Advanced Higher

Mathematics

HSNe21S05

Exam Solutions – 2005

Question 1

(a) $f(x) = x^3 \tan 2x \Rightarrow f'(x) = 3x^2 \tan 2x + x^3 . 2\sec^2 2x$

$= 3x^2 \tan 2x + 2x^3 \sec^2 2x$

$= x^2(3\tan 2x + 2x\sec^2 2x)$

(b) $y = \dfrac{1+x^2}{1+x} \Rightarrow \dfrac{dy}{dx} = \dfrac{2x.(1+x) - (1+x^2).1}{(1+x)^2}$

$= \dfrac{2x + 2x^2 - 1 - x^2}{(1+x)^2}$

$= \dfrac{x^2 + 2x - 1}{(1+x)^2}.$

Question 2

$2x^2 - 2x^2 - 4x + x^2 = 0.$

$\therefore \quad x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0$ or $x = 4$

Question 3

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$$

$$\therefore\ e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + \ldots$$

$$= 1 + x^2 + \frac{x^4}{2} + \ldots$$

So $e^{x^2+x} = e^{x^2}.e^x$

$$= \left(1 + x^2 + \frac{x^4}{2} + \ldots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + x^2 + x^3 + \frac{x^4}{2} + \frac{x^4}{2} + \ldots$$

$$= 1 + x + \frac{3x^2}{2} + \frac{7x^3}{6} + \frac{25x^4}{24} + \ldots$$

Question 4

$$S_n = 8n - n^2$$

$\therefore \quad u_1 = S_1 = 8 - 1 = 7$

$u_2 = S_2 - S_1 = S_2 - u_1 = 16 - 4 - 7 = 5$

$u_3 = S_3 - S_2 = 24 - 9 - 12 = 3.$

$u_2 - u_1 = u_3 - u_2 = 2 \Rightarrow$ Arithmetic Sequence

Method 1 From standard grade 7 5 3

$\therefore \quad u_n = -2n + 9$ or $9 - 2n$

Method 2 $u_n = a + (n-1)d$ where $a = 7$ $d = -2$

$= 7 + (n-1) \times -2$

$= 7 - 2n + 2$

$= 9 - 2n.$

Method 3 From lead-in $S_n - S_{n-1} = u_n$

$\therefore \quad u_n = 8n - n^2 - [8(n-1) - (n-1)^2]$

$= 8n - n^2 - (8n - 8 - n^2 + 2n - 1)$

$= \cancel{8n} - \cancel{n^2} - \cancel{8n} + 8 + \cancel{n^2} - 2n + 1$

$= 9 - 2n.$

Question 5

$$\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$$

Let $u = 1+x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

$x = u - 1$

When $x = 0$ $u = 1+0 = 1$

$x = 3$ $u = 1+3 = 4$

$$= \int_1^4 \frac{u-1}{\sqrt{u}}\,du$$

$$= \int_1^4 u^{1/2} - u^{-1/2}\,du = \left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^4$$

$$= \left(\frac{2}{3}4^{3/2} - 2.4^{1/2}\right) - \left(\frac{2}{3}.1^{3/2} - 2.1^{1/2}\right)$$

$$= \left(\frac{2}{3}\times 8 - 4\right) - \left(\frac{2}{3} - 2\right)$$

$$= \frac{16}{3} - \frac{2}{3} - 4 + 2$$

$$= \frac{14}{3} - 2$$

$$= \frac{8}{3}$$

Question 6

Using Gaussian elimination...

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & \lambda & 1 & 0 \\ 3 & 3 & 9 & 5 \end{array}\right)$$

$$\downarrow$$

$$\begin{array}{r} \\ r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 3r_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & \lambda-2 & -3 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right)$$

Using back substitution

$$3z = 2 \Rightarrow z = \frac{2}{3}.$$

and

$$(\lambda-2)y - 3z = -2$$
$$(\lambda-2)y - 2 = -2$$
$$(\lambda-2)y = 0$$
$$y = 0$$

and

$$x + y + 2z = 1$$
$$x + 0 + \frac{4}{3} = 1 \Rightarrow x = -\frac{1}{3}$$

When $\lambda = 2$ rows 2 and 3 are equivalent...

