Advanced Higher

Mathematics

HSNe21T03

Exam Solutions – 2003 (Typed)

Question A1

(a) $f(x) = x(1+x)^{10}$

Using the product rule…

$$\begin{aligned} f'(x) &= 1.(1+x)^{10} + x.10(1+x)^9 \\ &= (1+x)^{10} + 10x(1+x)^9 \\ &= (1+x)^9(1+x+10x) \\ &= (1+x)^9(1+11x) \end{aligned}$$

(b) $y = 3^x$

$\ln y = \ln 3^x$

$\ln y = x \ln 3$

Therefore $\dfrac{1}{y}\dfrac{dy}{dx} = 1.\ln 3$

$$\begin{aligned} \frac{dy}{dx} &= y \ln 3 \\ &= 3^x \ln 3 \end{aligned}$$

Question A2

Method 1

$$\begin{aligned} S_n &= \sum_{k=1}^{n} u_k \\ &= \sum_{k=1}^{n}(11-2k) \\ &= \sum_{k=1}^{n} 11 - 2\sum_{k=1}^{n} k \\ &= 11n - 2\left(\tfrac{1}{2}n(n+1)\right) \\ &= 11n - n(n+1) \\ &= 11n - n^2 - n \\ &= 10n - n^2 \end{aligned}$$

Method 2

From $u_k = 11-2k$ …

$$\begin{aligned} u_1 &= 11-2 = 9 \\ u_2 &= 11-4 = 7 \\ u_3 &= 11-6 = 5 \end{aligned}$$

Hence an arithmetic series with $a = u_1 = 9$ and $d = u_2 - u_1 = -2$.

$$\begin{aligned} \therefore\ S_n &= \tfrac{n}{2}[2a + (n-1)d] \\ &= \tfrac{n}{2}[18 + (n-1)(-2)] \\ &= \tfrac{n}{2}(18 - 2n + 2) \\ &= \tfrac{n}{2}(20 - 2n) \\ &= n(10-n) \\ &= 10n - n^2 \end{aligned}$$

Therefore when $S_n = 21$:

$$\begin{aligned} 10n - n^2 &= 21 \\ n^2 - 10n + 21 &= 0 \\ (n-7)(n-3) &= 0 \end{aligned}$$

So $n = 7$ or $n = 3$

Question A3

$$y^3 + 3xy = 3x^2 - 5$$

$$\tfrac{d}{dx}\left(y^3 + 3xy\right) = \tfrac{d}{dx}\left(3x^2 - 5\right)$$

$$3y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 6x$$

$$\left(3y^2 + 3x\right)\frac{dy}{dx} = 6x - 3y$$

So $\dfrac{dy}{dx} = \dfrac{6x-3y}{3y^2+3x} = \dfrac{3\left(2x-y\right)}{3\left(y^2+x\right)} = \dfrac{2x-y}{y^2+x}$

Therefore at $\mathrm{A}(2,1)$, $m_{\text{tangent}} = \dfrac{2(2)-1}{1^2+2} = \dfrac{3}{3} = 1$

So the equation for the tangent at A is:

$$y - 1 = 1(x-2)$$

$$y - 1 = x - 2$$

$$x - y - 1 = 0$$

Question A4

If $z = x + iy$ then $x + i = x + iy + i = x + \left(y+1\right)i$.

So $|z+i| = \left|x + \left(y+1\right)i\right|$

$$= \sqrt{x^2 + \left(y+1\right)^2}$$

Since $|z+i| = 2$ then $\sqrt{x^2 + \left(y+1\right)^2} = 2$

So $x^2 + \left(y+1\right)^2 = 4$, which is the equation of the circle centre $(0,-1)$ and radius 2 units.

Question A5

Let $x = 1 + \sin\theta$, then $\frac{dx}{d\theta} = \cos\theta$, i.e. $d\theta = \frac{dx}{\cos\theta}$.

