Advanced Higher

Mathematics

HSNe21S06

Exam Solutions – 2006

Question 1

$$\begin{vmatrix} 2 & x \\ -1 & 3 \end{vmatrix} = 6 + x \qquad \therefore \quad \begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}^{-1} = \frac{1}{6+x}\begin{pmatrix} 3 & -x \\ 1 & 2 \end{pmatrix}$$

Matrix is singular when $x = -6$.

Question 2

(a)
$$\frac{d}{dx}\left(2\tan^{-1}\sqrt{1+x}\right) = \frac{2}{1+1+x} \times \frac{d}{dx}\sqrt{1+x}$$
$$= \frac{\cancel{2}}{2+x} \times \frac{1}{\cancel{2}}(1+x)^{-\frac{1}{2}} \times 1$$
$$= \frac{1}{(2+x)(1+x)^{1/2}}.$$

(b)
$$\frac{d}{dx}\left(\frac{1+\ln x}{3x}\right) = \frac{\frac{1}{x} \times 3x - (1+\ln x) \times 3}{(3x)^2}$$
$$= \frac{3 - 3(1+\ln x)}{9x^2}$$
$$= \frac{\overset{1}{\cancel{3}}(1-1-\ln x)}{\underset{3}{\cancel{9}}x^2}$$
$$= -\frac{\ln x}{3x^2}$$

Question 3

$$\frac{1}{1-i} = \frac{1}{1-i} \times \frac{1+i}{1+i} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

$$\therefore \quad -i + \frac{1}{1-i} = -i + \frac{1}{2} + \frac{1}{2}i = \frac{1}{2} - \frac{1}{2}i$$

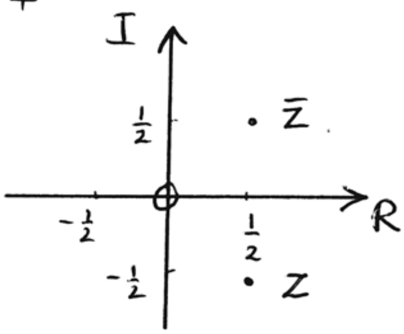
$$\therefore \quad x = \frac{1}{2} \text{ and } y = -\frac{1}{2}$$

$$|z| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\arg(z) = -\tan^{-1} 1 = -\frac{\pi}{4}$$

$$\bar{z} = \frac{1}{2} + \frac{1}{2}i$$

Argand diagram...



Question 4

$$xy - x = 4$$

$$\therefore \quad y + x\frac{dy}{dx} - 1 = 0$$

$$x\frac{dy}{dx} = 1 - y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1-y}{x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1-y}{x}\right) = \frac{-\frac{dy}{dx} \times x - (1-y).1}{x^2}$$

$$= \frac{-x\frac{dy}{dx} - 1 + y}{x^2}$$

$$= \frac{-\cancel{x}\left(\frac{1-y}{\cancel{x}}\right) - 1 + y}{x^2}$$

$$= \frac{-(1-y) - 1 + y}{x^2} = \frac{-1+y-1+y}{x^2}$$

$$= \frac{2(y-1)}{x^2}.$$

Question 5

Let fixed point be λ so as $n \to \infty$ $x_{n+1} = x_n = \lambda$

$$\lambda = \frac{1}{2}\left(\lambda + \frac{2}{\lambda^2}\right)$$

$$\therefore \quad 2\lambda = \frac{\lambda^3 + 2}{\lambda^2}$$

$$2\lambda^3 = \lambda^3 + 2$$

$$\lambda^3 = 2$$

$$\therefore \quad \lambda = \sqrt[3]{2}$$

$\therefore$ fixed point is $\sqrt[3]{2}$.

Question 6

$$\int \frac{12x^3 - 6x}{x^4 - x^2 + 1}\,dx = 3\int \frac{4x^3 - 2x}{x^4 - x^2 + 1}\,dx$$

$$= 3\ln\left|x^4 - x^2 + 1\right| + c.$$

Question 7

(a) $n^3 - n = n(n^2 - 1) = n(n+1)(n-1) = (n-1) \times n \times (n+1)$

Product of three consecutive natural numbers is divisible by 6.

$\therefore$ $n^3 - n$ is always divisible by 6 is true.

(b) When $n = 2$ $\quad n^3 + n + 5 = 2^3 + 2 + 5$

$= 15$ which is not prime

$\therefore$ $n^3 + n + 5$ is always prime is false by this counter-example.

