2014

Q1a $f(x) = \frac{x^2-1}{x^2+1}$

$$f'(x) = \frac{u'v - v'u}{v^2}$$

$$= \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2}$$

$$= \underline{\underline{\frac{4x}{(x^2+1)^2}}}$$

b $y = \tan^{-1}(3x^2)$

$$\frac{dy}{dx} = \frac{1}{1+(3x^2)^2} \cdot 6x$$

$$= \underline{\underline{\frac{6x}{1+9x^4}}}$$

Q2 $\left(\frac{2}{x} + \frac{1}{4x^2}\right)^{10}$ term in $\frac{1}{x^{13}}$

$$u_r = \binom{10}{r}\left(\frac{2}{x}\right)^{10-r}\left(\frac{1}{4x^2}\right)^r$$

$$= \binom{10}{r}\frac{2^{10-r} \cdot 4^{-r}}{x^{10-r+2r}}$$

$$= \underline{\underline{\binom{10}{r}\frac{2^{10-3r}}{x^{10+r}}}} \qquad \text{as } 4 = 2^{2(-r)}$$

$\frac{1}{x^{13}} \Rightarrow 10 + r = 13$

$\underline{\underline{r = 3}}$

i.e. $u_3 = \binom{10}{3}\frac{2^1}{x^{13}}$

$$= \underline{\underline{\frac{240}{x^{13}}}}$$

Q3 $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 5 & 6 & 8 & 11 \\ 4 & 3 & -\lambda & 4 \end{array}\right)$

$$\begin{array}{l} \\ R_2 - 5R_1 \\ R_3 - 4R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -\lambda-4 & -4 \end{array}\right)$$

$$\begin{array}{l} \\ \\ R_3 + R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -\lambda-1 & -3 \end{array}\right)$$

So, $(\lambda + 1) z = 3$

$$z = \frac{3}{\lambda + 1}$$

Solutions $\forall \lambda, \ \lambda \neq -1$

$\lambda = 2 \Rightarrow z = 1, \ y = -2, \ x = 3$

Q4 $x = \ln(1+t^2)$ $y = \ln(1+2t^2)$ $\frac{dy}{dx}(t)$

$$\frac{dx}{dt} = \frac{2t}{1+t^2} \qquad \frac{dy}{dt} = \frac{4t}{1+2t^2}$$

$$\frac{dy}{dx} = \frac{4t}{1+2t^2} \cdot \frac{1+t^2}{2t}$$

$$= \underline{\underline{\frac{2(1+t^2)}{1+2t^2}}}$$

Q5 $\underline{u} \cdot (\underline{v} \times \underline{w})$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$= \begin{vmatrix} 5 & 13 & 0 \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix} - 13\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= 5(-13) - 13(-5)$$

$$= \underline{\underline{0}}$$

$\Rightarrow$ $\underline{u}$, $\underline{v}$ and $\underline{w}$ lie in the same plane
(O, A, B and C are co-planar)

Q6 $e^y = x^3\cos^2 x, \quad x > 0 \qquad \frac{dy}{dx} = \frac{a}{x} + b\tan x$

$e^y \cdot \frac{dy}{dx} = 3x^2\cos^2 x + x^3 2\cos x \cdot (-\sin x)$

$x^3\cos^2 x \frac{dy}{dx} = 3x^2\cos^2 x - 2x^3 \sin x \cos x$

$\frac{dy}{dx} = \frac{3x^2\cos^2 x}{x^3\cos^2 x} - \frac{2x^3\sin x\cos x}{x^3\cos x\cos x}$

$= \frac{3}{x} - 2\tan x$

Q7 $A = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ prove $A^n = \begin{pmatrix} 2^n & a(2^n-1) \\ 0 & 1 \end{pmatrix}$ $n \geq 1$

$n=1$ LHS $= A^1 = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ RHS $= A^1 = \begin{pmatrix} 2^1 & a(2^1-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$

LHS = RHS $\Rightarrow$ true for $n=1$

Assume true for $n=k$ $\Rightarrow$ $A^k = \begin{pmatrix} 2^k & a(2^k-1) \\ 0 & 1 \end{pmatrix}$ (1)

Consider $n=k+1$

$A^{k+1} = A(A^k)$

$= \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k & a(2^k-1) \\ 0 & 1 \end{pmatrix}$ using (1)

$= \begin{pmatrix} 2\cdot 2^k + 0 & 2a(2^k-1)+a \\ 0+0 & 0+1 \end{pmatrix}$

$= \begin{pmatrix} 2^{k+1} & a(2^{k+1}-1) \\ 0 & 1 \end{pmatrix}$

true for $n=k$ $\Rightarrow$ true for $n=k+1$

since true for $n=1$ $\Rightarrow$ by induction $\forall\, n \geq 1$

Q8 $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$ $\quad y(0) = 4 \quad \frac{dy}{dx}(0) = 3$