Now $\theta = 0 \;\Rightarrow\; x = 1 + \sin 0 = 1$, and $\theta = \frac{\pi}{2} \;\Rightarrow\; x = 1 + \sin\frac{\pi}{2} = 2$. So…

$$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)^3}\,d\theta$$

$$= \int_1^2 \frac{\cos\theta}{x^3}\frac{dx}{\cos\theta}$$

$$= \int_1^2 \frac{1}{x^3}\,dx$$

$$= \left[\frac{x^{-2}}{-2}\right]_1^2 = -\tfrac{1}{2}\left[\frac{1}{x^2}\right]_1^2$$

$$= -\tfrac{1}{2}\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = -\tfrac{1}{2}\left(\tfrac{1}{4} - 1\right) - \tfrac{1}{2}\left(-\tfrac{3}{4}\right) = \tfrac{3}{8}$$

Question A6

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 3 & a & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{l} r_1' = r_1 \\ r_2' = r_2 - 3r_1 \\ r_3' = r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & a-3 & -8 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

Using back substitution…

$$-2z = -2$$
$$z = 1$$

So $(a-3)y - 8z = -2$

$$(a-3)y - 8 = -2$$
$$y = \frac{6}{a-3}$$

And $x + y + 3z = 1$

$$x + \frac{6}{a-3} + 3 = 1$$
$$x = 1 - 3 - \frac{6}{a-3}$$
$$x = -2 - \frac{6}{a-3}$$

When $a = 3$ there is an inconsistency since $r_2' \Rightarrow z = \frac{1}{4}$ and $r_3' \Rightarrow z = 1$

Question A7

$y=\dfrac{x}{1+x^2}$

Using the quotient rule…

$$\frac{dy}{dx}=\frac{1.(1+x^2)-x.2x}{(1+x^2)^2}=\frac{1+x^2-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}$$

Stationary points exist where $\dfrac{dy}{dx}=0$, i.e. where $\dfrac{1-x^2}{(1+x^2)^2}=0$, i.e. $1-x^2=0$

Method 1

$$1-x^2=0$$
$$x^2=1$$
$$x=\pm 1$$

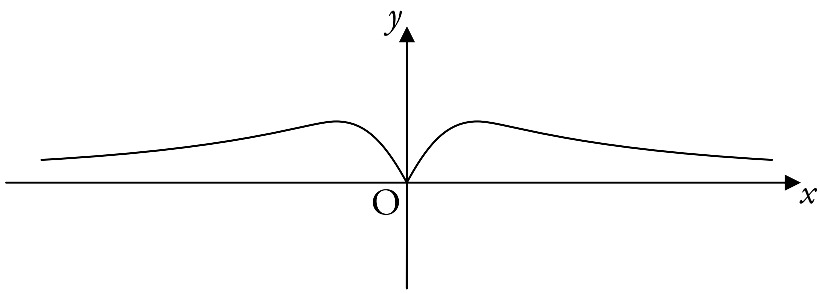
Method 2

$$1-x^2=0$$
$$(x+1)(x-1)=0$$
$$x=\pm 1$$

When $x=1$, $y=\dfrac{1}{1+1^2}=\frac{1}{2}$. And when $x=-1$, $y=\dfrac{-1}{1+(-1)^2}=-\frac{1}{2}$.

Therefore there are stationary points at $\left(1,\frac{1}{2}\right)$ and $\left(-1,-\frac{1}{2}\right)$.

The graph of $y=\left|\dfrac{x}{1+x^2}\right|$ is shown below:



The critical points are the maximum turning points at $\left(1,\frac{1}{2}\right)$ and $\left(-1,\frac{1}{2}\right)$, and a cusp at the origin.

Question A8

Given that $p(n)=n(n+1)$…

Statement A: True

If n is odd, then $(n+1)$ is even, alternatively if n is even then $(n+1)$ is odd.

Odd $\times$ even is always even.

Statement B: False

Counterexample: If $n=1$ then $p(1)=1\times 2=2$ which is not a multiple of 3.