Question 8

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

A.E. $m^2 + 2m + 2 = 0$

$$m^2 + 2m = -2$$

$$(m+1)^2 = -1$$

$$m+1 = \pm i$$

$$m = -1 \pm i$$

C.F. $y = e^{-x}(A\cos x + B\sin x)$

$$\therefore \frac{dy}{dx} = -e^{-x}(A\cos x + B\sin x) + e^{-x}(-A\sin x + B\cos x)$$

When $\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\}$ $0 = e^{-0}(A\cos 0 + B\cancel{\sin 0}^{0})$

$$\therefore A = 0.$$

When $\left.\begin{array}{l} x=0 \\ \frac{dy}{dx}=2 \end{array}\right\}$ $2 = -e^{0}(B\cancel{\sin 0}^{0}) + e^{0}(B\cos 0)$

$$2 = B$$

$\therefore$ Solution $y = 2e^{-x}\sin x$

Question 9

Using Gaussian elimination . . .

$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & -2 & 4 & -1 \end{array}\right)$$

$$\downarrow$$

$$\begin{array}{l} \\ r_2' = 2r_2 - r_1 \\ r_3' = 2r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & -3 & 6 & -3 \end{array}\right)$$

$$\downarrow$$

$$\begin{array}{l} \\ \\ r_3'' = r_2' + r_3' \end{array} \left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = t$

by back substitution

$$3y - 6z = 3$$

$$3y - 6t = 3 \Rightarrow 3y = 3 + 6t$$

$$y = 1 + 2t$$

$$2x - y + 2z = 1$$

$$2x - (1 + 2t) + 2t = 1.$$

$$2x - 1 \cancel{-2t} \cancel{+2t} = 1$$

$$2x = 2$$

$$x = 1.$$

Question 10

$$x = T^3 - 90T^2 + 2400T$$

$$\therefore \frac{dx}{dT} = 3T^2 - 180T + 2400$$

At stationary points $3T^2 - 180T + 2400 = 0$

$$\therefore T^2 - 60T + 800 = 0$$

$$(T-20)(T-40) = 0$$

$$T = 20 \text{ or } T = 40$$

$$\frac{d^2x}{dT^2} = 6T - 180$$

When $T = 20$ $\frac{d^2x}{dT^2} = 120 - 180 = -60 < 0$ $\therefore$ Maximum T. Pt at $T = 20$

When $T = 40$ $\frac{d^2x}{dT^2} = 240 - 180 = 60 > 0$ $\therefore$ Minimum T. Pt at $T = 40$

When $T = 10$ $x = 16\,000$

$T = 20$ $x = 20\,000$

$T = 40$ $x = 16\,000$

$T = 60$ $x = 36\,000$.

$\therefore$ Temperature is 60°C to remove maximum amount of impurity.

Question 11

$$1+\cot^2\theta = 1+\frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$= \operatorname{cosec}^2\theta.$$

Given $y=\cot^{-1}x$ then $x=\cot y$

$$\therefore \frac{dx}{dy} = \frac{d}{dy}(\cot y) \quad \left(\text{or } \frac{d}{dy}(\tan y)^{-1}\right)$$

$$= \frac{d}{dy}\left(\frac{\cos y}{\sin y}\right)$$

$$= \frac{-\sin y.\sin y - \cos y.\cos y}{\sin^2 y}$$

$$= \frac{-\sin^2 y - \cos^2 y}{\sin^2 y}.$$

$$= -\frac{1}{\sin^2 y}.$$

$$= -\operatorname{cosec}^2 y$$

$$= -1-\cot^2 y$$

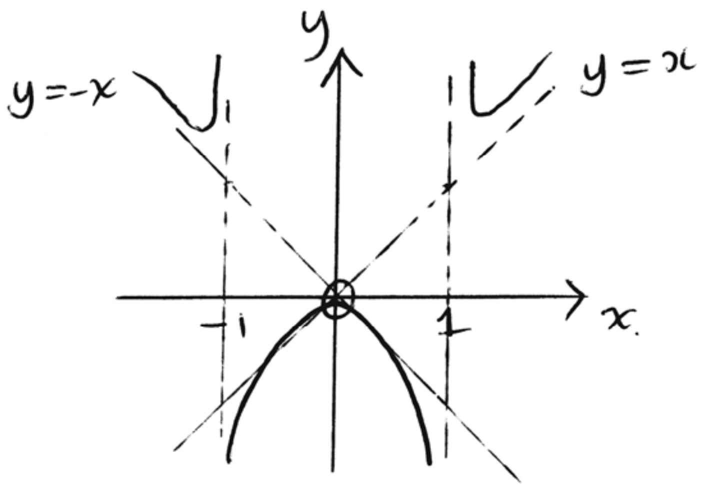
but $y=\cot^{-1}x$

$$= -1-\cot^2(\cot^{-1}x)$$

$$= -1-x^2$$

$$\therefore \frac{dy}{dx} = -\frac{1}{1+x^2}.$$

Question 12



Other asymptotes..