AE $4m^2 - 4m + 1 = 0$

$(2m-1)^2 = 0$

$m = \frac{1}{2}$

$y = Ae^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x}$

$y(0) = 4 \quad 4 = Ae^0 + B \cdot 0 \cdot e^0 \Rightarrow A = 4$

$y = 4e^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x}$

$\frac{dy}{dx} = 2e^{\frac{1}{2}x} + B(e^{\frac{1}{2}x} + \frac{1}{2}xe^{\frac{1}{2}x})$

$\frac{dy}{dx}(0) = 3 \quad 3 = 2e^0 + B(e^0 + 0)$

$3 = 2 + B$

$B = 1$

$\Rightarrow y = 4e^{\frac{1}{2}x} + xe^{\frac{1}{2}x}$

Q9 $\cos 3x$

$$= 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \dots$$

$$= 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4$$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$

$e^{2x}$

$$= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!}$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$e^{2x}\cos 3x = \left(1 + 2x + 2x^2 + \tfrac{4}{3}x^3\right)\left(1 - \tfrac{9}{2}x^2\right)$$

$$= 1 + 2x + 2x^2 + \tfrac{4}{3}x^3 - \tfrac{9}{2}x^2 - 9x^3$$

$$= 1 + 2x - \tfrac{5}{2}x^2 - \tfrac{23}{3}x^3$$

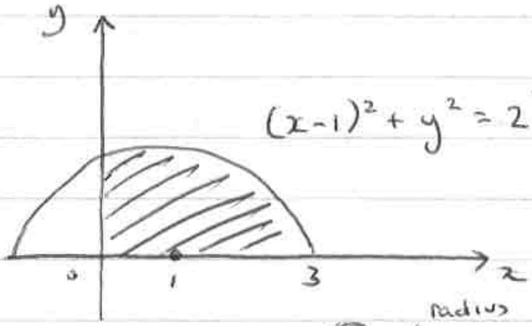
Q10 $V = \pi \int y^2 \, dx$

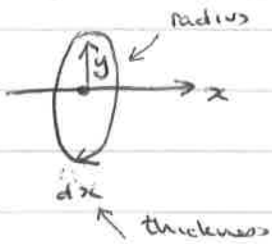
$$= \pi \int_0^3 4 - (x-1)^2 \, dx$$

$$= \pi \left[ 4x - \tfrac{1}{3}(x-1)^3 \right]_0^3$$

$$= \pi \left[ \left( 4(3) - \tfrac{1}{3}(3-1)^3 \right) - \left( 0 - \tfrac{1}{3}(0-1)^3 \right) \right]$$

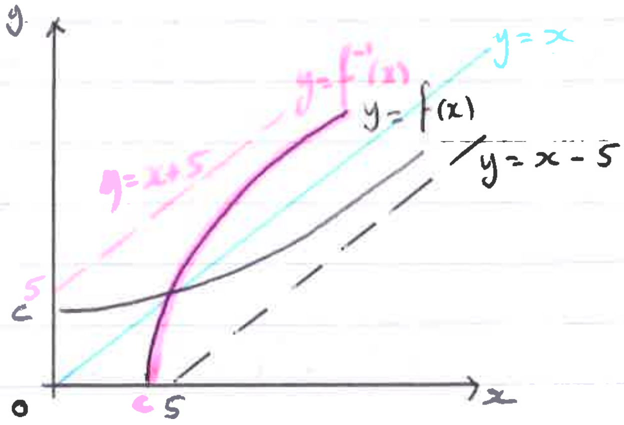
$$= 9\pi \text{ units}^3$$



Q11.

(a)




(b) $\underline{\underline{y = x - 3}}$

(c) $x = f(f(x))$

$f^{-1}(x) = f^{-1}(f(f(x)))$

$f^{-1}(x) = f(x)$

$y = f^{-1}(x)$ intersects $y = f(x)$ on the line $y = x$

$\Rightarrow$ at $\underline{\underline{\text{least one}}}$ solution.