$$\begin{array}{lcccc} r_2' & 0 & 0 & -3 & -2 \\ r_3' & 0 & 0 & 3 & 2 \end{array}$$

$\therefore$ There are an infinite number of solutions

Question 7

$$A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ 1 & 2 & 5 \end{pmatrix}$$

$$\therefore A^2 + A = \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ 1 & 2 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2I$$

so $k = 2$

Consequently ...

$$A^2 + A = A(A+I) = 2I$$

$$\therefore \tfrac{1}{2}A(A+I) = I$$

$$A.\left(\tfrac{1}{2}A + \tfrac{1}{2}I\right) = I$$

$$\therefore A^{-1} = \tfrac{1}{2}A + \tfrac{1}{2}I$$

so $p = \frac{1}{2}$ and $q = \frac{1}{2}$

Question 8

Method 1. Let $z = t$

$\therefore \quad x - 4y + 2z = 1 \quad \Rightarrow \quad x - 4y = 1 - 2t \quad — ①$

$x - y - z = -5 \quad \Rightarrow \quad x - y = t - 5 \quad — ②$

$② - ① \qquad 3y = 3t - 6$

$y = t - 2$

and $\quad x - 4(t-2) = 1 - 2t$

$x = 1 - 2t + 4t - 8$

$= 2t - 7$

$\therefore$ Parametric equations of line are

$x = 2t - 7$

$y = t - 2$

$z = t.$

Method 2. $\pi_1 : x - 4y + 2z = 1 \Rightarrow \underset{\sim}{n}_1 = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$

$\pi_2 : x - y - z = -5 \Rightarrow \underset{\sim}{n}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$$\therefore \underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -4 & 2 \\ 1 & -1 & -1 \end{vmatrix} = \underset{\sim}{i} \begin{vmatrix} -4 & 2 \\ -1 & -1 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} 1 & -4 \\ 1 & -1 \end{vmatrix}$$

$$= 6\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$$

$\therefore$ Direction of line is $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Let $z = 0$ $\quad \left.\begin{aligned} x - 4y &= 1 \\ x - y &= -5 \end{aligned}\right\}$ $\quad 3y = -6 \Rightarrow y = -2$

$x = -7$

Point on line is $(-7, -2, 0)$ with direction $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ Parametric equations are

$$x = 2t - 7$$

$$y = t - 2$$

$$z = t.$$

Substituting these expressions into

$$x + 2y - 4z$$

$$= 2t - 7 + 2(t-2) - 4t$$

$$= 2t - 7 + 2t - 4 - 4t$$

$$= -11$$

$\therefore$ line lies in the plane with equation

$$x + 2y - 4z = -11$$

Question 9

Let $z = a + ib \Rightarrow \bar{z} = a - ib$

$\therefore \quad z + 2i\bar{z} = (a+ib) + 2i(a-ib)$

$= a + ib + 2ai - 2bi^2 \quad (i^2 = -1)$

$= a + 2b + (2a+b)i$

$= 8 + 7i$

$\therefore$ Equating coefficients

$a + 2b = 8$ — ①

$2a + b = 7$ — ②

② ×2 $\quad 4a + 2b = 14$ — ③

③ − ① $\quad 3a = 6$

$a = 2$

Substitute $a = 2$ into eq^{n} ①

$2 + 2b = 8$

$2b = 6$

$b = 3$

$\therefore \quad z = 2 + 3i$

Question 10

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

When $\underline{n=1}$ $\quad$ L.H.S. $= \frac{1}{1\times 2\times 3} = \frac{1}{6}$.

R.H.S. $= \frac{1}{4} - \frac{1}{2\times 2\times 3} = \frac{1}{4} - \frac{1}{12}$

$= \frac{3-1}{12} = \frac{2}{12} = \frac{1}{6}$

LHS = RHS $\therefore$ True for $n=1$

<u>Assume true for $n=k$</u>

$$\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$$

<u>For $n=k+1$</u>

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \left[\frac{k+3-2}{2(k+1)(k+2)(k+3)} \right]$$

$$= \frac{1}{4} - \frac{\cancel{k+1}}{2\cancel{(k+1)}(k+2)(k+3)} = \frac{1}{4} - \frac{1}{2(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{1}{2((k+1)+1)((k+1)+2)}$$