Question A9

Given $\omega = \cos\theta + i\sin\theta$, then $\dfrac{1}{\omega} = \dfrac{1}{\cos\theta + i\sin\theta}$.

Method 1

$$\begin{aligned}\frac{1}{\cos\theta + i\sin\theta} &= (\cos\theta + i\sin\theta)^{-1} \\ &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - \sin\theta\end{aligned}$$

Method 2

$$\begin{aligned}\frac{1}{\cos\theta + i\sin\theta} &= \frac{1}{\cos\theta + i\sin\theta}\,\frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \\ &= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + i^2\sin^2\theta} \\ &= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} \\ &= \cos\theta - i\sin\theta\end{aligned}$$

Therefore $\omega^k = (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

$$\omega^{-k} = (\cos\theta + i\sin\theta)^{-k} = \cos k\theta - i\sin k\theta$$

So $\omega^k + \omega^{-k} = \cos k\theta + i\sin k\theta + \cos k\theta - i\sin k\theta$

$$= 2\cos k\theta$$

Now $(\omega + \omega^{-1})^4 = \omega^4 + 4\omega^3\omega^{-1} + 6\omega^2\omega^{-2} + 4\omega\omega^{-3} + \omega^{-4}$

$$\begin{aligned}&= \omega^4 + 4\omega^2 + 6 + 4\omega^{-2} + \omega^{-4} \\ &= (\omega^4 + \omega^{-4}) + 4(\omega^2 + \omega^{-2}) + 6\end{aligned}$$

Also $(\omega + \omega^{-1})^4 = (2\cos\theta)^4 = 16\cos^4\theta$, and so:

$$\begin{aligned}16\cos^4\theta &= (\omega^4 + \omega^{-4}) + 4(\omega^2 + \omega^{-2}) + 6 \\ &= 2\cos 4\theta + 4(2\cos 2\theta) + 6 \\ &= 2\cos 4\theta + 8\cos 2\theta + 6 \\ \cos^4\theta &= \tfrac{2}{16}\cos 4\theta + \tfrac{8}{16}\cos 2\theta + \tfrac{6}{16} \\ &= \tfrac{1}{8}\cos 4\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}\end{aligned}$$

Question A10

(a) From $I_n = \int_0^1 x^n e^{-x}\,dx$, then $I_1 = \int_0^1 xe^{-x}\,dx$ and using integration by parts…

$$\int_0^1 xe^{-x}\,dx = \left[x.-e^{-x}\right]_0^1 - \int_0^1 1.-e^{-x}\,dx$$
$$= -\left[xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx$$
$$= -\left(1.e^{-1}\right) - \left[e^{-x}\right]_0^1$$
$$= -e^{-1} - \left(e^{-1} - e^{0}\right)$$
$$= 1 - 2e^{-1} = 1 - \tfrac{2}{e}$$

(b) $I_n = \int_0^1 x^n e^{-x}\,dx$

$$= \left[x^n.\left(-e^{-x}\right)\right]_0^1 - \int_0^1 nx^{n-1}.\left(-e^{-x}\right)dx$$
$$= -\left[x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1}e^{-x}\,dx$$
$$= -e^{-1} + nI_{n-1}$$
$$= nI_{n-1} - e^{-1}$$

(c) Using the results from (a) and (b)…

$$I_3 = 3I_2 - e^{-1} \text{ where } I_2 = 2I_1 - e^{-1}$$

So $I_3 = 3\left(2I_1 - e^{-1}\right) - e^{-1}$

$$= 6I_1 - 4e^{-1}$$
$$= 6\left(1 - \tfrac{2}{e}\right) - \tfrac{4}{e}$$
$$= 6 - \tfrac{12}{e} - \tfrac{4}{e}$$
$$= 6 - \tfrac{16}{e}$$