- $x = -1$
- $y = -x$

Question 13

$A^n B = B A^n$ where $AB = BA$

For $n = 1$: $AB = BA$ $\therefore$ Result true for $n = 1$.

Assume true for $n = k$ $\qquad A^k B = BA^k$

For $n = k+1$

$$
\begin{aligned}
A^{k+1} B &= A . A^k B \quad \text{but } A^k B = BA^k \\
&= A . BA^k \quad \text{since } AB = BA \\
&= BAA^k \\
&= BA^{k+1}
\end{aligned}
$$

$\therefore$ Result true for $n = k+1$.

Since result is true for $n = 1$ and $n = k+1$ then by the principle of Mathematical induction it is true for all integers $n \geq 1$.

Question 14

(a)
$$f(x) = x^2 \sin x$$

$$\therefore f(-x) = (-x)^2 \sin(-x) = x^2 \times (-\sin x)$$
$$= -x^2 \sin x$$
$$= -f(x)$$

$\therefore f$ is odd.

(b)
$$\int x^2 \sin x \, dx = -x^2 \cos x - \int -2x \cos x \, dx$$
$$= -x^2 \cos x + 2\int x \cos x \, dx$$
$$= -x^2 \cos x + 2\left(x \sin x - \int \sin x \, dx\right)$$
$$= -x^2 \cos x + 2x \sin x - 2\int \sin x \, dx$$
$$= -x^2 \cos x + 2x \sin x + 2\cos x + c.$$
$$= (2 - x^2)\cos x + 2x \sin x + c.$$

(c) Bounded Area $= 2\int_0^{\pi/4} x^2 \sin x \, dx$

$$= 2\left[(2 - x^2)\cos x + 2x \sin x\right]_0^{\pi/4}$$
$$= 2\left[\left(\left(2 - \frac{\pi^2}{16}\right)\cos\frac{\pi}{4} + 2 \times \frac{\pi}{4} \sin\frac{\pi}{4}\right) - (2\cos 0 + 0)\right]$$
$$= 2\left[\frac{1}{\sqrt{2}}\left(2 - \frac{\pi^2}{16}\right) + 2 \times \frac{\pi}{4} \times \frac{1}{\sqrt{2}} - 2\right]$$
$$= \sqrt{2}\left(2 - \frac{\pi^2}{16}\right) + \frac{\pi}{\sqrt{2}} - 4$$

Question 15

From $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1} = t$ —— $*$

then normal $\underset{\sim}{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\therefore$ Equation of plane is of the form

$$2x + y - z = d$$

Since $P(1,1,0)$ lies on the plane

$$2(1) + 1 - 0 = d \quad \therefore \quad d = 3$$

Eq^n of plane : $2x + y - z = 3.$

From $*$ $\quad x = 2t - 1 \quad y = t + 2 \quad z = -t$

Substitute these into equation of plane

$$2(2t-1) + t + 2 - (-t) = 3$$
$$4t - 2 + t + 2 + t = 3$$
$$6t = 3$$
$$t = \frac{1}{2}$$

$\therefore \quad x = 0 \quad y = \frac{5}{2} \quad z = -\frac{1}{2}$

and Q is $\left(0, \frac{5}{2}, -\frac{1}{2}\right)$.

Shortest distance is $d_{PQ} = \sqrt{1^2 + \left(-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$$= \sqrt{1 + \frac{9}{4} + \frac{1}{4}} = \sqrt{\frac{14}{4}} = \frac{\sqrt{14}}{2} \text{ units}$$