Q12 $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^{3/2}}$ $\qquad x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$ i.e. $dx = \sec^2\theta \, d\theta$

$\therefore$ LIMITS : $x = 0$ $\quad \tan\theta = 0$

$\theta = 0$

So, $\displaystyle\int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^{3/2}}$ $\qquad x = 1$ $\quad \tan\theta = 1$

$\theta = \frac{\pi}{4}$

$= \displaystyle\int_0^{\pi/4} \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}} \, d\theta$ $\qquad \therefore \frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2}$

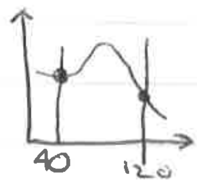
$\tan^2 + 1 = \sec^2$

$= \displaystyle\int_0^{\pi/4} \frac{d\theta}{\sec\theta}$

$= \displaystyle\int_0^{\pi/4} \cos\theta \, d\theta = [\sin\theta]_0^{\pi/4} = \sin\frac{\pi}{4} - \sin 0 = \underline{\underline{\frac{1}{\sqrt{2}}}}$

Q13 $F = 15 + e^x(\sin x - \cos x - \sqrt{2})$ $\qquad x = \dfrac{\pi(s-40)}{80}$

$$\boxed{\frac{dF}{ds} = \frac{dF}{dx} \cdot \frac{dx}{ds}}$$




$$\frac{dF}{dx} = e^x(\sin x - \cos x - \sqrt{2}) + e^x(\cos x + \sin x)$$
$$= e^x(2\sin x - \sqrt{2})$$

$$\frac{dx}{ds} = \frac{\pi}{80}$$

So, $\dfrac{dF}{ds} = \dfrac{\pi}{80} e^x (2\sin x - \sqrt{2})$

$\frac{dF}{ds} = 0$ @ S.P i.e. $\frac{\pi}{80} e^x (2\sin x - \sqrt{2}) = 0$

$\Rightarrow \quad 2\sin x - \sqrt{2} = 0$ as $e^x \neq 0$

$\sin x = \frac{1}{\sqrt{2}}$

$x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

$\because x = \dfrac{\pi(s-40)}{80}$ so $s = \dfrac{80x}{\pi} + 40$

So, $x = \frac{\pi}{4} \Rightarrow s = 60$ (in range)

$x = \frac{3\pi}{4} \Rightarrow s = 100$ (in range)

$\therefore$ need to test at 40, 60, 100, 120 ($= s$)

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

$x = 0 \quad x = \frac{\pi}{4} \quad x = \frac{3\pi}{4} \quad x = \pi$

$x = 0 \Rightarrow F = 14 - \sqrt{2}$ $(= 12.586)$

$x = \frac{\pi}{4} \Rightarrow F = 11.898$

$x = \frac{3\pi}{4} \Rightarrow F = 15$

$x = \pi \Rightarrow F = 5.415$

$\therefore$ Max when $x = \frac{3\pi}{4}$ i.e. max $F = 15$ km/L @ $S = 100$ km/h

Min when $x = \pi$ i.e. min $F = 5.42$ km/L @ $S = 120$ km/h

Q14 $1 + r + r^2 + r^3 + \ldots$ $\quad |r| < 1$

(a) $\therefore a = 1$ $\quad r = r$

common ratio

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{1}{1-r}$$

$$\frac{1}{2-3r}$$

$$= \frac{1}{2\left(1-\frac{3r}{2}\right)}$$

$$= \frac{\frac{1}{2}}{1-\frac{3r}{2}} \rightarrow S_\infty \text{ of series with } a = \frac{1}{2} \quad r = \frac{3r}{2}$$

$$\frac{1}{2-3r} = \frac{1}{2} + \frac{3}{4}r + \frac{9}{8}r^2 + \frac{27}{16}r^3 + \ldots$$

converge for $\left|\frac{3r}{2}\right| < 1$

$-1 < \frac{3r}{2}$ $\qquad \frac{3r}{2} < 1$

$r > -\frac{2}{3}$ $\qquad r < \frac{2}{3}$

converges for $-\frac{2}{3} < r < \frac{2}{3}$

(b) $$\frac{1}{3r^2-5r+2} = \frac{1}{(3r-2)(r-1)} = \frac{A}{3r-2} + \frac{B}{r-1}$$

$$A(r-1) + B(3r-2) = 1$$

let $r=1$ $$B = 1$$

let $r=\frac{2}{3}$ $$-\frac{1}{3}A = 1$$
$$A = -3$$

i.e. $$-\frac{3}{3r-2} + \frac{1}{r-1}$$

$$= \frac{3}{2-3r} - \frac{1}{1-r}$$

$$= \frac{\frac{3}{2}}{1-\frac{3r}{2}} - \frac{1}{1-r}$$

$$= \overset{*1}{\left(\frac{3}{2} + \frac{9}{4}r + \frac{27}{8}r^2\right)} - \overset{*2}{\left(1+r+r^2\right)}$$

$$= \frac{1}{2} + \frac{5}{4}r + \frac{19}{8}r^2$$

$*1$ converges $-\frac{2}{3} < r < \frac{2}{3}$

$*2$ converges $-1 < r < 1$

So, overall, converges $-\frac{2}{3} < r < \frac{2}{3}$

Q15 $I = \int \underset{i}{e^x} \underset{d}{\cos x}\, dx$

(a)