$\therefore$ True for $n=k+1$

Since true for $n=1$ and $n=k+1$ then by Mathematical induction true for all $n\geq 1$.

$$\lim_{n\to\infty} \sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \lim_{n\to\infty} \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

$$= \frac{1}{4}$$

Question 11

(a) $y = \dfrac{x^3}{x-2}$ Vertical Asymptote $x = 2$

(b) $y = \dfrac{x^3}{x-2} = x^2 + 2x + 4 + \dfrac{8}{x-2}$

$$\begin{array}{r} x^2+2x+4 \\ x-2\,\big)\overline{x^3 \qquad\quad} \\ \underline{x^3-2x^2} \\ 2x^2 \\ \underline{2x^2-4x} \\ 4x \\ \underline{4x-8} \\ 8 \end{array}$$

$\therefore \dfrac{dy}{dx} = 2x + 2 - 8(x-2)^{-2}$

$= 2x + 2 - \dfrac{8}{(x-2)^2} = 0 = \dfrac{(2x+2)(x-2)^2 - 8}{(x-2)^2}$

or. $y = \dfrac{x^3}{x-2} \Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2(x-2) - x^3.1}{(x-2)^2}$

$= \dfrac{3x^3 - 6x^2 - x^3}{(x-2)^2}$

$= \dfrac{2x^3 - 6x^2}{(x-2)^2}$

$= \dfrac{2x^2(x-3)}{(x-2)^2} = 0$ at st. pts.

$\therefore$ $x = 0$ or $x = 3$

When $x = 0$ $y = 0$ When $x = 3$ $y = \dfrac{27}{+1} = +27$

Stationary pts at $(0, 0)$ and $(3, +27)$

(c) $y = \left|\dfrac{x^3}{x-2}\right| + 1$ has stationary points $(0, 1)$ and $(3, 28)$

Question 12

(a) $z^4 = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta.$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$

(b) $z^4 = (\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

(c) $\cos 4\theta = \mathrm{Re}(z^4)$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$$\therefore \quad \frac{\cos 4\theta}{\cos^2\theta} = \frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{\cos^2\theta}$$

$$= \cos^2\theta - 6\sin^2\theta + \frac{\sin^4\theta}{\cos^2\theta}$$

$$= \cos^2\theta - 6(1-\cos^2\theta) + \frac{\sin^4\theta}{\cos^2\theta}$$

$$= \cos^2\theta - 6 + 6\cos^2\theta + \frac{\sin^4\theta}{\cos^2\theta}$$

but $\sin^4\theta = (\sin^2\theta)^2 = (1-\cos^2\theta)^2$

$= 1 - 2\cos^2\theta + \cos^4\theta$

$$\therefore \frac{\sin^4\theta}{\cos^2\theta} = \frac{1 - 2\cos^2\theta + \cos^4\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} - 2 + \cos^2\theta$$

$$= \sec^2\theta - 2 + \cos^2\theta$$

$$\therefore \frac{\cos 4\theta}{\cos^2\theta} = 7\cos^2\theta - 6 + \sec^2\theta - 2 + \cos^2\theta$$

$$= 8\cos^2\theta + \sec^2\theta - 8$$

$\therefore p = 8 \quad q = 1 \quad r = -8$

Question 13

$$\frac{1}{x^3+x}=\frac{1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$\therefore\ A(x^2+1)+(Bx+C)x=1$$

Let $x=0$ $\quad A=1$

Let $x=1$ $\quad 2+B+C=1 \Rightarrow B+C=-1$

Let $x=-1$ $\quad 2+B-C=1 \Rightarrow B-C=-1$

$$2B=-2$$
$$B=-1$$
$$\therefore\ C=0$$

$$\therefore\ \frac{1}{x^3+x}=\frac{1}{x}-\frac{x}{x^2+1}$$

$$I(k)=\int_1^k \frac{1}{x^3+x}\,dx=\int_1^k \frac{1}{x}-\frac{x}{x^2+1}\,dx$$

$$=\left[\ln x-\tfrac{1}{2}\ln|x^2+1|\right]_1^k$$

$$=\left[\ln k-\tfrac{1}{2}\ln(k^2+1)\right]-\left[\cancel{\ln 1}^{\,=0}-\tfrac{1}{2}\ln 2\right]$$