Question A11

$$\frac{dV}{dt} = V(10-V)$$

$$\int \frac{dV}{V(10-V)} = \int dt$$

$$\int \frac{\frac{1}{10}}{V} + \frac{\frac{1}{10}}{10-V}\, dV = \int dt$$

$$\tfrac{1}{10}\int \frac{1}{V} + \frac{1}{10-V}\, dV = \int dt$$

$$\tfrac{1}{10}\left(\ln V - \ln(10-V)\right) = t + c$$

$$\tfrac{1}{10}\ln V - \tfrac{1}{10}\ln(10-V) = t + c$$

And so $\ln V - \ln(10-V) = 10(t+c)$

$$\ln\left(\frac{V}{10-V}\right) = 10(t+c)$$

If $V(0) = 5$ then $\ln\left(\frac{5}{10-5}\right) = 10(0+c)$

$$\ln 1 = 10c$$

$$c = 0$$

And so…

$$\ln\left(\frac{V}{10-V}\right) = 10t$$

$$\frac{V}{10-V} = e^{10t}$$

$$V = e^{10t}(10-V)$$

$$V = 10e^{10t} - Ve^{10t}$$

$$V\left(1+e^{10t}\right) = 10e^{10t}$$

$$V = \frac{10e^{10t}}{1+e^{10t}}$$

$$V = \frac{10}{\frac{1}{e^{10t}}+1}$$

As $t \to \infty$, $e^{10t} \to \infty$, so $\frac{1}{e^{10t}} \to 0 \;\therefore\; V \to \frac{10}{1} = 10$

Question B1

From $\frac{x-3}{4}=\frac{y-2}{-1}=\frac{z+1}{2}=t$, we get the parametric equations…

$$\begin{aligned} x-3&=4t \quad \text{i.e.} \quad x=4t+3 \\ y-2&=-t \quad \text{i.e.} \quad y=2-t \\ z+1&=2t \quad \text{i.e.} \quad z=2t-1 \end{aligned}$$

Substituting these into the equation of the plane…

$$\begin{aligned} 2x+y-z&=4 \\ 2(4t+3)+2-t-(2t-1)&=4 \\ 8t+6+2-t-2t+1&=4 \\ 5t+9&=4 \\ 5t&=-5 \\ t&=-1 \end{aligned}$$

When $t=-1$, $x=-4+3=-1$

$y=2-(-1)=3$

$z=-2-1=-3$

And so the point of intersection is $(-1, 3, -3)$.

Question B2

Given $A^2=4A-3I$, then…

Method 1

$$\begin{aligned} A^4&=\left(A^2\right)^2 \\ &=(4A-3I)^2 \\ &=16A^2-12AI-12IA+9I^2 \\ &=16A^2-24A+9I \\ &=16(4A-3I)-24A+9I \\ &=64A-48I-24A+9I \\ &=40A-39I \end{aligned}$$

Method 2

$$\begin{aligned} A^3&=AA^2 \\ &=A(4A-3I) \\ &=4A^2-3AI \\ &=4(4A-3I)-3A \\ &=16A-12I-3A \\ &=13A-12I \\ A^4&=AA^3 \\ &=A(13A-12I) \\ &=13A^2-12AI \\ &=13(4A-3I)-12A \\ &=52A-39I-12A \\ &=40A-39I \end{aligned}$$

Hence $p=40$ and $q=-39$.

Question B3

Given $x_{n+1} = \frac{1}{2}\left\{x_n + \frac{7}{x_n}\right\}$, at fixed points $x_{n+1} = x_n = \lambda$. So…

$$\begin{aligned}
\lambda &= \tfrac{1}{2}\left\{\lambda + \tfrac{7}{\lambda}\right\} \\
2\lambda &= \lambda + \tfrac{7}{\lambda} \\
2\lambda^2 - \lambda^2 &= 7 \\
\lambda^2 &= 7 \\
\lambda &= \pm\sqrt{7}
\end{aligned}$$

Therefore the fixed points are $\sqrt{7}$ and $-\sqrt{7}$.