$$= e^x \cos x - \int e^x (-\sin x)\, dx$$

$$= e^x \cos x + \int \underset{i}{e^x} \underset{d}{\sin x}\, dx$$

$$= e^x \cos x + \left(e^x \sin x - \int e^x \cos x\, dx\right)$$

$$= e^x(\cos x + \sin x) - I$$

$$2I = e^x(\cos x + \sin x)$$

$$I = \frac{1}{2} e^x(\cos x + \sin x) + C$$

(b) $I_n = \int \underset{i}{e^x} \underset{d}{\cos nx}\, dx \qquad n \neq 0$

$$= e^x \cos nx + n \int \underset{i}{e^x} \underset{d}{\sin nx}\, dx$$

$$= e^x \cos nx + n\left(e^x \sin nx - n \int e^x \cos nx\, dx\right)$$

$$= e^x(\cos nx + n \sin nx) - n^2 I_n$$

$$(n^2+1) I_n = e^x(\cos nx + n \sin nx)$$

$$I_n = \frac{1}{n^2+1} \cdot e^x(\cos nx + n \sin nx) + C$$

(c) $\int e^x \cos nx \, dx = \frac{1}{n^2+1} e^x (\cos nx + n \sin nx) + c$

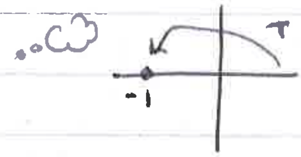
$$\int_0^{\frac{\pi}{2}} e^x \cos 8x \, dx = \left[\frac{1}{8^2+1} e^x (\cos 8x + 8 \sin 8x)\right]_0^{\pi/2}$$

$$= \frac{1}{65}\left(e^{\pi/2}(\cos 4\pi + 8 \sin 4\pi) - e^0(\cos 0 + 8 \sin 0)\right)$$

$$= \underline{\underline{\frac{1}{65}\left(e^{\frac{\pi}{2}} - 1\right)}} \qquad (= 0.05862\ldots)$$

Q16

(a) $-1 = \cos\pi + i\sin\pi$




$$z^4 + 1 = 0$$

$$z^4 = -1$$

$$z^4 = \cos\pi + i\sin\pi$$

$$z = (\cos\pi + i\sin\pi)^{1/4}$$

$$= \left(\cos\frac{\pi + n2\pi}{4} + i\sin\frac{\pi + n2\pi}{4}\right) \qquad n = 0, 1, 2, 3$$

$$z_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$z_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$z_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

$$z_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \underline{\underline{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i}}$$

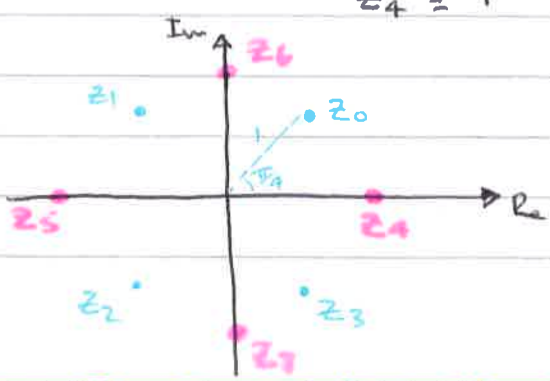
(b) $z^4 - 1 = 0$

$(z^2-1)(z^2+1) = 0$ i.e. $(z-1)(z+1)(z^2+1) = 0$

$z_4 = 1 \qquad z_5 = -1 \qquad z_6 = \pm i$

(let $z_6 = i$ and $z_7 = -i$)

(c)



(d) $z^8 - 1 = 0 \Rightarrow (z^4 - 1)(z^4 + 1) = 0$

has sol$^n$ $z_4 - z_7$ has sol$^n$ $z_0 - z_3$

$\Rightarrow z^8 - 1 = 0$ has sol$^n$ $z_0 - z_7$

$\Rightarrow z_0 \ldots z_7$ are sol$^n$ to $\underline{\underline{z^8 - 1 = 0}}$

(e) $z^6 + z^4 + z^2 + 1 = 0$

$z^8 - 1 = 0$

$\Rightarrow (z^2 - 1)(z^6 + z^4 + z^2 + 1) = 0$

has the same sol$^n$ $z_0 \ldots z_7$

since $z^2 - 1 = 0$ has sol$^n$s $z = 1, z = -1$

$\Rightarrow z^6 + z^4 + z^2 + 1 = 0$ has the rest

i.e. $\underline{\underline{z = i}}$, $\underline{\underline{z = -i}}$, $\underline{\underline{z = \pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i}}$