac{1}{n \times 2^{n+1}} + \left(\frac{2n-1}{2n}\right) I_n$$

(c) $I_1 = \int_0^1 \frac{1}{(1+x^2)} \, dx$

$I_2 = \int_0^1 \frac{1}{(1+x^2)^2} \, dx$

$I_3 = \int_0^1 \frac{1}{(1+x^2)^3} \, dx$. ✓

using $I_{n+1} = \frac{1}{n \times 2^{n+1}} + \left(\frac{2n-1}{2n}\right) I_n$

$n = 1 \quad I_2 = \frac{1}{4} + \frac{1}{2} I_1$

$n = 2 \quad I_3 = \frac{1}{16} + \frac{3}{4} I_2$

$$\int_0^1 \frac{1}{(1+x^2)^3} \, dx = \frac{1}{16} + \frac{3}{4} \int_0^1 \frac{1}{(1+x^2)^2} \, dx$$

$$= \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{1}{2} \int_0^1 \frac{1}{(1+x^2)} \, dx \right)$$

$$= \frac{1}{16} + \frac{3}{16} + \frac{3}{8} \left[\tan^{-1}(x) \right]_0^1$$

$$= \frac{1}{4} + \frac{3}{8} \left[\frac{\pi}{4} - 0 \right]$$

$$= \frac{1}{4} + \frac{3\pi}{32}$$