Question B4

$$\begin{aligned}
f(x) &= \sin^2 x = (\sin x)^2 & \Rightarrow \quad & f(0) = 0 \\
f'(x) &= 2\sin x\cos x = \sin 2x & \Rightarrow \quad & f'(0) = 0 \\
f''(x) &= 2\cos 2x & \Rightarrow \quad & f''(0) = 2 \\
f'''(x) &= -4\sin 2x & \Rightarrow \quad & f'''(0) = 0 \\
f^{(4)}(x) &= -8\cos 2x & \Rightarrow \quad & f^{(4)}(0) = -8
\end{aligned}$$

Using the Maclaurin series expansion…

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots$$

We have $\sin^2 x = 0 + x.0 + \dfrac{x^2}{2!}.2 + \dfrac{x^3}{3!}.0 + \dfrac{x^4}{4!}.(-8)$

$$\begin{aligned}
&= x^2 - \tfrac{8}{24}x^4 \\
&= x^2 - \tfrac{1}{3}x^4
\end{aligned}$$

Since $\sin^2 x + \cos^2 x = 1$

$$\begin{aligned}
\cos^2 x &= 1 - \sin^2 x \\
&= 1 - \left(x^2 - \tfrac{1}{3}x^4\right) \\
&= 1 - x^2 + \tfrac{1}{3}x^4
\end{aligned}$$

Question B5

(a) Given $\sum_{r=1}^{n} 3\left(r^2 - r\right) = (n-1)n(n+1)$

For $n = 1$ …

$$\text{LHS} = 3\left(1^2 - 1\right) = 3 \times 0 = 0$$

$$\text{RHS} = (1-1)\times 1 \times (1+1) = 0$$

Hence the statement is true for $n = 1$

Assume the statement is true for $n = k$, i.e.

$$\sum_{r=1}^{k} 3\left(r^2 - r\right) = (k-1)k(k+1)$$

For $n = k+1$ …

$$\begin{aligned}
\sum_{r=1}^{k+1} 3\left(r^2 - r\right) &= \sum_{r=1}^{k} 3\left(r^2 - r\right) + 3\left((k+1)^2 - (k+1)\right) \\
&= (k-1)k(k+1) + 3(k+1)^2 - 3(k+1) \\
&= (k+1)\left(k(k-1) + 3(k+1) - 3\right) \\
&= (k+1)\left(k^2 - k + 3k + \cancel{3} - \cancel{3}\right) \\
&= (k+1)\left(k^2 + 2k\right) \\
&= (k+1)k(k+2) \\
&= (k+1-1)(k+1)(k+1+1)
\end{aligned}$$

Hence the statement is true for $n = k+1$

Since the statement is true for $n = 1$ and $n = k+1$, then by the Principle of Mathematical Induction, it is true for all natural numbers $n \geq 1$.

(b) $\sum_{r=11}^{40} 3\left(r^2 - r\right) = \sum_{r=1}^{40} 3\left(r^2 - r\right) - \sum_{r=1}^{10} 3\left(r^2 - r\right)$

Using the result from part (a)…

$$\begin{aligned}
\sum_{r=11}^{40} 3\left(r^2 - r\right) &= (39 \times 40 \times 41) - (9 \times 10 \times 11) \\
&= 63960 - 990 \\
&= 62970
\end{aligned}$$

Question B6

From $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$ the Auxiliary Equation is…

$$m^2 - 4m + 4 = 0$$
$$(m-2)^2 = 0$$
$$m = 2$$

with resulting Complementary Function…

$$y = (A + Bx)e^{2x}$$

Particular Integral…

$$y = ae^x; \qquad \frac{dy}{dx} = ae^x; \qquad \frac{d^2y}{dx^2} = ae^x$$

Therefore $ae^x - \cancel{4ae^x} + \cancel{4ae^x} = e^x$

$$a = 1$$

Therefore the general solution is…

$$y = (A + Bx)e^{2x} + e^x$$

When $x = 0,\ y = 2$ so $Be^0 + 2(1+0)e^0 + e^0 = 1$

$$B + 2 + 1 = 1$$
$$B = -2$$

Therefore the particular solution is $y = (1-2x)e^{2x} + e^x$.