$$=\ln k-\tfrac{1}{2}\ln(k^2+1)+\tfrac{1}{2}\ln 2$$

$$=\ln k-\tfrac{1}{2}\ln(k^2+1)+\ln\sqrt{2}$$

$$=\ln k\sqrt{2}-\ln\sqrt{k^2+1}$$

$$=\ln\left(\frac{k\sqrt{2}}{\sqrt{k^2+1}}\right).$$

$$I(k) = \ln\left(\frac{k\sqrt{2}}{\sqrt{k^2+1}}\right)$$

$$\therefore e^{I(k)} = \frac{k\sqrt{2}}{\sqrt{k^2+1}}$$

$$\lim_{k\to\infty} e^{I(k)} = \lim_{k\to\infty} \frac{k\sqrt{2}}{\sqrt{k^2+1}} \longrightarrow \frac{k}{k}\sqrt{2} \longrightarrow \sqrt{2}$$

Question 14

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin x$$

A.E. $m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 2$ or $m = 1$

$\therefore$ C.F. $y = Ae^{2x} + Be^{x}$.

P.I Try $y = a\sin x + b\cos x$

$\therefore \frac{dy}{dx} = a\cos x - b\sin x$.

and $\frac{d^2y}{dx^2} = -a\sin x - b\cos x$

$\therefore -a\sin x - b\cos x - 3a\cos x + 3b\sin x + 2a\sin x + 2b\cos x$
$= 20\sin x$

so $(a+3b)\sin x + (-3a+b)\cos x = 20\sin x$

equating coefficients

$a + 3b = 20$

$-3a + b = 0$

1 $\times 3$ $3a + 9b = 60$

$10b = 60 \Rightarrow b = 6$

$\therefore a + 18 = 20 \Rightarrow a = 2$

G.S. $y = Ae^{2x} + Be^{x} + 2\sin x + 6\cos x$.

Given $y = Ae^{2x} + Be^{x} + 2\sin x + 6\cos x$.

$\left.\begin{array}{l} y=0 \\ x=0 \end{array}\right\} \quad A + B + 6 = 0 \Rightarrow A + B = -6, \quad \text{—— } ①$

$$\frac{dy}{dx} = 2Ae^{2x} + Be^{x} + 2\cos x - 6\sin x$$

$\left.\begin{array}{l} \frac{dy}{dx}=0 \\ x=0 \end{array}\right\} \quad 2A + B + 2 = 0 \Rightarrow 2A + B = -2 \quad \text{—— } ②$

$② - ① \qquad A = 4.$

$$4 + B = -6 \Rightarrow B = -10$$

$\therefore$ Particular solution is

$$y = 4e^{2x} - 10e^{x} + 2\sin x + 6\cos x$$

Question 15

(a) $f(x) = \sqrt{\sin x} = (\sin x)^{1/2}$

$\Rightarrow f'(x) = \frac{1}{2}(\sin x)^{-1/2} \times \cos x = \dfrac{\cos x}{2\sqrt{\sin x}}$

(b)

$$f(x) = \sqrt{g(x)} = [g(x)]^{1/2}$$

$$\Rightarrow f'(x) = \tfrac{1}{2}[g(x)]^{-1/2} \times g'(x)$$

$$= \frac{g'(x)}{2\sqrt{g(x)}} \quad \therefore\ k = 2$$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\int \frac{-2x}{2\sqrt{1-x^2}}\,dx$$

$g(x) = 1 - x^2$
$g'(x) = -2x.$

$$= -\sqrt{1-x^2} + C$$

(c)

$$\int_0^{1/2} \sin^{-1}x\,dx = \int_0^{1/2} 1.\sin^{-1}x\,dx$$

use above result

$$= x\sin^{-1}x\Big|_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1}x\Big|_0^{1/2} + \sqrt{1-x^2}\Big|_0^{1/2}$$

$$= \left(\tfrac{1}{2}\sin^{-1}\tfrac{1}{2} - 0\right) + \left(\sqrt{1-\tfrac{1}{4}} - \sqrt{1}\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \sqrt{\frac{3}{4}} - 1$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \quad (\approx 0{\cdot}128)$$