QP is perpendicular to line $\therefore$ shortest distance

Question 16

(a) $r = \frac{u_2}{u_1} = u_2 \div u_1 = \frac{x(x+1)^2}{(x-2)^2} \div \frac{x(x+1)}{(x-2)}$

$$= \frac{\cancel{x}(x+1)^{\cancel{2}}}{(x-2)^{\cancel{2}}} \times \frac{(\cancel{x-2})}{\cancel{x}(\cancel{x+1})}$$

$$= \frac{x+1}{x-2}$$

nth term of sequence $\frac{x(x+1)^n}{(x-2)^n}$

Pattern from given sequence

(b) $S_n = \frac{a(1-r^n)}{1-r}$ where $1-r^n = 1-\left(\frac{x+1}{x-2}\right)^n$

$$= \frac{(x-2)^n-(x+1)^n}{(x-2)^n}$$

$$1-r = 1-\frac{x+1}{x-2} = -\frac{3}{x-2}$$

$$\therefore\ S_n = a \times (1-r^n) \times \frac{1}{1-r}$$

$$= \frac{x(x+1)}{\cancel{x-2}} \times \frac{(x-2)^n-(x+1)^n}{(x-2)^n} \times \left(-\frac{\cancel{x-2}}{3}\right)$$

$$= -\frac{1}{3} \cdot \frac{x(x+1)\left[(x-2)^n-(x+1)^n\right]}{(x-2)^n}$$

(c) S_∞ exists if $-1 < r < 1$. ie $-1 < \frac{x+1}{x-2} < 1$

Left hand inequality

$$-1 < \frac{x+1}{x-2}$$

$$\therefore \frac{x+1}{x-2} + 1 > 0$$

$$\frac{2x-1}{x-2} > 0$$

either $2x-1 > 0 \Rightarrow x > \frac{1}{2}$

when $x-2 > 0 \Rightarrow x > 2$

Impossible since $x < 2$

or $2x-1 < 0 \Rightarrow x < \frac{1}{2}$

when $x-2 < 0 \Rightarrow x < 2$.

ie. $x < \frac{1}{2}$.

Right hand inequality

$$\frac{x+1}{x-2} < 1$$

$$\therefore \frac{x+1}{x-2} - 1 < 0$$

$$\frac{3}{x-2} < 0$$

When $x-2 < 0$

$x < 2$.

$\therefore$ Range of values is $x < \frac{1}{2}$.

and $S_\infty = \frac{a}{1-r} = \frac{x(x+1)}{\cancel{x-2}} \times \left(-\frac{\cancel{(x-2)}}{3}\right)$

$$= -\frac{1}{3}x(x+1)$$

Question 17

(a)
$$\int \sin^2 x \cos^2 x \, dx = \int \cos^2 x (1 - \cos^2 x) \, dx$$
$$= \int \cos^2 x \, dx - \int \cos^4 x \, dx$$

(b)
$$\int_0^{\pi/4} \cos^4 x \, dx = \int_0^{\pi/4} \cos x \cos^3 x \, dx$$
$$= \sin x \cos^3 x \Big|_0^{\pi/4} - \int_0^{\pi/4} \sin x . 3\cos^2 x . (-\sin x) \, dx$$
$$= \sin x \cos^3 x \Big|_0^{\pi/4} + 3 \int \sin^2 x \cos^2 x \, dx$$
$$= \left(\sin \tfrac{\pi}{4} \left(\cos \tfrac{\pi}{4} \right)^3 - 0 \right) + 3 \int \sin^2 x \cos^2 x \, dx$$
$$= \left(\frac{1}{\sqrt{2}} \right)^4 + 3 \int \sin^2 x \cos^2 x \, dx$$
$$= \frac{1}{4} + 3 \int \sin^2 x \cos^2 x \, dx.$$

(c)
$$\int_0^{\pi/4} \cos^2 x \, dx = \int_0^{\pi/4} \tfrac{1}{2} \cos 2x + \tfrac{1}{2} \, dx$$
$$= \left[\frac{1}{4} \sin 2x + \frac{1}{2} x \right]_0^{\pi/4}$$
$$= \frac{1}{4} \sin \frac{\pi}{2} + \frac{\pi}{8} - 0 = \frac{1}{4} + \frac{\pi}{8}$$
$$= \frac{\pi + 2}{8}.$$

(d)

$$3\int_0^{\pi/4} \sin^2 x \cos^2 x \, dx = 3\int_0^{\pi/4} \cos^2 x \, dx - 3\int \cos^4 x \, dx$$

$$\therefore \int_0^{\pi/4} \cos^4 x \, dx - \frac{1}{4} = 3\int_0^{\pi/4} \cos^2 x \, dx - 3\int \cos^4 x \, dx$$

$$\therefore \quad 4\int_0^{\pi/4} \cos^4 x \, dx = 3\int_0^{\pi/4} \cos^2 x \, dx + \frac{1}{4}$$

$$= 3\left(\frac{\pi + 2}{8}\right) + \frac{1}{4}$$

So

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{3(\pi+2)}{32} + \frac{1}{16}$$

$$= \frac{3\pi + 6 + 2}{32}$$

$$= \frac{3\pi + 8}{32}.$$